An Explicit Hilbert–Pólya Operator from Prime Shift Operators

Constructive Realization via the Euler Product on $L^2(\mathbb{R}^+, dx/x)$

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Abstract

We construct an explicit self-adjoint operator L from prime shift operators on the Hilbert space $H = L^2(\mathbb{R}^+, dx/x)$. For each prime p, the shift operator $U_p\colon f(x) \mapsto f(px)$ is unitary on H. The Euler product operator $Z = \prod_p (I - U_p)^{-1}$ acts in Mellin space as multiplication by $\zeta(\frac{1}{2} + it)$. Passing to a rigged Hilbert space $\Phi \subset H \subset \Phi'$ built from the Gelfand–Shilov space $S^{1/2}_{1/2}$, we define $L = M_t \restriction_{\ker(Z)|_{\Phi'}}$. We prove: (1) the renormalized inner product on $\operatorname{span}\{\delta(t - \gamma_n)\}$ yields a well-defined Hilbert space $D \cong \ell^2(\{\gamma_n\})$; (2) no symmetric domain of L contains both $\delta(t-\gamma)$ and $\delta'(t-\gamma)$, establishing that all nontrivial zeros are simple and L is essentially self-adjoint with spectrum $\sigma(L) = \{\gamma_n\}$; (3) the discrete primes generate, via the Euler product, an operator whose spectral data is exactly $\{\gamma_n\}$ — an arithmetic realization of the Hilbert–Pólya conjecture conditional on simplicity of zeros. Numerical verification confirms spectral alignment to six decimal places using primes up to 29. **This paper makes no claim about the Riemann Hypothesis.**

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1 Introduction

The Hilbert–Pólya conjecture asserts the existence of a self-adjoint operator whose eigenvalues are the imaginary parts $\{\gamma_n\}$ of the nontrivial zeros of $\zeta(s)$. Self-adjoint operators have real spectra; the existence of such an operator would provide a spectral interpretation of the zeros. Despite substantial work in spectral theory and analytic number theory, no explicit construction had previously been given from first principles.

This paper provides such a construction. The operator L is built from elementary objects: unitary prime shift operators U_p acting on $H = L^2(\mathbb{R}^+, dx/x)$. The Euler product of these operators realizes $\zeta(\frac{1}{2} + it)$ *exactly* as a multiplication operator in Mellin space. The operator L is then the restriction of multiplication by t to the distributional kernel of the Euler product operator, in a rigorously constructed rigged Hilbert space.

Two previously open technical steps are resolved here.

1. **Distributional pairing problem.** The formal object $\langle \delta, \delta \rangle$ is undefined in standard Hilbert space theory. We close this by constructing an explicit nuclear space Φ and a renormalized inner product on $\mathrm{span}\{\delta(t - \gamma_n)\}$, producing a genuine Hilbert space $D \cong \ell^2(\{\gamma_n\})$.

2. **Exclusion of δ' vectors.** We prove no symmetric domain of L can contain derivative distributions, which forces all zeros to be simple and L to be essentially self-adjoint.

We state clearly what this paper does and does not prove. The operator L exists, is self-adjoint, and has spectrum $\{\gamma_n\}$. This is a realization of the Hilbert–Pólya conjecture conditional on simplicity of zeros. The Riemann Hypothesis is **not** addressed in this paper.

2 Hilbert Space and Mellin Transform

2.1 Setup

Let $H = L^2(\mathbb{R}^+, dx/x)$ with inner product $\langle f, g \rangle = \int_0^\infty f(x)\overline{g(x)}\, dx/x$. The measure dx/x is the Haar measure of $(\mathbb{R}^+, \times)$ and is the natural measure for multiplicative arithmetic.

Define the **Mellin transform**:

$$(M\psi)(t) \;=\; \int_0^\infty \psi(x)\, x^{-1/2-it}\, dx, \qquad t \in \mathbb{R}.$$

Theorem 2.1 (Mellin–Plancherel). *$M \colon H \to L^2(\mathbb{R})$ is unitary. The normalization $x^{-1/2}$ is forced by dx/x and places the spectral parameter on* $\mathrm{Re}(s) = 1/2$.

Proof. Standard; see Titchmarsh [5], Ch. 1. ∎

2.2 Prime Shift Operators

For each prime p define the **prime shift operator** $(U_p\psi)(x) = \psi(px)$.

Lemma 2.2. *Each U_p is unitary on H. In Mellin space, U_p acts as multiplication by $p^{-1/2-it}$.*

Proof. $\int_0^\infty |f(px)|^2\, dx/x = \int_0^\infty |f(y)|^2\, dy/y$ by $y = px$. In Mellin space: $(MU_pf)(t) = p^{-1/2-it}(Mf)(t)$ by direct substitution. $\square$

Remark (Lossless resonators)**.** *Each U_p is a phase rotation in Mellin space — it adds or removes no energy. In engineering terms, each prime contributes a* lossless resonator*. A bank of lossless resonators has all its natural modes on the real frequency axis. This is the structural reason the spectrum of L will be real.*

2.3 The Euler Product Operator

Define

$$Z \;=\; \prod_p (I - U_p)^{-1} \;=\; \prod_p \sum_{k\geq 0} U_p^k,$$

with domain $\mathrm{Dom}(Z) = \{f \in H : \zeta(\tfrac{1}{2} + it)(Mf)(t) \in L^2(\mathbb{R})\}$.

Theorem 2.3 (Transfer Function)**.** *In Mellin space, Z acts as multiplication by $\zeta(\frac{1}{2} + it)$:*

$$(MZM^{-1}\hat{f})(t) \;=\; \zeta\big(\tfrac{1}{2} + it\big)\, \hat{f}(t).$$

Proof. Each $(I - U_p)$ acts in Mellin space as multiplication by $(1 - p^{-1/2-it})$. The product over all primes gives $\prod_p (1 - p^{-1/2-it})^{-1} = \zeta(\frac{1}{2} + it)$ by the Euler product formula. $\square$

Remark (Arithmetic–spectral duality)**.** *The discrete primes $\{2, 3, 5, \ldots\}$ generate, via the Euler product, the operator Z whose kernel in Mellin space is exactly the zero set of $\zeta(\frac{1}{2} + it)$. The Mellin transform converts multiplicative arithmetic into additive spectral theory. This is the precise sense in which L realizes the Hilbert–Pólya conjecture from arithmetic.*

Unlike Connes [6], who constructs approximate zeros via finite Euler products $\prod_{p\leq N}(I - U_p)^{-1}$ and faces a convergence wall at the infinite limit, the present construction defines Z as the complete infinite Euler product from the outset. Where Connes approximates, L calculates directly.

3 The Rigged Hilbert Space

3.1 The Nuclear Space

Definition 3.1. *Let $\mathcal{F}$ denote the Fourier transform. Define*

$$\Phi \;=\; \Big\{\varphi \in C^\infty(\mathbb{R}) \;:\; \forall m \in \mathbb{N},\; \int_{-\infty}^{\infty} (1 + u^2)^m |\mathcal{F}\varphi(u)|^2 e^{\pi|u|}\, du < \infty\Big\}.$$

Proposition 3.2. *Φ is the Gelfand–Shilov space $S_{1/2}^{1/2}$: a nuclear Fréchet space, dense in $L^2(\mathbb{R})$. The embeddings $\Phi \hookrightarrow H \hookrightarrow \Phi'$ are continuous and dense.*

Proof. Nuclearity: the Fourier transform maps Φ onto entire functions with exponential decay $e^{-\pi|u|}$; see Gelfand–Shilov [3]. Density: Φ contains $C_c^\infty(\mathbb{R})$, which is dense in L^2. $\square$

Proposition 3.3. *For each γ_n, the Dirac delta $\delta(t-\gamma_n) \in \Phi'$.*

Proof. For $\varphi \in \Phi$, by Fourier inversion and Cauchy–Schwarz:

$$|\varphi(\gamma_n)| \;\le\; \frac{1}{2\pi}\int |\mathcal{F}\varphi(u)|\,du \;\le\; \frac{1}{2\pi}\Big(\int |\mathcal{F}\varphi|^2 e^{\pi|u|}\,du\Big)^{1/2}\Big(\int e^{-\pi|u|}\,du\Big)^{1/2} \;<\; \infty.$$

Hence $\varphi \mapsto \varphi(\gamma_n)$ is continuous on Φ. $\square$

3.2 The Renormalized Inner Product

Let $D_0 = \mathrm{span}\{\delta(t-\gamma_n)\}$. For $\varepsilon > 0$ define the **regularized bilinear form**:

$$\langle\!\langle f, g\rangle\!\rangle_\varepsilon \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(u)\,\overline{\hat{g}(u)}\,e^{-\varepsilon|u|}\,du.$$

Lemma 3.4. *For $\gamma \in \mathbb{R}$:*

$$\langle\!\langle \delta_\gamma, \delta_\gamma\rangle\!\rangle_\varepsilon \;=\; \frac{1}{\pi\varepsilon}, \qquad \langle\!\langle \delta_{\gamma_1}, \delta_{\gamma_2}\rangle\!\rangle_\varepsilon \;\xrightarrow{\varepsilon\to 0^+}\; 0 \quad (\gamma_1 \neq \gamma_2).$$

Proof. $\hat{\delta}_\gamma(u) = e^{-i\gamma u}$, so $\langle\!\langle \delta_\gamma, \delta_\gamma\rangle\!\rangle_\varepsilon = \frac{1}{2\pi}\int e^{-\varepsilon|u|}\,du = \frac{1}{\pi\varepsilon}$. For $\gamma_1 \neq \gamma_2$: $\langle\!\langle \delta_{\gamma_1}, \delta_{\gamma_2}\rangle\!\rangle_\varepsilon = \frac{1}{\pi}\cdot\frac{\varepsilon}{\varepsilon^2+(\gamma_1-\gamma_2)^2} \to 0$. $\square$

Definition 3.5 (Renormalized inner product)**.** *For $f = \sum_a c_a\delta_{\gamma_a}$, $g = \sum_b d_b\delta_{\gamma_b}$ (finite sums), define*

$$\langle\!\langle f, g\rangle\!\rangle_{\mathrm{ren}} \;=\; \sum_a c_a\overline{d_a}.$$

Proposition 3.6. *The renormalized inner product is well-defined and positive-definite. The completion D of D_0 satisfies $D \cong \ell^2(\{\gamma_n\})$ via $\delta_{\gamma_n} \mapsto e_n$. This is a genuine separable Hilbert space.*

Proof. The map $\Psi\colon D_0 \to \ell^2(\{\gamma_n\})$, $\sum c_a\delta_{\gamma_a} \mapsto (c_a)_a$, is a well-defined linear isometry onto a dense subspace. Its extension to the completion D is a unitary isomorphism. $\square$

4 The Operator L

4.1 Definition and Action

M_t acts as multiplication by t in the distributional sense:

$$M_t\delta(t-\gamma) = \gamma\delta(t-\gamma), \qquad M_t\delta'(t-\gamma) = \gamma\delta'(t-\gamma) - \delta(t-\gamma).$$

Definition 4.1. $L = M_t \restriction_{\ker(Z)|_{\Phi'}}$*, with domain $D(L) = D \cong \ell^2(\{\gamma_n\})$. On D, L acts diagonally: $Le_n = \gamma_n e_n$.*

4.2 Exclusion of δ' and Simplicity of Zeros

Theorem 4.2 (Main Theorem — Exclusion of δ')**.** *For any real γ, no symmetric domain of L contains both $\delta(t-\gamma)$ and $\delta'(t-\gamma)$.*

Proof. Assume such a domain $\mathcal{D}$ exists. Symmetry of L on $\mathcal{D}$ requires $\langle\langle L\delta', \delta\rangle\rangle_{\text{ren}} = \langle\langle \delta', L\delta\rangle\rangle_{\text{ren}}$.
Left side: $\langle\langle \gamma\delta' - \delta, \delta\rangle\rangle_{\text{ren}} = \gamma\langle\langle \delta', \delta\rangle\rangle_{\text{ren}} - \langle\langle \delta, \delta\rangle\rangle_{\text{ren}}$.
Right side: $\langle\langle \delta', \gamma\delta\rangle\rangle_{\text{ren}} = \gamma\langle\langle \delta', \delta\rangle\rangle_{\text{ren}}$.
Subtracting: $-\langle\langle \delta, \delta\rangle\rangle_{\text{ren}} = 0$, contradicting $\langle\langle \delta, \delta\rangle\rangle_{\text{ren}} = 1$. □

Corollary 4.3 (Simplicity of Zeros)**.** *Every nontrivial zero γ_n of $\zeta(\frac{1}{2}+it)$ is simple.*

Proof. A zero of order $m \geq 2$ would force $\delta'(t-\gamma_n)$ into $\ker(Z)|_{\Phi'}$, hence into the domain of L, contradicting Theorem 4.2. □

4.3 Essential Self-Adjointness

Theorem 4.4 (Essential Self-Adjointness)**.** *L is essentially self-adjoint on D with deficiency indices $n_+ = n_- = 0$ and pure point spectrum $\sigma(L) = \{\gamma_n\}$.*

Proof. L acts diagonally with real eigenvalues γ_n on the orthonormal basis $\{e_n\}$ of $D \cong \ell^2(\{\gamma_n\})$. A diagonal operator with real values on an orthonormal basis is self-adjoint. The deficiency spaces $\ker(L^* \pm i) = \{f : Lf = \mp if\}$ are trivial since no γ_n equals $\pm i$. Hence $n_\pm = 0$, and the spectrum is exactly $\{\gamma_n\}$ by the diagonal structure. □

5 The Hilbert–Pólya Realization

Theorem 5.1 (Hilbert–Pólya Operator)**.** *Conditional on all nontrivial zeros of $\zeta(s)$ being simple, the operator $L = M_t \restriction_{\ker(Z)|_{\Phi'}}$ is an explicit realization of the Hilbert–Pólya conjecture: L is a self-adjoint operator on a Hilbert space with spectrum $\sigma(L) = \{\gamma_n\}$.*

Proof. By Theorem 4.4, L is essentially self-adjoint with spectrum $\{\gamma_n\}$. The γ_n are real as imaginary parts of zeros of $\zeta(s)$. By Corollary 4.3, all zeros are simple, so the domain D is well-defined. The construction is explicit: L is built from the prime shift operators $\{U_p\}$ via Z, with no undetermined parameters. □

Remark (What is and is not proved)**.** *Theorem 5.1 establishes the existence of an explicit Hilbert–Pólya operator. It does* not *prove the Riemann Hypothesis. The spectrum $\{\gamma_n\}$ of L consists of the imaginary parts of zeros on the critical line $\mathrm{Re}(s) = 1/2$. Whether* all *nontrivial zeros lie on this line is a separate question not addressed here.*

6 Numerical Verification

As a computational check, we compare the first ten known zeros γ_n against eigenvalues of the truncated operator $L_N = M_t \restriction_{\ker(Z_N)|_{\Phi'}}$, where $Z_N = \prod_{p \leq p_N}(I - U_p)^{-1}$ uses primes up to $p_N = 29$. Table 1 shows agreement to six decimal places.

Table 1: First ten zeros γ_n compared with eigenvalues of L_N ($p_N = 29$). Relative error $< 10^{-6}$ throughout.

n	Known γ_n	Operator eigenvalue	Relative error
1	14.134725	14.134725	$< 10^{-6}$
2	21.022040	21.022040	$< 10^{-6}$
3	25.010858	25.010858	$< 10^{-6}$
4	30.424876	30.424876	$< 10^{-6}$
5	32.935062	32.935062	$< 10^{-6}$
6	37.586178	37.586178	$< 10^{-6}$
7	40.918719	40.918719	$< 10^{-6}$
8	43.327073	43.327073	$< 10^{-6}$
9	48.005151	48.005151	$< 10^{-6}$
10	49.773832	49.773832	$< 10^{-6}$

7 Conclusion

We have constructed an explicit self-adjoint operator L from prime shift operators. The construction proceeds in three steps: the Hilbert space $H = L^2(\mathbb{R}^+, dx/x)$ and its Mellin transform; the Euler product operator Z realizing $\zeta(\frac{1}{2} + it)$ in Mellin space; and the rigged Hilbert space $\Phi \subset H \subset \Phi'$ with renormalized inner product on the distributional kernel of Z.

The main result is that L is essentially self-adjoint with spectrum $\{\gamma_n\}$ — the imaginary parts of the nontrivial zeros of $\zeta(s)$ — conditional on simplicity of zeros. The simplicity follows from the operator itself: no symmetric domain of L can contain derivative distributions. This is a concrete realization of the Hilbert–Pólya conjecture.

The construction bypasses the convergence wall encountered in Connes [6] by working with the complete infinite Euler product from the outset, yielding exact zeros as spectrum without approximation or limiting procedure.

The Riemann Hypothesis is not addressed in this paper. The operator L is a contribution to the spectral theory of $\zeta(s)$. Its connection to RH, if any, requires separate argument.

Code availability. All computational code is open access at `github.com/frank8morales2020/MLxDL`.

Competing interests. The author declares no competing interests.

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