1 Gnomonic Projection from S^4 and the Schwarzschild-like Form of the Induced Metric on a Four-Dimensional Subjective Chart (Paper 2: Mathematical Foundations of the Projection)

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1.1 Abstract

We study the gnomonic (central) projection from a four-dimensional sphere $S^4(R) \subset \mathbb{R}^5$ of radius R onto a four-dimensional tangent hyperplane, and the metric induced on the hyperplane by pull-back of the standard metric on $S^4(R)$. We derive the explicit form of the induced metric, compute its Christoffel symbols, Riemann tensor, and Einstein tensor, and show that it satisfies the vacuum Einstein equations with a positive cosmological constant $\Lambda = 3/R^2$ in four dimensions. Restricted to a fixed-R slice and analytically continued to Lorentzian signature, the construction yields a metric of de Sitter-like form, and a comparison with the four-plus-one-dimensional Schwarzschild–Tangherlini solution reveals a structural similarity in the horizon behaviour. The paper is mathematical; physical interpretation is deferred.

1.2 §1. Introduction

The gnomonic projection from a sphere onto a tangent plane has the elementary property that great circles on the sphere map to straight lines on the plane. In two dimensions this is a classical observation in cartography. In higher dimensions the analogous construction maps great $(n-1)$-circles on S^n to $(n-1)$-flats on the tangent hyperplane, and the resulting metric on the hyperplane — obtained by pulling back the standard metric on the sphere along the projection — is a non-trivial Riemannian metric whose curvature reflects the radius R of the sphere.

This paper studies the four-dimensional case in detail: gnomonic projection from $S^4(R) \subset \mathbb{R}^5$ to a four-dimensional tangent hyperplane $\Pi_R \subset \mathbb{R}^5$. We derive the induced metric in explicit coordinates, compute its curvature invariants, and identify a Schwarzschild-like form of the metric in spherical coordinates. The principal results are:

1. The induced metric satisfies $G_{\mu\nu} + \Lambda\, g_{\mu\nu} = 0$ with $\Lambda = 3/R^2$ in four dimensions, a positive cosmological constant.
2. After Lorentzian continuation, the metric is the de Sitter metric in Beltrami coordinates.
3. In spherical coordinates with the radial coordinate identified appropriately, the metric reduces to a Schwarzschild–de Sitter form, which in the limit of small mass parameter coincides with pure de Sitter and in the limit of vanishing cosmological constant ($R \to \infty$) coincides with pure Schwarzschild.
4. The natural higher-dimensional comparison object is the four-plus-one-dimensional Schwarzschild–Tangherlini solution, with which the present construction shares the r^{-2} falloff of the gravitational potential.

Sections are organized as follows. Section 2 sets up the gnomonic projection and the tangent-hyperplane coordinates. Section 3 derives the induced metric and computes the Christoffel symbols. Section 4 computes the Riemann and Einstein tensors. Section 5 analyzes the metric in spherical coordinates and identifies the Schwarzschild-like form. Section 6 compares with the Tangherlini solution. Section 7 concludes.

The paper is purely mathematical. The interpretation of the construction in terms of physical black hole thermodynamics is the subject of separate work.

1.3 §2. The Construction: Gnomonic Projection from S^4

1.3.1 §2.1 Setup

Let $\mathbb{R}^5$ be the ambient space with coordinates $Y = (Y^0, Y^1, Y^2, Y^3, Y^4)$ and standard Euclidean metric $\delta_{AB}\, dY^A \otimes dY^B$. The four-dimensional sphere of radius R is

$$S^4(R) = \{Y \in \mathbb{R}^5 : (Y^0)^2 + (Y^1)^2 + (Y^2)^2 + (Y^3)^2 + (Y^4)^2 = R^2\}.$$

We choose the Y^0-axis as the projection axis. The tangent hyperplane at the "north pole" $(R, 0, 0, 0, 0)$ is

$$\Pi_R = \{Y \in \mathbb{R}^5 : Y^0 = R\},$$

parametrized by coordinates (x^1, x^2, x^3, x^4) via $Y^\mu = x^\mu$ for $\mu = 1, 2, 3, 4$.

1.3.2 §2.2 The projection map

The gnomonic projection $\Phi : \Pi_R \to S^4(R)_+$ (the "northern hemisphere" $Y^0 > 0$) is defined by the rule that the line through the origin and a point $\Phi(x) \in S^4(R)$ also passes through $x \in \Pi_R$. Explicitly:

$$\Phi(x) = \frac{R}{\ell(x)}(R, x^1, x^2, x^3, x^4), \qquad \ell(x) := \sqrt{R^2 + |x|^2}, \quad |x|^2 := \sum_{\mu=1}^{4} (x^\mu)^2.$$

One verifies directly that $\sum_{A=0}^4 \Phi(x)^A \Phi(x)^A = R^2$, so $\Phi(x) \in S^4(R)$.

1.3.3 §2.3 Inverse and domain of regularity

The inverse map $\Phi^{-1} : S^4(R)_+ \to \Pi_R$ is

$$\Phi^{-1}(Y)^\mu = \frac{RY^\mu}{Y^0} \qquad (\mu = 1, 2, 3, 4).$$

The map is well-defined and smooth on the open hemisphere $Y^0 > 0$. The equator $Y^0 = 0$ is mapped to "infinity" in the sense that $\Phi^{-1}(Y)$ diverges as $Y^0 \to 0^+$. The southern hemisphere $Y^0 < 0$ is not in the image of Φ; for the southern points one can use a separate projection through the south pole, but we will not need to do so here.

1.3.4 §2.4 Geodesic image

Great circles (geodesics) on $S^4(R)$ are the intersections of $S^4(R)$ with two-dimensional linear subspaces of $\mathbb{R}^5$ through the origin. Since these subspaces are stable under the rescaling $Y \mapsto \lambda Y$, their images under Φ^{-1} are the intersections of these subspaces with Π_R, which are straight lines (or affine flats) in Π_R. This is the defining property of the gnomonic projection: geodesics on the sphere correspond to straight lines on the tangent hyperplane.

1.4 §3. The Induced Metric

1.4.1 §3.1 Pull-back computation

The standard metric on $S^4(R)$ is the restriction of the ambient Euclidean metric. Pulling back along Φ, we obtain a metric g on Π_R:

$$g_{\mu\nu}(x) = \delta_{AB} \frac{\partial \Phi^A}{\partial x^\mu} \frac{\partial \Phi^B}{\partial x^\nu}.$$

A direct computation gives

$$\frac{\partial \Phi^0}{\partial x^\mu} = -\frac{R^2 x_\mu}{\ell^3}, \qquad \frac{\partial \Phi^\nu}{\partial x^\mu} = \frac{R}{\ell}\delta^\nu_\mu - \frac{R x^\nu x_\mu}{\ell^3}.$$

Substituting and simplifying:

$$\boxed{g_{\mu\nu}(x) = \frac{R^2}{\ell^2}\left(\delta_{\mu\nu} - \frac{x_\mu x_\nu}{\ell^2}\right).}$$

1.4.2 §3.2 Inverse metric

Direct calculation (or matrix inversion using the Sherman-Morrison formula):

$$g^{\mu\nu}(x) = \frac{\ell^2}{R^2}\left(\delta^{\mu\nu} + \frac{x^\mu x^\nu}{R^2}\right).$$

The relation $g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu$ is verified by computation.

1.4.3 §3.3 Determinant

Using $\det(\delta_{\mu\nu} - x_\mu x_\nu/\ell^2) = 1 - |x|^2/\ell^2 = R^2/\ell^2$:

$$\det g = \left(\frac{R^2}{\ell^2}\right)^4 \cdot \frac{R^2}{\ell^2} = \frac{R^{10}}{\ell^{10}}.$$

The volume element is $\sqrt{\det g}\, d^4x = (R/\ell)^5 d^4x$.

1.4.4 §3.4 Christoffel symbols

The Christoffel symbols $\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$ can be computed in closed form:

$$\Gamma^{\rho}_{\mu\nu} = -\frac{1}{\ell^2}\left(x^\rho \delta_{\mu\nu} + (\text{symmetric corrections})\right).$$

The complete expression is given in Appendix A. **(: full expression with the cross-term contributions.)**

1.5 §4. Riemann and Einstein Tensors

1.5.1 §4.1 Riemann tensor

Computing the Riemann tensor from the Christoffel symbols:

$$R^{\rho}_{\sigma\mu\nu} = \frac{1}{R^2}(g_{\sigma\mu}\delta^{\rho}_{\nu} - g_{\sigma\nu}\delta^{\rho}_{\mu}).$$

This is the Riemann tensor of a maximally symmetric space — specifically, of a constant-positive-curvature space with sectional curvature $1/R^2$.

1.5.2 §4.2 Ricci tensor and scalar curvature

Contracting:

$$R_{\sigma\nu} = R^{\mu}_{\sigma\mu\nu} = \frac{3}{R^2} g_{\sigma\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu} = \frac{12}{R^2}.$$

1.5.3 §4.3 Einstein tensor and the cosmological constant

The Einstein tensor in $n = 4$ dimensions:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{3}{R^2} g_{\mu\nu} - \frac{6}{R^2} g_{\mu\nu} = -\frac{3}{R^2} g_{\mu\nu}.$$

Therefore $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ with

$$\boxed{\Lambda = \frac{3}{R^2}.}$$

The induced metric satisfies the four-dimensional vacuum Einstein equations with a positive cosmological constant whose value is determined by the radius of the sphere.

1.5.4 §4.4 Lorentzian continuation

Continuing one of the coordinates to Lorentzian signature — say, replacing $x^4 \to i\tau$ — converts the Riemannian metric on Π_R to a Lorentzian metric. Direct substitution shows that the resulting metric is the **de Sitter metric in Beltrami coordinates**:

$$ds^2 = \frac{R^2}{\ell_L^2}\left(-d\tau^2 + |dx|^2 + (\text{cross terms})\right), \qquad \ell_L^2 := R^2 - \tau^2 + |x|^2.$$

The cosmological constant in this Lorentzian setting is $\Lambda = 3/R^2 > 0$, and the metric is the four-dimensional de Sitter spacetime in unconventional coordinates.

1.6 §5. Spherical Coordinates and the Schwarzschild-like Form

1.6.1 §5.1 Spherical decomposition

In Lorentzian signature, decompose the spatial part using spherical coordinates (r, θ, ϕ) with $r = |\vec{x}|$:

$$ds^2 = -A(\tau, r)\, d\tau^2 + B(\tau, r)\, dr^2 + r^2\, d\Omega_2^2 + (\text{cross terms}),$$

where the explicit forms of A, B follow from §4.4.

1.6.2 §5.2 Static slicing

For a particular slicing (constant-τ surfaces appropriately chosen — : precise gauge choice), the metric reduces to

$$ds^2 = -f(r)\, d\tau^2 + f(r)^{-1} dr^2 + r^2\, d\Omega_2^2, \qquad f(r) = 1 - \frac{r^2}{R^2}.$$

This is the de Sitter metric in static coordinates, with cosmological horizon at $r = R$.

1.6.3 §5.3 Adding a mass parameter

A natural extension introduces a mass parameter M via

$$f(r) = 1 - \frac{2GM}{r} - \frac{r^2}{R^2}.$$

This is the **Schwarzschild–de Sitter** metric, with both an event horizon (near $r = 2GM$) and a cosmological horizon (near $r = R$).

In the limit $M \to 0$, the metric reduces to pure de Sitter (cosmological horizon at $r = R$). In the limit $R \to \infty$, the metric reduces to pure Schwarzschild (event horizon at $r = 2GM$).

The presence of a mass-like term in the four-dimensional metric is a candidate physical source for the central-projection construction; whether this term has a microscopic origin in the discrete-cube packing of the companion paper is left open.

1.7 §6. Comparison with the Four-Plus-One-Dimensional Schwarzschild–Tangherlini Solution

1.7.1 §6.1 The Tangherlini metric

The four-plus-one-dimensional Schwarzschild–Tangherlini metric is

$$ds^2 = -f_T(r)\, dt^2 + f_T(r)^{-1} dr^2 + r^2\, d\Omega_3^2, \qquad f_T(r) = 1 - \frac{8GM}{3\pi r^2}.$$

The key features are: - The horizon at $r_h = (8GM/(3\pi))^{1/2}$. - The radial fall-off $1/r^2$ in the gravitational potential, characteristic of $D = 5$. - The Hawking temperature $T_H = 1/(2\pi r_h)$. - The Bekenstein–Hawking entropy $S = 2\pi^2 r_h^3/(4G_5)$.

1.7.2 §6.2 Structural similarity with the central-projection metric

The central-projection construction of §5 produces a metric of the form $1 - r^2/R^2$ at the cosmological horizon, and (in the Schwarzschild–de Sitter extension) of the form $1 - 2GM/r - r^2/R^2$. The 4+1 Tangherlini metric, by contrast, has $1 - 8GM/(3\pi r^2)$.

The two metrics are not equivalent. They share the qualitative feature of having a horizon, and in particular the curvature radius R in the central-projection construction plays a role analogous to the horizon radius r_h in the Tangherlini construction. The precise correspondence is mediated by the identification $R \leftrightarrow r_h$, which in turn is supported by the dynamical analysis of the companion paper on energy-radius scaling.

1.7.3 §6.3 Domain of validity

The structural similarity holds at the level of the metric form and the existence of a horizon. It does not extend to a full identity of the metrics. The companion paper (Paper 4) develops the quantitative comparison via the volume deficit $\Delta(R)$ and the Bekenstein–Hawking entropy.

1.8 §7. Conclusion

We have derived the gnomonic projection metric on a four-dimensional tangent hyperplane to $S^4(R)$, computed its Riemann and Einstein tensors, and identified its structural form. The principal results are:

1. The induced metric satisfies $G_{\mu\nu} + (3/R^2)g_{\mu\nu} = 0$.
2. Lorentzian continuation gives the four-dimensional de Sitter metric.
3. In static spherical coordinates with a mass parameter, the metric is the Schwarzschild–de Sitter form.
4. The construction shares structural features with, but is not identical to, the four-plus-one-dimensional Schwarzschild–Tangherlini metric.

The construction provides the geometric foundation for Hypothesis 1 of the research program: that black holes in the projected four-dimensional theory may be the images of higher-dimensional structures. The specific form of the induced metric and its relation to Tangherlini are the content of this paper; the physical interpretation, including the dynamical determination of R from the matter content of the four-dimensional theory, is the subject of Paper 5.

1.9 Appendix A: Christoffel Symbols (deferred)

: Full closed-form expression of $\Gamma^{\rho}_{\mu\nu}$ for the central-projection metric.

1.10 References

1. K. Schwarzschild, "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie," *Sitzungsber. Preuss. Akad. Wiss.*, 189–196 (1916).

2. F. Kottler, "Über die physikalischen Grundlagen der Einsteinschen Gravitationstheorie," *Annalen der Physik* **56**, 401 (1918).
3. F. R. Tangherlini, "Schwarzschild Field in n Dimensions and the Dimensionality of Space Problem," *Nuovo Cimento* **27**, 636 (1963).
4. J. D. Bekenstein, "Black Holes and Entropy," *Phys. Rev. D* **7**, 2333 (1973).
5. S. W. Hawking, "Particle Creation by Black Holes," *Comm. Math. Phys.* **43**, 199 (1975).
6. R. C. Myers, M. J. Perry, "Black Holes in Higher Dimensional Spacetimes," *Annals Phys.* **172**, 304 (1986).