The fine-structure constant as a quantum expectation: a probabilistic model based on bounded continued fractions

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Abstract

The fine-structure constant α is traditionally a free parameter of quantum electrodynamics, inserted by hand into the coupling term $-e\bar{\psi}\gamma^\mu\psi A_\mu$. In this work we propose a different view: α^{-1} is the expectation value of a quantum system whose Hilbert space consists of all finite continued fractions with partial quotients bounded above by 45.

We have found an expression for α^{-1} of the form

$$\alpha^{-1} = A(\pi) - \frac{1}{24A(\pi)} - \frac{1}{A(\pi)^2\pi^2 K},$$

where $A(\pi) = 4\pi^3 + \pi^2 + \pi$ and K is a real number. For the CODATA 2022 value, solving the self-consistency equation yields $K \approx 9.9327912864$, whose continued fraction expansion is

$$K = 10 - \cfrac{1}{14 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{3}}}}}}.$$

(This derivation is discussed in detail in the source code, DOI: 10.5281/zenodo.19802606).

We elevate K to an operator $\hat{K}$ acting on a state $|\{q_1, \ldots, q_d\}\rangle$ as the continued fraction $K(\{q_i\}) = 10 - 1/(q_1 + 1/(q_2 + \ldots))$.

The dimensionless structure operator of the system is

$$\hat{S} = A(\pi)\,\mathbf{1} \;-\; \frac{1}{A'''(\pi)A(\pi)}\,\mathbf{1} \;-\; \frac{1}{A(\pi)^2\pi^2\hat{K}},$$

where $A'''(\pi) = 24$ is the third derivative of the polynomial, not a free parameter. The ground state $|\Psi\rangle$ is a superposition of all sequences of quotients with probability distribution $P(q) \propto e^{-q/(2E_{\text{int}})}$, where $E_{\text{int}} = 5.0$ is calibrated to the historical variance of measurements.

Numerical simulations (1 million measurements obtained by collapsing the state $|\Psi\rangle$, depth 20, precision 200 decimal digits) yield:

- Mean value of $\hat{S}$: $\langle \hat{S} \rangle = 137.035999167828$
- Difference from CODATA 2022: -9.17×10^{-9}
- Standard deviation: 1.40×10^{-8}

The specific sequence $[14, 1, 7, 3, 1, 3]$ was never observed in the simulations, consistent with its theoretical probability $\sim 1.2 \times 10^{-10}$.

The model incorporates five structural seals that make it falsifiable: (1) independence of quotients as an axiom of Hilbert space, (2) interpretation of CODATA values as ensemble averages, (3) phenomenological postulate of the $\{4, 1, 1\}$ polynomial, (4) dimensionless operator without temporal dynamics, (5) subtractive chirality $K = 10 - F$ for numerical stability. A future experiment requiring a quotient $q > 45$ would invalidate the model.

1 Introduction

For more than a century, the fine-structure constant α has been a mysterious parameter of quantum electrodynamics. Its inverse,

$$\alpha^{-1} = 137.035999177(21),$$

is known with extraordinary precision, yet no theoretical derivation from first principles exists.

In previous work we noted that the polynomial

$$A(\pi) = 4\pi^3 + \pi^2 + \pi$$

already approximates α^{-1} to within about 3×10^{-4} [2, 3]. Adding two correction terms yields the three-parameter expression reported in the Abstract.

However, examining historical CODATA values (2006–2022) reveals that the partial quotients change: the sequence $[14, 1, 7, 3, 1, 3]$ appears only for the 2022 value. For other years the quotients are completely different (e.g., $[1, 2, 1, 1, 8, 9, 3, 1, 45, 1]$ for 2014). The only invariant is that **all partial quotients are small** (never exceeding 45).

This observation forces a paradigm shift: there is no fixed continued fraction; there is a **probability distribution** over all continued fractions with bounded quotients. The fine-structure constant is not a number – it is the expectation value of a quantum operator whose eigenstates are sequences of partial quotients.

2 The quantum model

2.1 Phenomenological postulate of the $\{4, 1, 1\}$ polynomial

Postulate 1 (Generator polynomial). *Let*

$$A(x) = 4x^3 + x^2 + x.$$

This polynomial is the empirical starting point of the model. It is known in the literature to approximate α^{-1} *to within* 3×10^{-4}. *The coefficients* $\{4, 1, 1\}$ *are not derived; they are the algebraic seed that initiates the entire construction. Any future theory that derives these coefficients from first principles would naturally supersede our model, but within the current framework they are taken as given.*

Definition 1 (Hilbert space). *Let $\mathcal{H}$ be the Hilbert space spanned by the orthonormal basis vectors $|\{q_1,\ldots,q_d\}\rangle$, where d is the depth (in practice $d=20$ is sufficient for numerical convergence) and the partial quotients q_i are positive integers.*

In principle, q_i can take any integer value ≥ 1. However, the probability distribution of the ground state (defined in Section 2.4) is $P(q) \propto e^{-q/(2E_{int})}$ with $E_{int}=5.0$. For this value, the probability of observing a quotient $q \geq 46$ is less than 2×10^{-5} (less than 0.002%). No historical measurement of α^{-1} (CODATA 2006–2022) has ever required a quotient exceeding 45.

*For practical computational reasons and reproducibility, we therefore restrict the quotients to the set $\{1,2,\ldots,45\}$. This restriction is **falsifiable**: a future measurement requiring a quotient $q>45$ would invalidate the model or require a revision of the statistical cutoff.*

***Independence axiom**: Each depth of the continued fraction corresponds to an independent orthogonal degree of freedom. This is not an approximation of classical continued fraction theory; it is the foundational axiom of our Hilbert space. The factorization of the probability distribution reflects the tensor product structure of $\mathcal{H}$.*

2.2 The operator $\hat{K}$ and subtractive chirality

Definition 2 (Operator $\hat{F}$). *For a basis vector $|\{q_1,\ldots,q_d\}\rangle$ we define the operator $\hat{F}$ as:*

$$\hat{F}|\{q_1,\ldots,q_d\}\rangle = \cfrac{1}{q_1+\cfrac{1}{q_2+\cdots+\cfrac{1}{q_d}}}|\{q_1,\ldots,q_d\}\rangle.$$

Lemma 1 (Domain of $\hat{F}$). *The spectrum of $\hat{F}$ is contained in the interval $(0,1)$.*

Proof. Every finite continued fraction with positive quotients lies strictly between 0 and 1. The upper bound 1 is approached only in the formal limit of an infinite continued fraction, but for finite depth and quotients ≥ 1 the value is always <1. The lower bound 0 is an asymptote as the first quotient becomes arbitrarily large. □

Definition 3 (Operator $\hat{K}$ – subtractive chirality). *We define the operator $\hat{K}$ as:*

$$\hat{K} = 10\,\mathbf{1} - \hat{F}.$$

*The choice of the **subtractive** form (as opposed to the additive form $9\mathbf{1}+\hat{F}$) is dictated by the probability distribution. Small quotients (the most probable) make $\hat{F}\approx 1$ and thus $\hat{K}\approx 9$. The required expectation value $\langle\hat{K}\rangle \approx 9.93$ lies between the bulk of the distribution and the upper bound 10, ensuring numerical stability. The additive form would place the most probable states near the upper bound, making the expectation unstable and dominated by rare fluctuations.*

Remark 1. *Why 10? The self-consistency equation (derived from the structure of $\hat{S}$) requires $\langle\hat{K}\rangle \approx 9.93$. Since $\hat{K} = m\mathbf{1}-\hat{F}$ with $\hat{F}\in(0,1)$, the spectrum of $\hat{K}$ is $(m-1,m)$. The only integer m such that $9.93\in(m-1,m)$ is $m=10$. The number 10 is not a free parameter: it is the topological ceiling imposed by compatibility between the required expectation value and the operator domain.*

2.3 The dimensionless structure operator $\hat{S}$

Definition 4 (Structure operator)**.** *The dimensionless operator $\hat{S}$ is defined as:*

$$\hat{S} = A(\pi)\,\mathbf{1} \;-\; \frac{1}{A'''(\pi)A(\pi)}\,\mathbf{1} \;-\; \frac{1}{A(\pi)^2\pi^2}\,\hat{K}^{-1},$$

where $A'''(\pi) = 24$ is the third derivative of the polynomial $A(x)$ evaluated at $x = \pi$, not a free parameter.

Remark 2. *$\hat{S}$ is not a physical Hamiltonian with dimensions of energy. It is a **dimensionless structure operator** or **topological generator**. Its Hilbert space does not describe the temporal evolution of a particle but maps the allowed states of the continued fraction geometry. Its expectation value directly yields the dimensionless constant α^{-1}, without any conversion factor.*

2.4 Ground state and interaction parameter

Definition 5 (Ground state)**.** *The ground state $|\Psi\rangle$ of the system is a coherent superposition of all admissible sequences of quotients:*

$$|\Psi\rangle = \sum_{\{q_1,\ldots,q_d\}} \sqrt{P(q_1,\ldots,q_d)}\,|\{q_1,\ldots,q_d\}\rangle,$$

with

$$P(q_1,\ldots,q_d) = \prod_{i=1}^{d} \frac{e^{-q_i/(2E_{int})}}{\sum_{k=1}^{45} e^{-k/(2E_{int})}}.$$

Remark 3 (Nature of E_{int})**.** *The parameter $E_{int} = 5.0$ is **not derived from higher mathematical principles**. It represents the effective interaction energy between the quantum system and the experimental apparatus. Its value was calibrated so that the standard deviation of the simulations ($\sigma \approx 1.40 \times 10^{-8}$) reproduces the order of magnitude of the fluctuations observed in historical CODATA values (2006–2022).*

***Mean invariance**: A sensitivity analysis shows that the expectation value $\langle\hat{S}\rangle$ is structurally insensitive to variations of E_{int} in the interval $[3, 10]$. This means that E_{int} determines exclusively the **width** of the distribution of measured values, not their **center**. The convergence to α^{-1}_{CODATA} is therefore a property of the algebraic structure of $\hat{S}$, independent of the calibration of E_{int}.*

$E_{int} = 5.0$ is the only phenomenological element of the model. All other numbers (π, $4, 1, 1$, 24, 10, 45) have structural or observational justifications.

3 Numerical simulations

We performed 10^6 measurements obtained by collapsing the state $|\Psi\rangle$ with depth $d = 20$ and interaction energy $E_{\mathrm{int}} = 5.0$, using a precision of 200 decimal digits (the `mpmath` library). For each collapse we:

1. Draw a random sequence of 20 quotients $q_1, \ldots, q_{20}$ from the distribution $P(q) \propto e^{-q/10}$.

2. Compute F from the continued fraction (bottom-up reconstruction).
3. Compute $K = 10 - F$.
4. Compute $\alpha_{\text{collapse}}^{-1} = A(\pi) - 1/(24A(\pi)) - 1/(A(\pi)^2\pi^2K)$.

3.1 Results

The results are summarized in Table 1.

Table 1: Results of 10^6 collapses (depth $d = 20$, $E_{\text{int}} = 5.0$, 200-digit precision)

Metric	Value
Mean $\langle\hat{S}\rangle$ (predicted α^{-1})	137.035999167828
Standard deviation	1.40×10^{-8}
Minimum	137.035999122117
Maximum	137.035999179475
Range	5.74×10^{-8}
Difference from CODATA 2022	$\mathbf{-9.17 \times 10^{-9}}$
Relative error	6.69×10^{-11}

The distribution of partial quotients is shown in Table 2. The probability decreases monotonically with q, and the largest observed quotient is 45. The specific sequence $[14, 1, 7, 3, 1, 3]$ never appeared in 10^6 collapses, as expected from its theoretical probability $\approx (1/45)^6 \approx 1.2 \times 10^{-10}$.

Table 2: Probability distribution of quotients (from 10^6 collapses, $6 \cdot 10^6$ total samples)

Quotient	Frequency	Probability (%)
1	18.15	18.15
2	14.84	14.84
3	12.20	12.20
4	9.95	9.95
5	8.14	8.14
6	6.64	6.64
7	5.44	5.44
8	4.47	4.47
9	3.64	3.64
10	2.99	2.99
$\vdots$	$\vdots$	$\vdots$
45	0.0026	0.0026

3.2 Depth convergence

Theorem 1 (Euler-Wallis convergence). *For any sequence of quotients $\{q_1, \dots, q_d\}$, the truncated continued fraction value K_d converges to the limit K_∞ (infinite depth). The* *error is bounded by:*

$$|K_d - K_\infty| \le \frac{1}{Q_d Q_{d+1}} < \frac{1}{Q_d^2},$$

where Q_d is the denominator of the d-th convergent, satisfying $Q_d = q_d Q_{d-1} + Q_{d-2}$ with $Q_0 = 1, Q_1 = q_1$.

In our Hilbert space, quotients have mean value $\langle q \rangle \approx 4.5$. Under these conditions, the denominators Q_d grow exponentially with a base much larger than the golden ratio $\phi \approx 1.618$ (which is the worst case, corresponding to $q_i = 1$ for all i). The structure operator contains the term $\frac{1}{A(\pi)^2\pi^2} \approx 10^{-8}$. The error on $\langle \hat{S} \rangle$ induced by truncation is therefore:

$$|\langle \hat{S} \rangle_d - \langle \hat{S} \rangle_\infty| \lesssim \frac{1}{A(\pi)^2\pi^2} \cdot \frac{1}{Q_d^2}.$$

For $d = 6$, the error is already below 10^{-16}. The choice $d = 20$ is therefore an **extreme safety margin** that makes the truncation error completely negligible compared to double precision ($\sim 10^{-15}$) and to the statistical fluctuations of the model ($\sim 10^{-8}$).

3.3 Ergodic convergence verification

The running mean stabilizes rapidly. The standard deviation of the last 1000 means is 5.96×10^{-13}, demonstrating that the expectation value $\langle \hat{S} \rangle = 137.035999167828$ is the ergodic center of mass of the defined Hilbert space.

4 Interpretation

The proposed model replaces the traditional view – α as a fixed number waiting to be discovered – with a statistical description:

- The fundamental object is the quantum state $|\Psi\rangle$, a superposition of all sequences of partial quotients bounded by 45.
- The operator $\hat{S}$ encodes the geometry of the physical vacuum through $A(\pi)$ and the cubic curvature $A'''(\pi) = 24$.
- The interaction energy $E_{\text{int}} = 5.0$ reflects the sensitivity of the experimental apparatus; it determines the width of the distribution of measured values.
- A measurement (a collapse) produces a specific sequence and hence a specific measured value of α^{-1}.
- The historical CODATA values (2006, 2010, 2014, 2018, 2022) are different realizations of the same quantum state. Their scatter ($\sim 2 \times 10^{-8}$) is compatible with the standard deviation of the model (1.40×10^{-8}).

The sequence $[14, 1, 7, 3, 1, 3]$ that accidentally appeared in 2022 is nothing but a rare fluctuation – it has no privileged status. Its theoretical probability is $\sim 1.2 \times 10^{-10}$, and it was not observed in 10^6 collapses.

5 Falsifiability and conclusions

The model is intrinsically falsifiable:

- A future experiment yielding a value of α^{-1} whose continued fraction expansion requires a quotient $q > 45$ would invalidate the model.
- To date, all CODATA values from 2006 to 2022 respect this bound.
- The code is public and reproducible (DOI: 10.5281/zenodo.19802606), executable on Google Colab.

Summary of the five structural seals

1. **Independence of quotients**: axiom of Hilbert space, not an approximation.
2. **CODATA as ensemble averages**: historical values are ergodic averages, not single collapses.
3. **Postulate of** $\{4, 1, 1\}$: the polynomial $A(x) = 4x^3 + x^2 + x$ is the empirical seed of the construction.
4. **Dimensionless operator**: $\hat{S}$ is not a physical Hamiltonian; no units are required.
5. **Subtractive chirality**: $K = 10 - F$, chosen for numerical stability and compatibility with the required expectation value.

Conclusions

These results suggest that the fine-structure constant is not describable by a single fixed continued fraction, but rather by the expectation value of a quantum operator defined on a space of bounded continued fractions. The model is falsifiable, reproducible, and contains only one phenomenological parameter ($E_{\text{int}} = 5.0$) that calibrates exclusively the width of fluctuations, not the central value.

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The code is available on Zenodo at DOI 10.5281/zenodo.19802606 (concept DOI; the latest version is always accessible at this address).

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