Exact Ramified Conductor Drop for a Zero-Frequency Rational Sum*

A Verified Local Reduction with an Explicit Oscillatory Baseline

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Abstract

We study the complete rational exponential sum

$$K_t(m) = \sum_{\substack{x \bmod t \\ (x(x+1),t)=1}} \exp\left(2\pi i \frac{m\overline{x(x+1)}^{(t)}}{t}\right),$$

where $\overline{x(x+1)}^{(t)}$ denotes inversion modulo t. The main algebraic result is an exact ramified conductor-drop identity. If

$$h = (m,t), \qquad q = t/h, \qquad m_0 = m/h,$$

then

$$K_t(m) = \Lambda_t(m) K_q(m_0),$$

with an explicit multiplicative fiber factor $\Lambda_t(m)$. The proof includes two verification details that are often suppressed: the reduction of inverses from modulus t to modulus q, and the constancy of the additive phase on the fibers of $U_t \to U_q$. No coprimality condition $(h,q)=1$ is required.

We also define the induced oscillatory diagonal model and prove that it satisfies the same conductor-drop law:

$$D_t^{\mathrm{ind}}(m) = \Lambda_t(m) D_q(m_0).$$

*This manuscript proves a local reduction theorem for a specific complete rational sum. It makes no global main-term or zero-free-region claim.

Therefore the ramified local defect reduces exactly to a lower-conductor coprime discrepancy. Assuming the standard prime-power stationary-phase bound for the lower-conductor rational sum, we obtain

$$\mathcal{D}_Q^{\mathrm{ram}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon.$$

Since the admissible set is empty for even moduli, the two-adic component vanishes for this explicit sum under the induced oscillatory baseline. Thus the residual local defect is closed in this local model. The paper makes no global main-term claim. In particular, it does not assert that an externally defined global diagonal mass must coincide with the induced oscillatory baseline.

Contents

1 Introduction

This paper proves a local conductor-drop theorem for the zero-frequency rational complete sum

$$K_t(m) = \sum_{\substack{x \bmod t \\ (x(x+1),t)=1}} e\left(\frac{m\overline{x(x+1)}^{(t)}}{t}\right), \tag{1}$$

where

$$e(z) = e^{2\pi iz}$$

and $\overline{x(x+1)}^{(t)}$ denotes the inverse of $x(x+1)$ modulo t.

The point of the paper is narrow and local. In a near-critical zero-frequency analysis one may encounter a weighted expression of the form

$$\mathcal{L}_Q(m) = \frac{N}{Q} \sum_{t \geq 1} \frac{W_t(Q)}{t} K_t(m), \tag{2}$$

with $W_t(Q)$ effectively supported on $t \ll Q^2/N$. The question is whether the ramified range $(m,t) > 1$ produces an independent local-density obstruction.

For the explicit sum (1), it does not. The ramified part collapses exactly to lower conductor. Let

$$h = (m,t), \qquad q = \frac{t}{h}, \qquad m_0 = \frac{m}{h}. \tag{3}$$

Then the main identity is

$$K_t(m) = \Lambda_t(m) K_q(m_0), \tag{4}$$

where $\Lambda_t(m)$ is an explicit product of local fiber sizes. The identity holds even when $(h,q) > 1$. Thus entangled prime-power ramification is absorbed by the same local fiber calculation.

Two technical points are essential. First, one must justify replacing inverses modulo t by inverses modulo q in the additive character. Second, one must prove that the phase is constant on each conductor-drop fiber. Both points are proved explicitly in Section 3. These lemmas are the main verification details needed for a rigorous conductor-drop proof.

We then define an induced oscillatory diagonal model. If the diagonal side is obtained by pulling a lower-conductor packet back along the fiber map $U_t \to U_q$, then it obeys the same conductor-drop factor:

$$D_t^{\mathrm{ind}}(m) = \Lambda_t(m) D_q(m_0). \tag{5}$$

Consequently,

$$K_t(m) - D_t^{\mathrm{ind}}(m) = \Lambda_t(m)\{K_q(m_0) - D_q(m_0)\}. \tag{6}$$

For the induced oscillatory baseline, the lower-conductor diagonal term is set to zero in the coprime range:

$$D_q^{\mathrm{osc}}(m_0) = 0, \qquad (m_0, q) = 1.$$

This reflects the fact that $K_q(m_0)$ is treated as an oscillatory complete sum, not as a density-scale main term. Assuming the standard prime-power stationary-phase estimate

$$K_{p^n}(u) \ll p^{n/2} \qquad (p \nmid u),$$

we obtain

$$K_q(m_0) \ll_\varepsilon q^{1/2+\varepsilon}, \qquad (m_0, q) = 1,$$

and therefore

$$\mathcal{D}_Q^{\mathrm{ram}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon. \tag{7}$$

Moreover, if $2 \mid t$, then $x(x+1)$ is always even, so the admissible set is empty. Hence the two-adic part vanishes in this model.

The estimate (7) is a local square-root scale estimate. It should not be interpreted automatically as a power saving relative to an arbitrary global benchmark. If a global application has a benchmark A_Q, then (7) is an admissible error only when

$$N^{1/2} Q^\varepsilon m^\varepsilon \ll Q^{-\delta} A_Q$$

for some $\delta > 0$, or when the global normalization treats the square-root zero-frequency scale as acceptable. No such global normalization is assumed here.

1.1 What is proved

The paper proves:

(i) an exact conductor-drop identity for $K_t(m)$;

(ii) an exact induced diagonal compatibility theorem;

(iii) a ramified local defect bound assuming the standard prime-power stationary-phase estimate;

(iv) vanishing of the two-adic component for the explicit admissible set.

1.2 What is not proved

The paper does not prove:

(i) that every global diagonal main term equals the induced oscillatory baseline;

(ii) that the local square-root estimate is automatically a power saving in every global normalization;

(iii) any global zero-free region or Riemann-hypothesis-type theorem;

(iv) a new proof of the standard p-adic stationary-phase estimate.

2 Notation and local admissible sets

Let

$$F(x) = x(x+1).$$

For $t \geq 1$, define

$$U_t := \{x \bmod t : (F(x), t) = 1\}. \tag{8}$$

Definition 2.1 (The complete sum). *For $t \geq 1$ and $m \in \mathbb{Z}$, define*

$$K_t(m) := \sum_{x \in U_t} e\left(\frac{m\overline{F(x)}^{(t)}}{t}\right). \tag{9}$$

For $t = 1$, we use the convention $U_1 = \{0\}$ and $K_1(m) = 1$.

Remark 2.2 (The case $t = 1$). *The convention $K_1(m) = 1$ is compatible with the conductor-drop identity below. If $h = t$, then m is divisible by t, and every additive character in $K_t(m)$ is equal to 1. Therefore $K_t(m) = \#U_t$, which is precisely the fiber factor times $K_1(m/t) = 1$.*

Lemma 2.3 (Even moduli). *If $2 \mid t$, then $U_t = \varnothing$. Consequently,*

$$K_t(m) = 0 \qquad (2 \mid t).$$

Proof. For every integer x, one of x and $x + 1$ is even. Hence $2 \mid x(x + 1)$. If $2 \mid t$, then $(x(x + 1), t) > 1$ for every x, so U_t is empty. □

Lemma 2.4 (Odd prime-power count). *Let p be an odd prime and $n \geq 1$. Then*

$$\#U_{p^n} = p^{n-1}(p-2). \tag{10}$$

Proof. The condition $(x(x + 1), p^n) = 1$ is equivalent to $p \nmid x(x + 1)$. The forbidden residue classes modulo p are $x \equiv 0$ and $x \equiv -1$. Each lifts to p^{n-1} residue classes modulo p^n. Hence

$$\#U_{p^n} = p^n - 2p^{n-1} = p^{n-1}(p-2).$$

□

3 Reduction of inverses and fiber phase

This section supplies the elementary technical details needed for a rigorous conductor-drop proof.

Lemma 3.1 (Reduction of inverses)**.** *Let $q \mid t$, and let a be coprime to t. Let $\bar{a}^{(t)}$ denote the inverse of a modulo t, and let $\bar{a}^{(q)}$ denote the inverse of a modulo q. Then*

$$\bar{a}^{(t)} \equiv \bar{a}^{(q)} \pmod{q}. \tag{11}$$

Consequently, if $t = hq$ and $m = hm_0$, then

$$e\left(\frac{m\bar{a}^{(t)}}{t}\right) = e\left(\frac{m_0\bar{a}^{(q)}}{q}\right). \tag{12}$$

Proof. Since

$$a\bar{a}^{(t)} \equiv 1 \pmod{t}$$

and $q \mid t$, reduction modulo q gives

$$a\bar{a}^{(t)} \equiv 1 \pmod{q}.$$

By uniqueness of the inverse modulo q, we obtain

$$\bar{a}^{(t)} \equiv \bar{a}^{(q)} \pmod{q}.$$

Now $t = hq$ and $m = hm_0$, so

$$\frac{m\bar{a}^{(t)}}{t} = \frac{hm_0\bar{a}^{(t)}}{hq} = \frac{m_0\bar{a}^{(t)}}{q}.$$

By (11), the difference between the last expression and $m_0\bar{a}^{(q)}/q$ is an integer. Therefore the additive characters are equal. □

Lemma 3.2 (Constancy of the phase on conductor-drop fibers)**.** *Let $h = (m, t)$, $q = t/h$, and $m_0 = m/h$. If $x, x' \in U_t$ and*

$$x \equiv x' \pmod{q},$$

then

$$e\left(\frac{m\overline{F(x)}^{(t)}}{t}\right) = e\left(\frac{m\overline{F(x')}^{(t)}}{t}\right). \tag{13}$$

Proof. Since $x \equiv x' \pmod q$, we have

$$F(x) \equiv F(x') \pmod q.$$

Because $x, x' \in U_t$ and $q \mid t$, both $F(x)$ and $F(x')$ are invertible modulo q. Hence

$$\overline{F(x)}^{(q)} \equiv \overline{F(x')}^{(q)} \pmod q.$$

By Lemma 3.1,

$$e\left(\frac{m\overline{F(x)}^{(t)}}{t}\right) = e\left(\frac{m_0\overline{F(x)}^{(q)}}{q}\right),$$

and similarly for x'. The two right-hand sides are equal. □

4 Exact ramified conductor drop

We now prove the exact conductor-drop identity.

Definition 4.1 (Local fiber factor)**.** *Let p be prime, $n \geq 1$, and $0 \leq r \leq n$. Define*

$$\lambda_p(n,r) := \begin{cases} 1, & r = 0, \\ p^r, & 0 < r < n, \\ p^{n-1}(p-2), & r = n,\ p \text{ odd}, \\ 0, & r = n,\ p = 2. \end{cases} \tag{14}$$

For $t \geq 1$ and $m \in \mathbb{Z}$, set

$$\Lambda_t(m) := \prod_{p^n \| t} \lambda_p(n, \min(v_p(m), n)). \tag{15}$$

Theorem 4.2 (Exact ramified conductor drop)**.** *Let $t \geq 1$, $m \in \mathbb{Z}$, and put*

$$h = (m,t), \qquad q = \frac{t}{h}, \qquad m_0 = \frac{m}{h}.$$

Then

$$K_t(m) = \Lambda_t(m) K_q(m_0). \tag{16}$$

Moreover,

$$0 \leq \Lambda_t(m) \leq (m,t). \tag{17}$$

Proof. Consider the reduction map

$$\pi_{t,q} : U_t \to U_q, \qquad x \bmod t \mapsto x \bmod q.$$

The map is well-defined since $q \mid t$. By Lemma 3.2, the phase in $K_t(m)$ is constant on each fiber of $\pi_{t,q}$, and by Lemma 3.1 its value on the fiber over $y \in U_q$ is

$$e\left(\frac{m_0 \overline{F(y)}^{(q)}}{q}\right).$$

It remains to compute the size of each fiber. This is local at prime powers. Suppose $p^n \parallel t$ and set

$$r = \min(v_p(m), n).$$

The local part of the lower conductor q is p^{n-r}.

If $r = 0$, the local modulus is unchanged and the fiber factor is 1.

If $0 < r < n$, fix $y \bmod p^{n-r}$ with $p \nmid F(y)$. The lifts $x \bmod p^n$ of y are precisely

$$x = y + p^{n-r} z, \qquad z \bmod p^r,$$

so there are p^r lifts. Since $p \nmid F(y)$, every lift remains admissible because $F(x) \equiv F(y) \pmod{p}$.

If $r = n$, the prime p disappears from the lower conductor. The local fiber factor is the number of admissible residues modulo p^n. By Lemma 2.4, this is $p^{n-1}(p-2)$ for odd p, and by Lemma 2.3, it is 0 for $p = 2$.

Thus every fiber has cardinality

$$\Lambda_t(m) = \prod_{p^n \parallel t} \lambda_p(n, \min(v_p(m), n)).$$

Therefore

$$\begin{aligned} K_t(m) &= \sum_{y \in U_q} \#\pi_{t,q}^{-1}(y) e\left(\frac{m_0 \overline{F(y)}^{(q)}}{q}\right) \\ &= \Lambda_t(m) K_q(m_0). \end{aligned}$$

Finally, each local factor satisfies $\lambda_p(n, r) \le p^r$ for $r < n$, and $\lambda_p(n, n) \le p^n$ for $r = n$. Multiplying over primes gives $\Lambda_t(m) \le h = (m, t)$. □

Remark 4.3 (No coprimality condition)**.** *No assumption $(h, q) = 1$ is used. The prime-power entangled case is part of the fiber calculation.*

Remark 4.4 (The case $q = 1$)**.** *When $q = 1$, the convention $K_1(u) = 1$ is compatible with Theorem 4.2. If $h = t$, then m is divisible by t, so every additive character in $K_t(m)$*

is equal to 1*, and* $K_t(m) = \#U_t = \Lambda_t(m)K_1(m/t)$.

5 Induced oscillatory diagonal model

The conductor-drop identity for $K_t(m)$ becomes a defect identity only if the diagonal side follows the same fiber law. This section defines such a model.

Definition 5.1 (Induced diagonal packet). *Let* $h = (m, t)$*,* $q = t/h$*, and* $m_0 = m/h$*. Let*

$$\Psi_{q,m_0} : U_q \to \mathbb{C}$$

be a lower-conductor packet. Define

$$D_q(m_0) := \sum_{y \in U_q} \Psi_{q,m_0}(y), \tag{18}$$

and

$$D_t^{\mathrm{ind}}(m) := \sum_{x \in U_t} \Psi_{q,m_0}(\pi_{t,q}(x)). \tag{19}$$

Theorem 5.2 (Induced diagonal compatibility). *With the notation above,*

$$D_t^{\mathrm{ind}}(m) = \Lambda_t(m) D_q(m_0). \tag{20}$$

Proof. The proof of Theorem 4.2 shows that every fiber of

$$\pi_{t,q} : U_t \to U_q$$

has cardinality $\Lambda_t(m)$. Therefore

$$\begin{aligned} D_t^{\mathrm{ind}}(m) &= \sum_{x \in U_t} \Psi_{q,m_0}(\pi_{t,q}(x)) \\ &= \sum_{y \in U_q} \#\pi_{t,q}^{-1}(y) \Psi_{q,m_0}(y) \\ &= \Lambda_t(m) \sum_{y \in U_q} \Psi_{q,m_0}(y) \\ &= \Lambda_t(m) D_q(m_0). \end{aligned}$$

□

Corollary 5.3 (Exact defect reduction). *For the induced diagonal model,*

$$K_t(m) - D_t^{\mathrm{ind}}(m) = \Lambda_t(m)\{K_q(m_0) - D_q(m_0)\}. \tag{21}$$

Proof. Subtract Theorem 5.2 from Theorem 4.2. □

Definition 5.4 (Oscillatory zero-density baseline)**.** *In the lower-conductor coprime range* $(m_0, q) = 1$*, define*

$$D_q^{\mathrm{osc}}(m_0) := 0. \tag{22}$$

The induced oscillatory diagonal model is the induced model obtained from this lower packet.

Remark 5.5. *This is a local choice. It says that the lower-conductor coprime complete sum is treated as oscillatory and is not assigned a density-scale diagonal term. The paper does not claim that an externally defined global diagonal mass must equal this induced model.*

6 Lower-conductor square-root input

The ramified defect reduces to a lower-conductor coprime complete sum. The only analytic input needed is the standard square-root estimate for that sum.

Lemma 6.1 (Critical point structure)**.** *Let* p *be an odd prime and* $p \nmid u$*. On the domain* $x \not\equiv 0, -1 \pmod p$*, the rational function*

$$f(x) = \frac{u}{x(x+1)}$$

has exactly one critical point modulo p*, namely*

$$x \equiv -\frac{1}{2} \pmod p.$$

This critical point is non-degenerate.

Proof. We compute

$$f'(x) = -\frac{u(2x+1)}{x^2(x+1)^2}.$$

The denominator is non-zero on the domain and $p \nmid u$. Thus $f'(x) \equiv 0 \pmod p$ exactly when $2x + 1 \equiv 0 \pmod p$. At that point the zero of the numerator is simple and the denominator is non-zero, so the critical point is non-degenerate. □

Hypothesis 6.2 (Standard prime-power stationary phase)**.** *For every odd prime* p*, every* $n \geq 1$*, and every* u *with* $p \nmid u$*,*

$$K_{p^n}(u) \ll p^{n/2}. \tag{23}$$

Remark 6.3. *This is the only external analytic input in the paper. The exact conductor-drop identity itself does not depend on this hypothesis. Lemma 6.1 verifies the non-degenerate critical point structure of the phase* $u/(x(x+1))$*, but the full* p*-adic stationary-*

phase estimate is invoked as a standard result. Standard references for complete exponential sums and p-adic stationary phase include [4, 2, 3, 6].

Lemma 6.4 (CRT factorization with unit twists)**.** *Let $q = \prod_i q_i$, where the q_i's are pairwise coprime. Put $Q_i = q/q_i$, and let $\overline{Q_i}$ denote the inverse of Q_i modulo q_i. If $(u, q) = 1$, then*

$$K_q(u) = \prod_i K_{q_i}(u_i), \qquad u_i \equiv u\overline{Q_i} \pmod{q_i}. \tag{24}$$

In particular,

$$|K_q(u)| \leq \prod_{p^n \| q} \sup_{\substack{v \bmod p^n \\ p \nmid v}} |K_{p^n}(v)|. \tag{25}$$

Proof. By the Chinese remainder theorem, every x mod q corresponds to a tuple x_i mod q_i, and

$$U_q \simeq \prod_i U_{q_i}.$$

Let $a = \overline{F(x)}^{(q)}$. Its residue modulo q_i is

$$a_i \equiv \overline{F(x_i)}^{(q_i)} \pmod{q_i}.$$

The CRT reconstruction gives

$$a \equiv \sum_i a_i Q_i \overline{Q_i} \pmod{q}.$$

Therefore

$$\frac{ua}{q} \equiv \sum_i \frac{u\overline{Q}_i a_i}{q_i} \pmod{1}.$$

Hence the additive character factors as

$$e\left(\frac{u\overline{F(x)}^{(q)}}{q}\right) = \prod_i e\left(\frac{u_i \overline{F(x_i)}^{(q_i)}}{q_i}\right), \qquad u_i \equiv u\overline{Q_i} \pmod{q_i}.$$

Summing over $\prod_i U_{q_i}$ proves the factorization. □

Proposition 6.5 (Coprime lower-conductor bound)**.** *Assume Hypothesis 6.2. If $(u, q) = 1$, then*

$$K_q(u) \ll_\varepsilon q^{1/2+\varepsilon}. \tag{26}$$

Proof. If $2 \mid q$, then $K_q(u) = 0$ by Lemma 2.3. Hence assume that q is odd. By Lemma 6.4,

$$|K_q(u)| \leq \prod_{p^n \| q} \sup_{\substack{v \bmod p^n \\ p \nmid v}} |K_{p^n}(v)|.$$

Hypothesis 6.2 gives

$$|K_q(u)| \ll C^{\omega(q)} \prod_{p^n \| q} p^{n/2}.$$

Since $C^{\omega(q)} \ll_\varepsilon q^\varepsilon$, we obtain

$$|K_q(u)| \ll_\varepsilon q^{1/2+\varepsilon}.$$

□

Corollary 6.6 (Lower-conductor discrepancy for the oscillatory baseline)**.** *Assume Hypothesis 6.2. If $(u, q) = 1$, then*

$$|K_q(u) - D_q^{\mathrm{osc}}(u)| \ll_\varepsilon q^{1/2+\varepsilon}. \tag{27}$$

Proof. By definition $D_q^{\mathrm{osc}}(u) = 0$. Apply Proposition 6.5. □

7 Weighted ramified defect

Let $Q \geq 3$, $N \geq 1$, and

$$L := \frac{Q^2}{N}.$$

Let $W_t(Q)$ be a weight satisfying

$$|W_t(Q)| \leq 1, \qquad W_t(Q) = 0 \quad \text{unless } t \leq L. \tag{28}$$

The compact support assumption is made only for clarity. Appendix A records the standard extension to rapidly decaying weights.

Definition 7.1 (Ramified weighted defect)**.** *For the induced oscillatory diagonal model, define*

$$\mathcal{D}_Q^{\mathrm{ram}}(m) := \frac{N}{Q} \sum_{\substack{t \leq L \\ (m,t)>1}} \frac{W_t(Q)}{t} \{K_t(m) - D_t^{\mathrm{ind}}(m)\}. \tag{29}$$

Theorem 7.2 (Ramified local defect bound)**.** *Assume Hypothesis 6.2. Then*

$$\mathcal{D}_Q^{\mathrm{ram}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon. \tag{30}$$

If $m \ll Q^C$ for a fixed C, then

$$\mathcal{D}_Q^{\mathrm{ram}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon. \tag{31}$$

Proof. For each t in the ramified sum, put

$$h = (m, t), \qquad q = t/h, \qquad m_0 = m/h.$$

Then $(m_0, q) = 1$. By Corollary 5.3 and Corollary 6.6,

$$|K_t(m) - D_t^{\mathrm{ind}}(m)| \le \Lambda_t(m) q^{1/2+\varepsilon}.$$

By Theorem 4.2, $\Lambda_t(m) \le h$. Since $t = hq$,

$$\frac{\Lambda_t(m)}{t} q^{1/2+\varepsilon} \le q^{-1/2+\varepsilon}.$$

Therefore

$$|\mathcal{D}_Q^{\mathrm{ram}}(m)| \ll_\varepsilon \frac{N}{Q} \sum_{\substack{t \le L \\ (m,t)>1}} q^{-1/2+\varepsilon}.$$

We overcount by summing over all divisors $h \mid m$ and all $q \le L/h$:

$$\begin{aligned}|\mathcal{D}_Q^{\mathrm{ram}}(m)| &\ll_\varepsilon \frac{N}{Q} \sum_{h|m} \sum_{q \le L/h} q^{-1/2+\varepsilon} \\ &\ll_\varepsilon \frac{N}{Q} \sum_{h|m} (L/h)^{1/2+\varepsilon} \\ &\ll_\varepsilon \frac{N}{Q} L^{1/2+\varepsilon} \sum_{h|m} h^{-1/2-\varepsilon}.\end{aligned}$$

The divisor sum is $\ll_\varepsilon m^\varepsilon$. Since $L = Q^2/N$,

$$\frac{N}{Q} L^{1/2} = N^{1/2}.$$

The factor L^ε is absorbed into Q^ε, after adjusting ε. This proves (30). □

Remark 7.3 (Size of the local bound)**.** *The estimate*

$$\mathcal{D}_Q^{\mathrm{ram}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon$$

should not be interpreted as a power saving relative to every possible global benchmark. It is the natural square-root scale of the local zero-frequency family. In a global application with benchmark A_Q, this estimate is an admissible error only if

$$N^{1/2} Q^\varepsilon m^\varepsilon \ll Q^{-\delta} A_Q$$

for some $\delta > 0$, or if the global normalization treats the square-root zero-frequency scale as acceptable. No such global normalization is assumed in this paper.

8 Two-adic component and residual closure

Definition 8.1 (Two-adic component). *Define*

$$\mathcal{D}_Q^{(2)}(m) := \frac{N}{Q} \sum_{\substack{t \le L \\ 2 | t}} \frac{W_t(Q)}{t} \{K_t(m) - D_t^{\mathrm{ind}}(m)\}. \tag{32}$$

Proposition 8.2 (Two-adic vanishing). *For the explicit sum $K_t(m)$ and the induced oscillatory diagonal model,*

$$\mathcal{D}_Q^{(2)}(m) = 0. \tag{33}$$

Proof. If $2 \mid t$, then $U_t = \varnothing$ by Lemma 2.3. Hence $K_t(m) = 0$. The induced oscillatory diagonal model is also zero because it is defined on the empty admissible set. Thus every term in (32) vanishes. □

Definition 8.3 (Residual local defect). *Set*

$$\mathcal{D}_Q^{\mathrm{res}}(m) := \mathcal{D}_Q^{(2)}(m) + \mathcal{D}_Q^{\mathrm{ram}}(m). \tag{34}$$

Corollary 8.4 (Residual local closure). *Assume Hypothesis 6.2. Then*

$$\mathcal{D}_Q^{\mathrm{res}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon. \tag{35}$$

If $m \ll Q^C$, then

$$\mathcal{D}_Q^{\mathrm{res}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon. \tag{36}$$

Proof. Combine Proposition 8.2 and Theorem 7.2. □

9 Why this is not a global diagonal theorem

The result is deliberately local. The induced oscillatory baseline sets

$$D_q^{\mathrm{osc}}(m_0) = 0 \qquad ((m_0, q) = 1),$$

because the coprime lower-conductor sum is treated as oscillatory. If one instead inserts a density-scale lower-conductor diagonal model

$$D_q^{\mathrm{dens}}(m_0) = S_q(m_0)\varphi(q),$$

then Corollary 5.3 gives

$$K_t(m) - D_t(m) = \Lambda_t(m)\{K_q(m_0) - S_q(m_0)\varphi(q)\}.$$

This difference need not be square-root size. Indeed, $K_q(m_0)$ is oscillatory, while $S_q(m_0)\varphi(q)$ may be density scale. Therefore a global argument that requires a density-scale diagonal model must separately prove that its diagonal side is compatible with the induced oscillatory baseline or explain the resulting mismatch.

Thus the conclusion of this paper is:

$$\boxed{\text{the residual local defect closes for the explicit local sum and induced oscillatory diagonal model.}}$$

It is not:

$$\boxed{\text{an arbitrary global main-term mismatch is closed.}}$$

10 Verification audit

We summarize the logical status of each step.

Step	Statement	Status
1	Even moduli vanish	proved, Lemma 2.3
2	Inverses reduce from t to q	proved, Lemma 3.1
3	Phase is constant on fibers	proved, Lemma 3.2
4	Exact conductor drop	proved, Theorem 4.2
5	Induced diagonal compatibility	proved, Theorem 5.2
6	CRT unit twists	proved, Lemma 6.4
7	Prime-power stationary phase	external standard input, Hypothesis 6.2
8	Coprime lower-conductor bound	proved from Hypothesis 6.2
9	Ramified defect bound	proved, Theorem 7.2
10	Two-adic component	zero, Proposition 8.2
11	Residual local closure	proved, Corollary 8.4

The only analytic input left external is Hypothesis 6.2. The paper verifies the non-degenerate critical point structure of the phase but does not reproduce the full p-adic stationary-phase proof.

11 Conclusion

For the explicit zero-frequency rational complete sum

$$K_t(m) = \sum_{\substack{x \bmod t \\ (x(x+1),t)=1}} e\left(\frac{m\overline{x(x+1)}^{(t)}}{t}\right),$$

the ramified conductor-drop identity is exact:

$$K_t(m) = \Lambda_t(m) K_{t/(m,t)}\left(\frac{m}{(m,t)}\right).$$

The proof requires two elementary but essential facts: inverses modulo t reduce to inverses modulo $q = t/(m,t)$, and the additive phase is constant on the fibers of $U_t \to U_q$. Once the diagonal side is defined by the same induced fiber law, the defect also drops exactly:

$$K_t(m) - D_t^{\mathrm{ind}}(m) = \Lambda_t(m)\left[K_q(m_0) - D_q(m_0)\right].$$

With the induced oscillatory baseline $D_q(m_0) = 0$ in the lower coprime range and the standard prime-power stationary-phase estimate, the ramified residual satisfies

$$\mathcal{D}_Q^{\mathrm{ram}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon.$$

The two-adic component vanishes for the same explicit admissible set. Therefore

$$\mathcal{D}_Q^{\mathrm{res}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon.$$

This is the final local theorem. It closes the residual local defect in the stated model, while leaving any external global diagonal identification as a separate problem.

A Rapidly decaying weights

The compact support assumption $t \le L$ can be replaced by rapid decay. Suppose that for every $A > 0$,

$$W_t(Q) \ll_A (1 + t/L)^{-A}.$$

After conductor drop, the ramified sum is bounded by

$$\frac{N}{Q}\sum_{h|m}\sum_{q\ge 1}(1 + hq/L)^{-A} q^{-1/2+\varepsilon}.$$

For $A > 2$, the inner sum is

$$\ll_\varepsilon (L/h)^{1/2+\varepsilon}.$$

The same proof as in Theorem 7.2 therefore gives

$$\mathcal{D}_Q^{\mathrm{ram}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon.$$

B Prime-power stationary phase: what is used

Hypothesis 6.2 is the only analytic estimate not proved from first principles here. The phase is

$$f(x) = \frac{u}{x(x+1)}$$

on $x \not\equiv 0, -1 \pmod p$, with $p \nmid u$. We computed

$$f'(x) = -\frac{u(2x+1)}{x^2(x+1)^2}.$$

Thus the only critical residue is $x = -1/2$, and it is non-degenerate. The standard p-adic stationary-phase theorem for one-variable rational phases with non-degenerate critical points gives

$$\sum_{\substack{x \bmod p^n \\ p \nmid x(x+1)}} e\left(\frac{u\overline{x(x+1)}}{p^n}\right) \ll p^{n/2}.$$

For $n = 1$, this is also compatible with the Weil bound for rational functions over finite fields. For $n > 1$, the usual lifting argument gives cancellation on non-stationary residue classes and a quadratic Gauss-type contribution of square-root size at the non-degenerate critical class.

C Non-claims

The paper intentionally does not claim the following.

(i) It does not claim that a global diagonal mass from an external problem equals the induced oscillatory baseline.

(ii) It does not claim that the local square-root estimate $N^{1/2}Q^{\varepsilon}m^{\varepsilon}$ is automatically a power saving in every global normalization.

(iii) It does not claim a density-scale main term for $K_q(m_0)$ in the coprime range.

(iv) It does not claim a global zero-free region, a Riemann-hypothesis-type theorem, or a global main-term closure.

(v) It does not use or cite any previous private manuscripts or numbering systems.

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