A Local Scale Gap for Inverse Rational Zero-Modes

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Abstract

We study the complete inverse rational sum

$$K_t(A) = \sum_{\substack{x \bmod t \\ (x(x+1),t)=1}} e_t\Big(A\,\overline{x(x+1)}\Big), \qquad e_t(z) = \exp(2\pi i z/t),$$

where A is a nonzero integer and $\overline{x(x+1)}$ denotes the inverse of $x(x+1)$ modulo the relevant modulus. We prove an exact conductor-drop identity, a primitive odd-conductor square-root estimate, and a weighted short-modulus estimate.

The main global consequence is deliberately modest. On prime-supported primitive ranges, the density quantity $\#U_p/p$ is bounded below by a positive absolute constant, while the normalized oscillatory quantity $K_p(A)/p$ has square-root size. Thus, under positive weights supported away from small primes, the density contribution and the oscillatory zero-mode contribution have different natural scales.

The paper does not claim a terminal closure theorem. It proves a local arithmetic scale gap and records precisely what additional mechanisms would be needed to turn this local gap into a statement about a concrete global problem.

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1 Introduction

This paper isolates a local arithmetic phenomenon which can be obscured inside a larger terminal or adjacent-modulus analysis.

Let

$$U_t := \{x \bmod t : (x(x+1), t) = 1\}.$$

For a nonzero integer A, define

$$K_t(A) = \sum_{x \in U_t} e_t\Big(A\,\overline{x(x+1)}\Big).$$

The phase is the inverse rational phase $A(x(x+1))^{-1}$ on the admissible residue set. This is the convention used throughout the paper.

There are two different scales in the problem.

First, the density scale is governed by $\#U_t$. For odd t,

$$\#U_t = t \prod_{p|t} \left(1 - \frac{2}{p}\right).$$

In particular, for an odd prime p,

$$\#U_p = p - 2.$$

Thus $\#U_p/p$ is bounded below by a positive absolute constant.

Second, the oscillatory scale is governed by $K_t(A)$. On primitive odd conductors, that is, when $(A,t) = 1$, the sum satisfies

$$K_t(A) \ll_\varepsilon t^{1/2+\varepsilon}.$$

Therefore

$$\frac{K_p(A)}{p}$$

has square-root normalized size on prime moduli.

The purpose of the paper is not to close a global terminal theorem. Instead, it proves the local arithmetic facts cleanly and derives a scale-gap criterion which any concrete global argument must confront.

1.1 Main local results

The first result is exact conductor drop. If t is odd, A is nonzero, $h = (A, t)$, $q = t/h$, and $A_0 = A/h$, then

$$K_t(A) = \Lambda_t(A) K_q(A_0),$$

with an explicit multiplicative factor $\Lambda_t(A)$. The proof uses the fact that modular inverses reduce correctly to lower moduli.

The second result is primitive square-root cancellation:

$$(A, q) = 1, \quad q \text{ odd} \qquad \Longrightarrow \qquad K_q(A) \ll_\varepsilon q^{1/2+\varepsilon}.$$

The prime-field case follows from the Weil bound for rational functions, and prime powers follow from one-dimensional non-degenerate p-adic stationary phase.

The third result is a weighted short-modulus bound:

$$\sum_{t \le L} \frac{|K_t(A)|}{t} \ll_\varepsilon L^{1/2+\varepsilon} |A|^\varepsilon.$$

1.2 Main scale-gap consequence

Let $\mathcal{P}$ be a set of odd primes not dividing A, and let $b_p \ge 0$ be weights. Define

$$D_{\mathcal{P}} := \sum_{p \in \mathcal{P}} b_p \frac{\# U_p}{p}, \qquad O_{\mathcal{P}} := \operatorname{Re} \sum_{p \in \mathcal{P}} b_p \frac{K_p(A)}{p}.$$

If all primes in $\mathcal{P}$ are at least Y, then

$$D_{\mathcal{P}} \gg \sum_{p \in \mathcal{P}} b_p, \qquad |O_{\mathcal{P}}| \ll Y^{-1/2} \sum_{p \in \mathcal{P}} b_p.$$

Thus, for sufficiently large Y, the density term cannot be matched to lower order by the oscillatory zero-mode term.

1.3 What is not claimed

This paper does not claim that a concrete terminal expression automatically reduces to the prime-supported model above. It also does not claim that no global theorem can overcome the gap. A concrete problem may avoid the scale gap by support collapse, diagonal renormalization, or a genuinely global cancellation theorem.

The result proved here is narrower:

inverse rational zero-modes have density scale and oscillatory scale separated on primitive support.

2 Notation and Basic Definitions

For a positive integer t, set

$$e_t(z) := \exp(2\pi i z/t).$$

If u is a unit modulo t, its inverse modulo t is denoted by $\bar{u}$. Thus $\bar{u}$ always means the inverse modulo the modulus currently under discussion.

For a nonzero integer A, $v_p(A)$ means $v_p(|A|)$.

Definition 2.1 (Admissible residue set)**.** For $t \geq 1$, define

$$U_t := \{x \bmod t : (x(x+1), t) = 1\}.$$

Definition 2.2 (Inverse rational zero-mode)**.** For a nonzero integer A, define

$$K_t(A) := \sum_{x \in U_t} e_t\Big(A\,\overline{x(x+1)}\Big). \tag{2.1}$$

The expression $\overline{x(x+1)}$ in (2.1) denotes the inverse of $x(x+1)$ modulo t. Since $x \in U_t$, this inverse exists.

Remark 2.3 (Phase convention)*.* The phase is always the inverse rational phase

$$A(x(x+1))^{-1}.$$

This convention is fixed once and for all. No result below concerns the quadratic polynomial phase.

Lemma 2.4 (Even moduli vanish)**.** *If $2 \mid t$, then $U_t = \varnothing$. Hence*

$$K_t(A) = 0.$$

Proof. For every integer x, one of x and $x+1$ is even. Thus $2 \mid x(x+1)$. If $2 \mid t$, then $(x(x+1), t) > 1$, so no residue class is admissible. □

Lemma 2.5 (Size of the admissible set)**.** *If t is odd, then*

$$\#U_t = t \prod_{p \mid t} \left(1 - \frac{2}{p}\right). \tag{2.2}$$

In particular, if p is an odd prime, then

$$\#U_p = p - 2.$$

Proof. The condition $(x(x+1), t) = 1$ is equivalent to requiring

$$x \not\equiv 0, -1 \pmod{p}$$

for every prime $p \mid t$. Since t is odd, these two forbidden classes are distinct modulo each such prime p. The Chinese remainder theorem gives (2.2). □

3 Exact Conductor Drop

We now prove the exact conductor-drop identity for the inverse rational zero-mode. The key point is that inverse residues are compatible with reduction to lower moduli.

Theorem 3.1 (Exact conductor drop)**.** *Let A be a nonzero integer and let t be odd. Write*

$$t = \prod_{p^k \| t} p^k.$$

For each $p^k \| t$, put

$$v_p := \min(v_p(A), k).$$

Define

$$h := (A,t) = \prod_{p^k \| t} p^{v_p}, \qquad q := \frac{t}{h}, \qquad A_0 := \frac{A}{h}.$$

Then $(A_0, q) = 1$, and

$$K_t(A) = \Lambda_t(A) K_q(A_0), \tag{3.1}$$

where

$$\Lambda_t(A) := \prod_{\substack{p^k \| t \\ v_p(A) < k}} p^{v_p(A)} \prod_{\substack{p^k \| t \\ v_p(A) \geq k}} p^{k-1}(p-2). \tag{3.2}$$

We use the convention $K_1(A_0) = 1$.

Proof. Let $h = (A,t)$, $q = t/h$, and $A = hA_0$. For $x \in U_t$, choose

$$u_t \equiv \overline{x(x+1)} \pmod{t}.$$

Then

$$x(x+1)u_t \equiv 1 \pmod{t}.$$

Since $q \mid t$, reducing modulo q gives

$$x(x+1)u_t \equiv 1 \pmod{q}.$$

Therefore $u_t \bmod q$ is the inverse of $x(x+1)$ modulo q. It follows that

$$e_t\Big(A\,\overline{x(x+1)}\Big) = e_{hq}(hA_0 u_t) = e_q(A_0 u_t) = e_q\Big(A_0\,\overline{x(x+1)}\Big),$$

where the inverse on the left is taken modulo t, while the inverse on the right is taken modulo q. Hence the phase factors through the reduction map

$$U_t \longrightarrow U_q.$$

It remains to compute the size of each fiber of this reduction map. This is multiplicative over the prime powers dividing t.

Fix $p^k \| t$ and put $v = \min(v_p(A), k)$. If $v < k$, then the local modulus in q is p^{k-v}. For a fixed admissible class $y \bmod p^{k-v}$, the lifts $x \bmod p^k$ with

$$x \equiv y \pmod{p^{k-v}}$$

are exactly p^v in number. Since $p \nmid y(y+1)$, every lift also satisfies $p \nmid x(x+1)$. Thus the local fiber size is p^v.

If $v = k$, then the p-part disappears from q. The local fiber size is the number of admissible residues modulo p^k, namely

$$p^k - 2p^{k-1} = p^{k-1}(p-2).$$

Multiplying these local fiber sizes over all prime powers $p^k \| t$ gives $\Lambda_t(A)$, and (3.1) follows. □

Corollary 3.2 (Conductor-drop bound)**.** *For odd t and nonzero integer A,*

$$|K_t(A)| \le \Lambda_t(A) \left| K_{t/(A,t)}\left(\frac{A}{(A,t)}\right) \right|.$$

Moreover,

$$\Lambda_t(A) \le (A,t).$$

Proof. The first statement is Theorem 3.1. For the second, compare local factors. If $v_p(A) < k$, the local factor is $p^{v_p(A)}$. If $v_p(A) \ge k$, the local factor is $p^{k-1}(p-2) \le p^k$. Hence the product is at most (A,t). □

4 Primitive Square-Root Estimate

We prove the primitive square-root estimate. The prime-field case uses the Weil bound for rational functions. The prime-power case uses one-dimensional non-degenerate p-adic stationary phase.

Proposition 4.1 (Primitive estimate)**.** *Let q be odd and $(A,q) = 1$. Then, for every $\varepsilon > 0$,*

$$K_q(A) \ll_\varepsilon q^{1/2+\varepsilon}.$$

More precisely, there is an absolute constant C such that

$$|K_q(A)| \le C^{\omega(q)} q^{1/2}. \tag{4.1}$$

Proof. Let

$$q = \prod_{p^k \| q} p^k.$$

By the Chinese remainder theorem, a residue class x mod q corresponds to a tuple

$$(x_p \bmod p^k)_{p^k \| q}.$$

Moreover,

$$(x(x+1), q) = 1$$

if and only if

$$p \nmid x_p(x_p+1) \qquad (p^k \| q)$$

for every prime divisor p of q. Thus the admissible set decomposes as a product of local admissible sets.

If

$$u \equiv \overline{x(x+1)} \pmod q,$$

then reducing the congruence

$$x(x+1)u \equiv 1 \pmod q$$

modulo p^k shows that u mod p^k is the inverse of $x_p(x_p+1)$ modulo p^k. Hence the inverse rational phase is compatible with the Chinese remainder decomposition.

Finally, the additive character e_q decomposes into a product of local additive characters. This introduces local unit coefficients in front of the rational phases. Since $(A,q) = 1$, all these local coefficients are p-adic units. Therefore it suffices to prove the local bound for $q = p^k$, uniformly in unit coefficients.

Let $q = p^k$, where p is odd and $p \nmid A$. On the p-adic unit domain $p \nmid x(x+1)$, set

$$f(x) = A(x(x+1))^{-1}.$$

This function is p-adic analytic on every admissible residue class. Its derivative is

$$f'(x) = -A(2x+1)(x(x+1))^{-2}.$$

Since $p \nmid A$, the critical equation modulo p is

$$2x + 1 \equiv 0 \pmod{p}.$$

Thus there is a unique critical residue

$$x_0 \equiv -\frac{1}{2} \pmod{p}.$$

This residue is admissible because

$$x_0(x_0+1) = -\frac{1}{4}$$

is a p-adic unit.

Differentiating again gives

$$f''(x) = A\left[2(2x+1)^2(x(x+1))^{-3} - 2(x(x+1))^{-2}\right].$$

At $x_0 = -1/2$, the first term vanishes and

$$(x_0(x_0+1))^{-2} = \left(-\frac{1}{4}\right)^{-2} = 16.$$

Hence

$$f''(x_0) = -32A.$$

Since p is odd and $p \nmid A$, this is a p-adic unit. The critical point is non-degenerate.

For $k \geq 2$, first-derivative cancellation eliminates all non-stationary residue classes modulo p, while the unique non-degenerate stationary class contributes $O(p^{k/2})$. For $k = 1$, the Weil bound for additive character sums over rational functions gives $O(p^{1/2})$. Therefore

$$K_{p^k}(A) \ll p^{k/2}.$$

Multiplying the local bounds over $p^k \| q$ gives

$$|K_q(A)| \leq C^{\omega(q)} q^{1/2} \ll_\varepsilon q^{1/2+\varepsilon}.$$

□

5 Weighted Short-Modulus Bound

Proposition 5.1 (Weighted short-modulus bound)**.** *Let A be a nonzero integer and let $L \geq 1$. Then*

$$\sum_{t \leq L} \frac{|K_t(A)|}{t} \ll_\varepsilon L^{1/2+\varepsilon}|A|^\varepsilon. \tag{5.1}$$

More precisely,

$$\sum_{t \leq L} \frac{|K_t(A)|}{t} \ll_\varepsilon L^{1/2+\varepsilon}\sigma_{-1/2}(|A|), \qquad \sigma_{-1/2}(n) := \sum_{d|n} d^{-1/2}.$$

Proof. By Lemma 2.4, even moduli contribute zero. For odd t, Corollary 3.2 and Proposition 4.1 give

$$|K_t(A)| \ll_\varepsilon (A,t)\left(\frac{t}{(A,t)}\right)^{1/2+\varepsilon}.$$

Since $(A,t) \leq t$, this implies

$$|K_t(A)| \ll_\varepsilon (t(A,t))^{1/2}t^\varepsilon.$$

Therefore

$$\frac{|K_t(A)|}{t} \ll_\varepsilon (A,t)^{1/2}t^{-1/2+\varepsilon}.$$

Using

$$(A,t)^{1/2} \leq \sum_{d|(A,t)} d^{1/2},$$

we get

$$\sum_{t \leq L} \frac{|K_t(A)|}{t} \ll_\varepsilon \sum_{d||A|} d^{1/2} \sum_{\substack{t \leq L \\ d|t}} t^{-1/2+\varepsilon}.$$

Writing $t = dn$, the inner sum satisfies

$$\sum_{\substack{t \leq L \\ d|t}} t^{-1/2+\varepsilon} = d^{-1/2+\varepsilon} \sum_{n \leq L/d} n^{-1/2+\varepsilon} \ll_\varepsilon d^{-1/2+\varepsilon}(L/d)^{1/2+\varepsilon} = L^{1/2+\varepsilon}d^{-1}.$$

After multiplying by $d^{1/2}$, the contribution of d is

$$\ll_\varepsilon L^{1/2+\varepsilon}d^{-1/2}.$$

Summing over $d \mid |A|$ proves the refined estimate, and $\sigma_{-1/2}(|A|) \ll_\varepsilon |A|^\varepsilon$ gives (5.1). □

6 Prime-Supported Scale Gap

We now state the clean scale-gap consequence. This is the main global-looking result of the paper, but it is still local in nature: it concerns a prescribed prime-supported model and does not assert that every terminal problem reduces to this model.

Let A be a nonzero integer. Let $\mathcal{P}$ be a finite set of odd primes such that $p \nmid A$ for every $p \in \mathcal{P}$. Let $(b_p)_{p\in\mathcal{P}}$ be nonnegative real weights.

Define

$$D_{\mathcal{P}} := \sum_{p\in\mathcal{P}} b_p \frac{\#U_p}{p},$$

and

$$O_{\mathcal{P}} := \mathrm{Re} \sum_{p\in\mathcal{P}} b_p \frac{K_p(A)}{p}.$$

Theorem 6.1 (Prime-supported scale gap)**.** *Let C be the absolute constant in the prime-modulus bound*

$$|K_p(A)| \le Cp^{1/2}$$

for $p \nmid A$. Assume that every $p \in \mathcal{P}$ satisfies $p \ge Y$. Then

$$D_{\mathcal{P}} \ge \frac{1}{3} \sum_{p\in\mathcal{P}} b_p,$$

and

$$|O_{\mathcal{P}}| \le CY^{-1/2} \sum_{p\in\mathcal{P}} b_p.$$

Consequently, if $Y > 36C^2$, then

$$D_{\mathcal{P}} - O_{\mathcal{P}} \ge \frac{1}{6} \sum_{p\in\mathcal{P}} b_p.$$

Proof. For an odd prime p,

$$\#U_p = p - 2,$$

and therefore

$$\frac{\#U_p}{p} = 1 - \frac{2}{p} \ge \frac{1}{3}.$$

This proves the lower bound for $D_{\mathcal{P}}$.

Since $p \nmid A$, Proposition 4.1 in the prime case gives

$$|K_p(A)| \le Cp^{1/2}.$$

Thus

$$\left|\frac{K_p(A)}{p}\right| \le Cp^{-1/2} \le CY^{-1/2}.$$

Therefore

$$|O_{\mathcal{P}}| \le \sum_{p\in\mathcal{P}} b_p \frac{|K_p(A)|}{p} \le CY^{-1/2} \sum_{p\in\mathcal{P}} b_p.$$

Combining this with the density lower bound, we obtain

$$D_{\mathcal{P}} - O_{\mathcal{P}} \ge D_{\mathcal{P}} - |O_{\mathcal{P}}| \ge \left(\frac{1}{3} - CY^{-1/2}\right) \sum_{p\in\mathcal{P}} b_p.$$

If $Y > 36C^2$, then $CY^{-1/2} < 1/6$, and the final assertion follows. □

Remark 6.2 (Meaning of the gap)*.* The conclusion is one-sided. It says that, on positive prime-supported primitive ranges away from small primes, the oscillatory zero-mode cannot approximate the density contribution to lower order. It does not assert that every global expression has this prime-supported form.

7 Weighted Forms of the Scale Gap

The prime-supported model of Theorem 6.1 is a concrete positive-weight instance of the more flexible weighted formalism below. Indeed, if $W_p = (Q/N)b_p$ on a prime set $\mathcal{P}$ and $W_t = 0$ otherwise, then the density functional below reduces to $D_{\mathcal{P}}$. The following corollary keeps the normalization N/Q, which is convenient in applications.

Let $L = Q^2/N$, and let $(W_t)_{t \le L}$ be a complex sequence. Define

$$D_{\text{dens}}(W) := \frac{N}{Q} \operatorname{Re} \sum_{t \le L} \frac{W_t}{t} \# U_t,$$

and

$$M_{\text{osc}}(W; A) := \frac{N}{Q} \operatorname{Re} \sum_{t \le L} \frac{W_t}{t} K_t(A).$$

Corollary 7.1 (Abstract weighted obstruction). *Let A be a nonzero integer. Suppose that $|W_t| \ll Q$, and that*

$$D_{\text{dens}}(W) \gg \frac{Q^2}{\log L}.$$

If

$$QN^{1/2+\varepsilon}|A|^{\varepsilon} = o\left(\frac{Q^2}{\log L}\right),$$

then

$$D_{\text{dens}}(W) - M_{\text{osc}}(W; A) \gg \frac{Q^2}{\log L}.$$

Proof. By Proposition 5.1,

$$\sum_{t \le L} \frac{|K_t(A)|}{t} \ll_{\varepsilon} L^{1/2+\varepsilon}|A|^{\varepsilon}.$$

Thus

$$|M_{\text{osc}}(W; A)| \le \frac{N}{Q} \sum_{t \le L} \frac{|W_t|}{t} |K_t(A)| \ll_{\varepsilon} NL^{1/2+\varepsilon}|A|^{\varepsilon}.$$

Since $L = Q^2/N$,

$$|M_{\text{osc}}(W; A)| \ll_{\varepsilon} QN^{1/2+\varepsilon}|A|^{\varepsilon}.$$

The scale-separation assumption makes this $o(Q^2/\log L)$. Therefore

$$D_{\text{dens}}(W) - M_{\text{osc}}(W; A) \ge D_{\text{dens}}(W) - |M_{\text{osc}}(W; A)| \gg \frac{Q^2}{\log L}.$$

□

Remark 7.2. Corollary 7.1 is conditional on the density lower bound. It does not prove that a concrete problem supplies such a bound.

Corollary 7.3 (A sufficient positivity condition). *Let $L = Q^2/N$. Suppose that $\operatorname{Re} W_t \ge 0$ for all $t \le L$. Assume that there exists a set*

$$\mathcal{P} \subseteq \{p \le L : p \text{ is an odd prime},\ p \nmid A\}$$

such that

$$\#\mathcal{P} \gg \frac{L}{\log L}$$

and

$$\operatorname{Re} W_p \geq cQ \qquad (p \in \mathcal{P})$$

for some fixed $c > 0$. *Then*

$$D_{\mathrm{dens}}(W) \gg \frac{Q^2}{\log L}.$$

Consequently, if the scale-separation hypothesis of Corollary 7.1 also holds, then

$$D_{\mathrm{dens}}(W) - M_{\mathrm{osc}}(W; A) \gg \frac{Q^2}{\log L}.$$

Proof. For $p \in \mathcal{P}$, Lemma 2.5 gives

$$\#U_p = p - 2, \qquad \frac{\#U_p}{p} = 1 - \frac{2}{p} \geq \frac{1}{3}.$$

Since $\#U_p/p$ is real and positive,

$$\operatorname{Re}\left(\frac{W_p}{p}\#U_p\right) = \frac{\#U_p}{p}\operatorname{Re} W_p \geq \frac{cQ}{3}.$$

The remaining terms have nonnegative real part by assumption. Hence

$$D_{\mathrm{dens}}(W) \geq \frac{N}{Q}\sum_{p\in\mathcal{P}} \operatorname{Re}\left(\frac{W_p}{p}\#U_p\right) \gg \frac{N}{Q}\cdot Q \cdot \frac{L}{\log L} = \frac{Q^2}{\log L},$$

because $L = Q^2/N$. The final assertion follows from Corollary 7.1. □

8 How a Concrete Problem Can Avoid the Gap

The scale gap proved above is not a universal obstruction. A concrete problem can avoid it if the structure of the problem supplies an additional mechanism. Three possible mechanisms are the following.

(1) **Support collapse.** The effective support of the modulus sum may be concentrated on t for which $t/(t, A)$ is small. Then the conductor after drop is small, and the primitive square-root scale no longer describes the main contribution.

(2) **Diagonal renormalization.** A term that initially looks like a density diagonal may actually be normalized by an oscillatory local factor. In that case the comparison is not between $\#U_t$ and $K_t(A)$, but between two quantities already placed on the same scale.

(3) **Global defect cancellation.** There may be cancellation in a global defect such as

$$\sum_t \frac{W_t}{t}\left(\#U_t - \operatorname{Re} K_t(A)\right).$$

Such a theorem would be global and cannot be inferred from the local conductor-drop identity alone.

The present paper proves none of these mechanisms. It identifies the local scale gap that must be overcome if none of them is present.

9 Logical Status

The status of the results is as follows:

Statement	Status
Exact conductor drop	unconditional
Primitive square-root estimate	standard Weil and p-adic stationary phase
Weighted short-modulus bound	unconditional from local estimates
Prime-supported scale gap	unconditional under stated support and positivity
Abstract weighted obstruction	conditional on density lower bound and scale separation
Terminal closure theorem	not claimed
Universal obstruction theorem	not claimed

This separation is intentional. The paper is a local arithmetic note with a scale-gap corollary, not a terminal closure argument.

10 Conclusion

The inverse rational zero-mode

$$K_t(A) = \sum_{x \in U_t} e_t\Big(A\,\overline{x(x+1)}\Big)$$

has exact conductor drop and primitive square-root cancellation. The density term $\#U_t$, however, remains density-scale on primitive prime-supported ranges. This produces a clean scale gap between density mass and oscillatory zero-mode mass.

The conclusion is deliberately modest:

$$\boxed{\text{local conductor drop does not by itself create density-scale matching.}}$$

A larger global theorem must supply an additional mechanism.

A Optional Support Calculation

This appendix is not used in the proof of the main theorems. It records a simple calculation that may be useful in adjacent-modulus comparison problems.

Suppose a short parameter t is represented by

$$t = q\bar{a} - (q+1)\bar{b},$$

where

$$\bar{a} \in (\mathbb{Z}/(q+1)\mathbb{Z})^\times, \qquad \bar{b} \in (\mathbb{Z}/q\mathbb{Z})^\times.$$

Reducing the identity modulo $q+1$ and modulo q, respectively, gives

$$\bar{a} \equiv -t \pmod{q+1}, \qquad \bar{b} \equiv -t \pmod{q}.$$

Thus such a unit pair exists if and only if

$$(t, q(q+1)) = 1.$$

This condition is independent of A.

If

$$c_A(t) := \frac{t}{(t, A)},$$

then

$$\#\{t \le L : c_A(t) \le H\} \le \min\{L, H\tau(|A|)\}.$$

Indeed, the bound by L is immediate. For the second bound, if $c_A(t) \le H$, then $t = hc$ with $h = (t, A) \mid A$ and $c \le H$. The pair (h, c) determines t.

Remark A.1. The visible condition $(t, q(q+1)) = 1$ does not imply $t \mid A$, nor does it imply that $t/(t, A)$ is small. Therefore this visible support condition alone does not provide support collapse.

B Stationary Phase Lemmas

We record the standard local estimates used above.

Lemma B.1 (First-derivative cancellation)**.** *Let p be an odd prime and $k \ge 2$. Let f be p-adic analytic on a residue class $x \equiv a \pmod p$. If*

$$f'(a) \not\equiv 0 \pmod p,$$

then

$$\sum_{\substack{x \bmod p^k \\ x \equiv a \pmod p}} e_{p^k}(f(x)) = 0.$$

Lemma B.2 (Non-degenerate stationary phase)**.** *Let p be an odd prime. Let f be p-adic analytic on a residue class $x \equiv a \pmod p$. Suppose*

$$f'(a) \equiv 0 \pmod p, \qquad f''(a) \not\equiv 0 \pmod p.$$

Then

$$\sum_{\substack{x \bmod p^k \\ x \equiv a \pmod p}} e_{p^k}(f(x)) \ll p^{k/2}.$$

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