Exact Ramified Conductor Drop for a Zero-Frequency Rational Sum

Residual Local Defects Collapse to Lower-Conductor Oscillation

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Abstract

We study the zero-frequency rational complete sum

$$K_t(m) = \sum_{\substack{x \bmod t \\ (x(x+1),t)=1}} \exp\left(2\pi i \frac{m\overline{x(x+1)}}{t}\right),$$

where the bar denotes inversion modulo the modulus. The main theorem is an exact ramified conductor-drop identity. If

$$h = (m,t), \qquad q = t/h, \qquad m_0 = m/h,$$

then

$$K_t(m) = \Lambda_t(m) K_q(m_0),$$

with an explicit multiplicative fiber factor $\Lambda_t(m)$. This identity includes the prime-power entangled case and does not require the coprimality condition $(h,q) = 1$. We then define the natural induced oscillatory diagonal model and prove its exact compatibility with the same conductor-drop factor. Consequently the ramified local defect is reduced exactly to a lower-conductor coprime oscillatory discrepancy. Using the standard square-root bound for the remaining coprime rational complete sum, we obtain

$$\mathcal{D}_Q^{\mathrm{ram}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon$$

for the weighted ramified zero-frequency defect. Since the admissible set is empty for even moduli, the two-adic component is also absent in this explicit model under

the induced oscillatory baseline. The result is a local theorem: it closes the residual local defect for the stated rational sum and diagonal model, without making any unconditional claim about an external global main-term problem.

Contents

1 Introduction

This paper isolates and resolves a local obstruction that appears when a zero-frequency complete rational sum is examined in a ramified range. The central object is

$$K_t(m) = \sum_{\substack{x \bmod t \\ (x(x+1),t)=1}} \exp\left(2\pi i \frac{m\overline{x(x+1)}}{t}\right), \tag{1}$$

where $\overline{x(x+1)}$ is the inverse of $x(x+1)$ modulo t. This is a complete exponential sum with rational phase. It is not, in the generic coprime range, a density object. The expected scale is square-root scale, in line with the standard theory of complete exponential sums and p-adic stationary phase; see, for example, [4, 2, 3, 6].

The problem motivating the present paper is the following. In a near-critical bilinear regime one may encounter a zero-frequency main term of the schematic form

$$\mathcal{L}_Q(m) = \frac{N}{Q} \sum_{t \geq 1} \frac{W_t(Q)}{t} K_t(m), \tag{2}$$

where $W_t(Q)$ is a smooth weight effectively supported on $t \ll Q^2/N$. A natural question is whether the ramified part of this expression can create an independent density-scale obstruction. The answer for (1) is no. The ramified part collapses exactly to a lower-conductor oscillatory sum.

The exact identity is the following. Put

$$h = (m,t), \qquad q = \frac{t}{h}, \qquad m_0 = \frac{m}{h}. \tag{3}$$

Then

$$K_t(m) = \Lambda_t(m) K_q(m_0), \tag{4}$$

where $\Lambda_t(m)$ is an explicit product of local fiber sizes. The identity is not an approximation and it does not require $(h,q)=1$. Thus prime-power entanglement in the ramified sector is not a separate obstruction for (1); it is absorbed by a uniform fiber multiplicity.

The second point is compatibility with the diagonal side. If the diagonal model is defined by the same lower-conductor pullback principle, then

$$D_t(m) = \Lambda_t(m) D_q(m_0). \tag{5}$$

Consequently

$$K_t(m) - D_t(m) = \Lambda_t(m)\{K_q(m_0) - D_q(m_0)\}. \tag{6}$$

For the induced oscillatory baseline, one takes $D_q(m_0) = 0$ in the coprime lower-conductor range. Then the remaining problem is just the standard bound

$$K_q(m_0) \ll_\varepsilon q^{1/2+\varepsilon}, \qquad (m_0, q) = 1. \tag{7}$$

This gives

$$\mathcal{D}_Q^{\mathrm{ram}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon. \tag{8}$$

If $m \ll Q^C$, then the factor m^ε may be absorbed into Q^ε.

The result should be read as a local theorem. It does not assert that an arbitrary global diagonal main term equals the induced oscillatory baseline. It proves that, for the explicit complete sum (1), the ramified local sum itself does not generate a new density obstruction. Any remaining global difficulty must arise from the interpretation of the diagonal model or from a different part of the global argument.

1.1 Organization

Section 2 fixes notation. Section 3 proves the exact ramified conductor-drop identity. Section 4 proves diagonal compatibility under the induced density model. Section 5 gives the lower-conductor square-root estimate. Section 6 applies these facts to the weighted ramified defect. Section 7 treats the two-adic sector. Section 9 records a verification audit and explains precisely what has and has not been proved.

2 Notation and basic definitions

For $z \in \mathbb{R}$, write

$$e(z) = e^{2\pi i z}.$$

If $(a, t) = 1$, then $\bar{a}$ denotes the inverse of a modulo t. We use $p^n \parallel t$ to mean that $p^n \mid t$ and $p^{n+1} \nmid t$. The function

$$F(x) = x(x + 1)$$

will be fixed throughout the paper.

Definition 2.1 (Admissible residue set)**.** *For a positive integer t, define*

$$U_t := \{x \bmod t : (F(x), t) = 1\}. \tag{9}$$

Definition 2.2 (The complete sum)**.** *For integers $t \geq 1$ and m, define*

$$K_t(m) := \sum_{x \in U_t} e\left(\frac{m\overline{F(x)}}{t}\right). \tag{10}$$

For $t = 1$, we use the convention $U_1 = \{0\}$ and $K_1(m) = 1$.

The set U_t is empty for even t. Indeed, $x(x + 1)$ is always even. This elementary fact is important enough to record.

Lemma 2.3 (Even moduli)**.** *If $2 \mid t$, then $U_t = \varnothing$. Hence*

$$K_t(m) = 0 \qquad (2 \mid t). \tag{11}$$

Proof. For every integer x, one of x and $x+1$ is even. Thus $2 \mid x(x+1)$, and so $(x(x+1),t) > 1$ whenever $2 \mid t$. $\square$

For odd prime powers the admissible fibers are simple.

Lemma 2.4 (Odd prime-power admissible count)**.** *Let p be an odd prime and $n \geq 1$. Then*

$$\#U_{p^n} = p^{n-1}(p-2). \tag{12}$$

Proof. The condition $(x(x+1),p^n) = 1$ is equivalent to $p \nmid x(x+1)$, i.e. $x \not\equiv 0,-1 \pmod p$. There are two forbidden residue classes modulo p, and each has p^{n-1} lifts modulo p^n. Therefore

$$\#U_{p^n} = p^n - 2p^{n-1} = p^{n-1}(p-2).$$

$\square$

3 Exact ramified conductor drop

The first main theorem gives the precise conductor-drop identity. It includes all prime-power entanglement.

Definition 3.1 (Local fiber factor)**.** *Let p be prime, $n \geq 1$, and $0 \leq r \leq n$. Define*

$$\lambda_p(n,r) := \begin{cases} 1, & r = 0, \\ p^r, & 0 < r < n, \\ p^{n-1}(p-2), & r = n,\ p \text{ odd}, \\ 0, & r = n,\ p = 2. \end{cases} \tag{13}$$

For integers m, t with $t \geq 1$, set

$$\Lambda_t(m) := \prod_{p^n \| t} \lambda_p(n, \min(v_p(m), n)). \tag{14}$$

Theorem 3.2 (Exact ramified conductor drop)**.** *Let $t \geq 1$, $m \in \mathbb{Z}$, and put*

$$h = (m,t), \qquad q = \frac{t}{h}, \qquad m_0 = \frac{m}{h}. \tag{15}$$

Then

$$K_t(m) = \Lambda_t(m) K_q(m_0). \tag{16}$$

Moreover,

$$0 \leq \Lambda_t(m) \leq (m,t). \tag{17}$$

Proof. Let $F(x) = x(x+1)$. Since $m = hm_0$ and $t = hq$, for every $x \in U_t$ we have

$$e\left(\frac{m\overline{F(x)}}{t}\right) = e\left(\frac{m_0\overline{F(x)}}{q}\right). \tag{18}$$

The inverse $\overline{F(x)}$ modulo t, reduced modulo q, is the inverse of $F(x)$ modulo q. Hence the phase depends only on the residue class of x modulo q.

It remains to compute the size of the fiber of the reduction map

$$\pi_{t,q} : U_t \to U_q, \qquad x \bmod t \mapsto x \bmod q. \tag{19}$$

We do this prime-power by prime-power. Suppose $p^n \parallel t$ and set

$$r = \min(v_p(m), n).$$

The local component of q at p is p^{n-r}.

If $r = 0$, then the modulus p^n remains fully present in q, so there is no lifting multiplicity and the local fiber factor is 1.

If $0 < r < n$, fix $y \bmod p^{n-r}$ with $p \nmid F(y)$. It has exactly p^r lifts modulo p^n. Since $p \nmid F(y)$, every such lift x also satisfies $p \nmid F(x)$. Thus the local fiber factor is p^r.

If $r = n$, then the p-part disappears completely from q, so the local fiber factor is the number of admissible residues modulo p^n. By Lemma 2.4, this is $p^{n-1}(p-2)$ for odd p. For $p = 2$, it is 0 by Lemma 2.3.

The Chinese remainder theorem multiplies these local fiber sizes, giving $\Lambda_t(m)$. Since the phase is constant on each fiber, summing over fibers gives

$$K_t(m) = \Lambda_t(m) \sum_{y \in U_q} e\left(\frac{m_0\overline{F(y)}}{q}\right) = \Lambda_t(m) K_q(m_0).$$

Finally, for each local factor one has $\lambda_p(n, r) \le p^r$ if $r < n$, and $\lambda_p(n, n) \le p^n$ if $r = n$. Multiplying over primes gives $\Lambda_t(m) \le h = (m, t)$. $\square$

Remark 3.3 (No coprimality condition). *Theorem 3.2 does not require $(h, q) = 1$. Thus the case in which the ramified divisor and the residual conductor share a prime is not a separate obstruction for the sum (10). It is already included in the prime-power fiber calculation.*

Remark 3.4 (Ramified vanishing at the prime two). *If $2 \mid t$, then $K_t(m) = 0$ by Lemma 2.3. This agrees with Theorem 3.2: either the lower conductor q remains even, so $K_q(m_0) = 0$, or the entire two-adic part is absorbed in h, in which case the factor $\lambda_2(n, n) = 0$ appears in $\Lambda_t(m)$.*

4 Induced diagonal compatibility

A diagonal model is compatible with Theorem 3.2 if it is induced from the lower conductor by the same fiber projection. We now formalize this point.

Definition 4.1 (Induced diagonal packet)**.** *Let $h = (m, t)$, $q = t/h$, and $m_0 = m/h$. Let*

$$\Psi_{q,m_0} : U_q \to \mathbb{C}$$

be a lower-conductor local packet and define

$$D_q(m_0) := \sum_{y \in U_q} \Psi_{q,m_0}(y). \tag{20}$$

The induced diagonal model at conductor t is

$$D_t^{\mathrm{ind}}(m) := \sum_{x \in U_t} \Psi_{q,m_0}(\pi_{t,q}(x)), \tag{21}$$

where $\pi_{t,q}$ is the reduction map (19).

Theorem 4.2 (Diagonal compatibility)**.** *With notation as above,*

$$D_t^{\mathrm{ind}}(m) = \Lambda_t(m) D_q(m_0). \tag{22}$$

Proof. By the proof of Theorem 3.2, every fiber of $\pi_{t,q} : U_t \to U_q$ has cardinality $\Lambda_t(m)$. Hence

$$\begin{aligned} D_t^{\mathrm{ind}}(m) &= \sum_{x \in U_t} \Psi_{q,m_0}(\pi_{t,q}(x)) \\ &= \sum_{y \in U_q} \#\pi_{t,q}^{-1}(y) \Psi_{q,m_0}(y) \\ &= \Lambda_t(m) \sum_{y \in U_q} \Psi_{q,m_0}(y) \\ &= \Lambda_t(m) D_q(m_0). \end{aligned}$$

□

Corollary 4.3 (Exact ramified defect reduction)**.** *Assume that the diagonal model at conductor t is induced in the sense of Definition 4.1. Then*

$$K_t(m) - D_t^{\mathrm{ind}}(m) = \Lambda_t(m)\{K_q(m_0) - D_q(m_0)\}. \tag{23}$$

Proof. Subtract Theorem 4.2 from Theorem 3.2. □

Definition 4.4 (Oscillatory zero-density baseline). *In the coprime lower-conductor range* $(m_0, q) = 1$, *define*

$$D_q^{\mathrm{osc}}(m_0) := 0. \tag{24}$$

The induced oscillatory diagonal model is the model obtained from Definition 4.1 with lower packet identically zero.

This definition reflects the fact that the coprime complete sum $K_q(m_0)$ is treated as an oscillatory object rather than as a density-scale main term.

5 The coprime lower-conductor bound

We now prove the lower-conductor estimate used in the ramified bound. The key input is a standard square-root bound for one-variable rational complete sums with non-degenerate critical points. We include the elementary stationary-phase structure to verify that the relevant phase has only one non-degenerate critical residue for odd primes.

Lemma 5.1 (Critical point structure). *Let* p *be an odd prime and* $p \nmid m_0$. *On the domain* $x \not\equiv 0, -1 \pmod{p}$, *the rational function*

$$f(x) = \frac{m_0}{x(x+1)} \tag{25}$$

has exactly one critical point modulo p, *namely*

$$x \equiv -\frac{1}{2} \pmod{p}. \tag{26}$$

This critical point is non-degenerate.

Proof. Differentiating formally gives

$$f'(x) = -\frac{m_0(2x+1)}{x^2(x+1)^2}. \tag{27}$$

The denominator is non-zero on the domain, and $p \nmid m_0$. Hence $f'(x) \equiv 0 \pmod{p}$ if and only if $2x + 1 \equiv 0 \pmod{p}$, giving (26). At this point the second derivative is non-zero modulo p. Indeed, the zero in (27) is simple because the derivative of $2x + 1$ is $2 \not\equiv 0 \pmod{p}$, while the denominator is non-zero at $x = -1/2$. □

Hypothesis 5.2 (Prime-power stationary-phase bound). *For every odd prime* p, *every* $n \geq 1$, *and every* m_0 *with* $p \nmid m_0$,

$$K_{p^n}(m_0) \ll p^{n/2}. \tag{28}$$

Remark 5.3. *Hypothesis 5.2 is a standard consequence of p-adic stationary phase for rational phases with non-degenerate critical points. Lemma 5.1 verifies the required non-degeneracy for the phase $m_0/(x(x+1))$. Standard accounts and related estimates may be found in [4, 2]. One may replace (28) by $K_{p^n}(m_0) \ll_\varepsilon p^{n/2+\varepsilon}$ without changing any conclusion of this paper.*

Proposition 5.4 (Coprime square-root bound)**.** *Let $(m_0, q) = 1$. Then*

$$K_q(m_0) \ll_\varepsilon q^{1/2+\varepsilon}. \tag{29}$$

Proof. If $2 \mid q$, then $K_q(m_0) = 0$ by Lemma 2.3. We may therefore assume that q is odd. Write

$$q = \prod_{p^n \| q} p^n.$$

The Chinese remainder theorem factors the complete sum into prime-power sums. The additive character on each prime-power component may acquire a unit twist, but because $(m_0, q) = 1$, the twisted coefficient remains coprime to the corresponding prime. Thus Hypothesis 5.2 gives

$$|K_q(m_0)| \le \prod_{p^n \| q} C p^{n/2} \ll_\varepsilon q^{1/2+\varepsilon},$$

where the harmless factor $C^{\omega(q)}$ is absorbed into q^ε. □

Corollary 5.5 (Lower-conductor discrepancy for the oscillatory baseline)**.** *For $(m_0, q) = 1$, with $D_q^{\mathrm{osc}}(m_0) = 0$,*

$$|K_q(m_0) - D_q^{\mathrm{osc}}(m_0)| \ll_\varepsilon q^{1/2+\varepsilon}. \tag{30}$$

6 Weighted ramified defect

Let $Q \ge 3$, $N \ge 1$, and

$$L := \frac{Q^2}{N}. \tag{31}$$

Let $W_t(Q)$ be a smooth or bounded weight satisfying

$$W_t(Q) = 0 \quad \text{unless} \quad t \le L \tag{32}$$

for simplicity. The same proof works with rapidly decaying weights.

Definition 6.1 (Ramified weighted defect)**.** *For an integer m, define*

$$\mathcal{D}_Q^{\mathrm{ram}}(m) := \frac{N}{Q} \sum_{\substack{t \le L \\ (m,t)>1}} \frac{W_t(Q)}{t} \left(K_t(m) - D_t^{\mathrm{ind}}(m) \right), \tag{33}$$

where D_t^{ind} is the induced oscillatory diagonal model.

Theorem 6.2 (Closure of the ramified local defect)**.** *Assume $|W_t(Q)| \ll 1$. Then*

$$\mathcal{D}_Q^{\mathrm{ram}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon. \tag{34}$$

In particular, if $m \ll Q^C$ for a fixed C, then

$$\mathcal{D}_Q^{\mathrm{ram}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon. \tag{35}$$

Proof. For each t in the ramified sum, put

$$h = (m,t), \qquad q = t/h, \qquad m_0 = m/h.$$

Then $(m_0, q) = 1$. By Corollary 4.3 and Corollary 5.5,

$$|K_t(m) - D_t^{\mathrm{ind}}(m)| \le \Lambda_t(m) q^{1/2+\varepsilon}.$$

By Theorem 3.2, $\Lambda_t(m) \le h$, and since $t = hq$,

$$\frac{\Lambda_t(m)}{t} q^{1/2+\varepsilon} \le q^{-1/2+\varepsilon}.$$

Thus

$$|\mathcal{D}_Q^{\mathrm{ram}}(m)| \ll_\varepsilon \frac{N}{Q} \sum_{\substack{t \le L \\ (m,t)>1}} q^{-1/2+\varepsilon}.$$

We overcount by summing over all divisors $h \mid m$ with $h > 1$ and all $q \le L/h$:

$$\begin{aligned} |\mathcal{D}_Q^{\mathrm{ram}}(m)| &\ll_\varepsilon \frac{N}{Q} \sum_{h|m} \sum_{q \le L/h} q^{-1/2+\varepsilon} \\ &\ll_\varepsilon \frac{N}{Q} \sum_{h|m} (L/h)^{1/2+\varepsilon} \\ &\ll_\varepsilon \frac{N}{Q} L^{1/2+\varepsilon} \sum_{h|m} h^{-1/2-\varepsilon}. \end{aligned}$$

The divisor sum is $\ll_\varepsilon m^\varepsilon$. Since $L = Q^2/N$,

$$\frac{N}{Q} L^{1/2} = N^{1/2}.$$

The remaining factor L^ε is absorbed into Q^ε after adjusting ε. This proves (34). □

Remark 6.3 (No gcd-bilinear obstruction in the explicit local sum)**.** *For the explicit sum* (10)*, the ramified local contribution is controlled by exact conductor drop and lower-conductor oscillation. A weighted gcd-bilinear obstruction may still arise in a separate global coefficient-distribution problem, but it is not forced by the ramified local sum itself.*

7 The two-adic sector

The two-adic sector is even simpler in the present model.

Definition 7.1 (Two-adic component)**.** *Define*

$$\mathcal{D}_Q^{(2)}(m) := \frac{N}{Q} \sum_{\substack{t \le L \\ 2 \mid t}} \frac{W_t(Q)}{t} \left(K_t(m) - D_t^{\mathrm{ind}}(m) \right). \tag{36}$$

Proposition 7.2 (Vanishing of the two-adic component)**.** *For the induced oscillatory diagonal model,*

$$\mathcal{D}_Q^{(2)}(m) = 0. \tag{37}$$

Proof. If $2 \mid t$, then $U_t = \varnothing$ by Lemma 2.3, hence $K_t(m) = 0$. The induced oscillatory diagonal model is also zero on this empty admissible set. Therefore every summand in (36) is zero. □

Definition 7.3 (Residual local defect)**.** *Let*

$$\mathcal{D}_Q^{\mathrm{res}}(m) := \mathcal{D}_Q^{(2)}(m) + \mathcal{D}_Q^{\mathrm{ram}}(m). \tag{38}$$

Corollary 7.4 (Residual local closure)**.** *For the complete sum* (10) *and the induced oscillatory diagonal model,*

$$\mathcal{D}_Q^{\mathrm{res}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon. \tag{39}$$

If $m \ll Q^C$, then

$$\mathcal{D}_Q^{\mathrm{res}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon. \tag{40}$$

Proof. Combine Proposition 7.2 and Theorem 6.2. □

8 Comparison with density-scale diagonal models

The result above depends on the induced oscillatory baseline. It is useful to state explicitly why this is the correct local conclusion and why it should not be confused with an arbitrary density-scale diagonal model.

Suppose one instead inserts a density-like lower-conductor diagonal term

$$D_q^{\mathrm{dens}}(m_0) = S_q(m_0)\varphi(q). \tag{41}$$

Then the reduced defect becomes

$$\Lambda_t(m)\{K_q(m_0) - S_q(m_0)\varphi(q)\}. \tag{42}$$

In general, $K_q(m_0)$ is oscillatory and has square-root scale, while $S_q(m_0)\varphi(q)$ is density scale unless $S_q(m_0)$ is itself small or zero. Therefore (42) may be large. This is not a failure of conductor drop; it is a mismatch between an oscillatory local object and a density-like diagonal model.

The local theorem proved here says precisely that the ramified complete sum does not require such a density term. It collapses to the same lower-conductor oscillatory object. If an external global theory demands a density-scale diagonal term, then the remaining problem is to justify that global diagonal choice, not to control an unaccounted ramified local sum.

9 Verification audit

We record the logical dependencies and the status of each step.

Step	Statement	Status
1	$U_t = \varnothing$ for even t	proved, Lemma 2.3
2	exact conductor drop	proved, Theorem 3.2
3	no condition $(h, q) = 1$ needed	included in Theorem 3.2
4	induced diagonal compatibility	proved, Theorem 4.2
5	lower coprime square-root estimate	standard input, Proposition 5.4
6	ramified weighted bound	proved, Theorem 6.2
7	two-adic component	zero, Proposition 7.2
8	residual local closure	proved, Corollary 7.4

The only analytic input not proved from first principles in this paper is the standard prime-power stationary-phase estimate stated as Hypothesis 5.2. This is a classical estimate for non-degenerate rational phases. The present paper verifies the necessary critical-point structure for the phase $m_0/(x(x+1))$ in Lemma 5.1.

The conclusion is local. It closes the residual local defect for the explicit rational sum (10) under the induced oscillatory diagonal model. It does not assert that every global diagonal construction must coincide with this induced model.

10 Conclusion

The main identity of the paper is

$$K_t(m) = \Lambda_t(m) K_{t/(m,t)}\left(\frac{m}{(m,t)}\right).$$

It is exact and includes entangled prime-power ramification. Once the diagonal side is defined by the same fiber-pushforward principle, the corresponding defect satisfies

$$K_t(m) - D_t^{\mathrm{ind}}(m) = \Lambda_t(m)\left(K_{t/(m,t)}\left(\frac{m}{(m,t)}\right) - D_{t/(m,t)}\left(\frac{m}{(m,t)}\right)\right).$$

For the induced oscillatory baseline, the lower-conductor diagonal term is zero in the coprime range, and the remaining lower-conductor complete sum has square-root size. Therefore

$$\mathcal{D}_Q^{\mathrm{ram}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon.$$

The two-adic part vanishes because the admissible set is empty for even moduli. Hence the residual local defect is closed:

$$\mathcal{D}_Q^{\mathrm{res}}(m) \ll_\varepsilon N^{1/2} Q^\varepsilon m^\varepsilon.$$

This is the final local conclusion. The ramified residual complete sum is not an independent density-scale obstruction; it collapses to lower-conductor oscillation.

11 Appendix A: prime-power fiber verification

This appendix expands the prime-power calculation used in Theorem 3.2. The point is to make clear that no hidden assumption $(h,q)=1$ enters the proof. Let $p^n \parallel t$ and let $r = \min(v_p(m), n)$. The local part of the lower conductor is p^{n-r}. We compare admissible residues modulo p^n with admissible residues modulo p^{n-r}.

If $r=0$, the local modulus has not changed. The reduction map is the identity at p^n, so the local fiber size is 1. If $0<r<n$, fix an admissible residue $y \bmod p^{n-r}$. The set of lifts is

$$x = y + p^{n-r} z, \qquad z \bmod p^r.$$

Since $p \nmid y(y+1)$, reducing x modulo p gives the same non-zero values y and $y+1$. Hence every lift remains admissible, and there are exactly p^r lifts.

If $r=n$, the lower conductor has no p-part. The fiber over the unique lower residue consists of all admissible residues modulo p^n. For odd p, the forbidden residues are exactly the two classes 0 and -1 modulo p, each with p^{n-1} lifts. This gives $p^{n-1}(p-2)$. For $p=2$, the two forbidden classes cover all residues modulo 2, because every product of consecutive integers is even. Hence the fiber size is 0.

The local factors above are independent of the lower residue. Therefore CRT multiplication gives a global constant fiber size. This independence is the decisive reason why the ramified local sum collapses to a scalar multiple of the lower-conductor sum rather than to a more complicated quotient-distribution expression.

12 Appendix B: stationary phase details

The lower-conductor estimate in Proposition 5.4 relies on the standard one-dimensional non-degenerate stationary-phase bound. We recall how the phase fits the standard template. For an odd prime p and $p \nmid m_0$, consider

$$f(x) = \frac{m_0}{x(x+1)}.$$

The domain excludes $x = 0, -1$. The derivative is

$$f'(x) = -\frac{m_0(2x+1)}{x^2(x+1)^2}.$$

Hence the only stationary residue modulo p is $x = -1/2$. Since the numerator has a simple zero there and the denominator is non-zero, the critical point is non-degenerate. The standard p-adic stationary-phase estimate then gives a contribution of size $O(p^{n/2})$ from the unique critical class and cancellation from the non-stationary classes. This is the estimate stated as Hypothesis 5.2.

For $n = 1$, the same conclusion follows from the Weil bound for rational functions over finite fields. For $n > 1$, the lifting argument is the usual p-adic analogue: non-stationary residue classes vanish after summing over a sufficiently high lift, while the non-degenerate quadratic expansion at the critical class produces a Gauss-type factor of square-root size. The references [4, 2] contain standard forms of these estimates.

13 Appendix C: rapidly decaying weights

The proof of Theorem 6.2 assumed compact support $t \leq L$ for simplicity. The same estimate holds if the weight satisfies, for every $A > 0$,

$$W_t(Q) \ll_A (1 + t/L)^{-A}.$$

Indeed, after conductor drop the relevant sum is dominated by

$$\frac{N}{Q}\sum_{h|m}\sum_{q\geq 1}(1+hq/L)^{-A}q^{-1/2+\varepsilon}.$$

For $A>2$, the inner sum is

$$\ll_\varepsilon (L/h)^{1/2+\varepsilon}.$$

Thus the same argument gives

$$\mathcal{D}_Q^{\mathrm{ram}}(m)\ll_\varepsilon N^{1/2}Q^\varepsilon m^\varepsilon.$$

This shows that the result is insensitive to the usual smooth truncations appearing in completed sums.

14 Appendix D: statements deliberately not made

The paper proves a local result. It is useful to state the non-claims explicitly. First, the paper does not prove that every possible global diagonal mass equals the induced oscillatory baseline. It proves that if the local diagonal side is induced by the same fiber-pushforward principle as the sum, then it obeys the exact compatibility relation in Theorem 4.2.

Second, the paper does not claim a density-scale main term for $K_q(m_0)$ in the coprime range. The opposite philosophy is used: the coprime lower-conductor sum is treated as oscillatory and bounded at square-root scale.

Third, the paper does not assert a global zero-free region or a Riemann-hypothesis-type theorem. The final conclusion is the local estimate (39) for the explicit rational sum (10) under the induced oscillatory diagonal model.

Fourth, no previous manuscripts or private numbering systems are used as input. All notation and assumptions needed for the local theorem are defined within the present paper.

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