Odlyzko's Sign-Alternation Theorem — Complete Proof

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A self-contained reconstruction of the Stanton–Zeilberger–O'Hara route

This manuscript gives a complete, self-contained reconstruction of the Stanton–Zeilberger–O'Hara route to the sign alternation theorem attributed to Odlyzko. The proof derives the KOH identity from the corrected O'Hara–Goodman product-decomposition corridor, turns that corridor into a q-enumeration recurrence, expands the recurrence into the Stanton–Zeilberger multiplicity form, and then proves the finite sign theorem and coefficientwise limiting step directly.

Abstract

For every integer $k \geq 0$ we prove, in a self-contained reconstruction of the known Stanton–Zeilberger–O'Hara route, that the Maclaurin coefficients of

$$\frac{1}{(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{k-1})}$$

alternate in sign. Equivalently, with $(q)_k = \prod_{j=1}^{k}(1-q^j)$, we prove $(1-q)^k/(q)_k \in \mathcal{A}$, where $\mathcal{A}$ denotes the cone of formal power series $\sum a_n q^n$ satisfying $(-1)^n a_n \geq 0$. Stanton and Zeilberger proved this theorem through the KOH identity extracted from O'Hara's constructive proof. The present manuscript records a complete internal reconstruction of that corridor: the KOH identity is derived from the corrected O'Hara–Goodman product decomposition, and the sign-cone bridge, alpha estimates, exceptional $G(0,1)$ correction, and limit $G(N,k) \to 1/(q)_k$ are proved explicitly. An appendix records the sharp exponent threshold for the limiting family $(1-q)^e/(q)_k$.

Scholarly positioning. The result treated here is the theorem of Stanton and Zeilberger on Odlyzko's sign alternation problem. The manuscript does not present the theorem as newly open. Its contribution is a self-contained reconstruction of the proof corridor: corrected O'Hara–Goodman product decomposition, q-enumeration, explicit iteration to KOH, finite sign theorem, and coefficientwise limiting exit.

Guide to the proof structure. The proof spine is

$$\begin{aligned}&\text{corrected O'Hara corridor} \Rightarrow q\text{-enumeration recurrence}\\ &\Rightarrow \text{KOH} \Rightarrow \text{finite sign theorem}\\ &\Rightarrow \text{coefficientwise limit} \Rightarrow \text{Odlyzko.}\end{aligned}$$

The corrected corridor is stated and proved using the internal-active deletion index. This avoids the endpoint ambiguity created by the endpoint correction.

Table of Contents

PART I. Statement and sign-cone algebra

Node OD1. Equivalent q-factorial form and sign cone

Theorem 0.1 (Equivalent form of Odlyzko's conjecture) *For every integer $k \geq 0$, the reciprocal product*

$$R_k(q) = \frac{1}{(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{k-1})}$$

has alternating coefficients if and only if

$$\frac{(1-q)^k}{(q)_k} \in \mathcal{A}, \qquad (q)_k = \prod_{j=1}^{k}(1-q^j).$$

Proof. For $j \geq 1$,

$$1+q+\cdots+q^{j-1} = \frac{1-q^j}{1-q}.$$

Hence

$$R_k(q) = \prod_{j=1}^{k} \frac{1-q}{1-q^j} = \frac{(1-q)^k}{(q)_k}.$$

The factor with $j = 1$ is equal to 1, and for $k = 0$ the product is empty. Membership in $\mathcal{A}$ means precisely that the coefficients alternate in sign starting with a nonnegative constant term. ☐

Definition 0.2 (Alternating sign cone)**.** Let

$$\mathcal{A} = \left\{ f(q) = \sum_{n \geq 0} a_n q^n : (-1)^n a_n \geq 0 \text{ for every } n \right\}.$$

Lemma 0.3 (Elementary closure properties) *The cone $\mathcal{A}$ is closed under finite sums, Cauchy products, even shifts q^{2r}, multiplication by $(1-q)^s$ for $s \geq 0$, and coefficientwise limits.*

Proof. The assertions for finite sums and even shifts are immediate. If $f = \sum a_i q^i$ and $g = \sum b_j q^j$ lie in $\mathcal{A}$, then the coefficient h_n of fg satisfies

$$(-1)^n h_n = \sum_{i+j=n} ((-1)^i a_i)((-1)^j b_j) \geq 0.$$

Thus $fg \in \mathcal{A}$. Also

$$(1-q)^s = \sum_{j=0}^{s} (-1)^j \binom{s}{j} q^j \in \mathcal{A}.$$

Finally, if $f_m = \sum a_{m,n} q^n \in \mathcal{A}$ and $a_{m,n} \to a_n$ coefficientwise, then $(-1)^n a_n$ is a limit of nonnegative real numbers. ☐

Node OD2. Gaussian polynomials and conventions

Definition 0.4 (Gaussian polynomial)**.** For $n, k \geq 0$ set

$$G(n,k) = \binom{n+k}{k}_q = \frac{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+k})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

We use the convention

$$G(n,k) = 0 \quad (n < 0,\ k \geq 0), \qquad G(n,0) = 1 \quad (n \geq 0).$$

Equivalently, $G(b,a)$ is the rank-generating function of partitions fitting in an $a \times b$ rectangle.

PART II. Reconstruction of the KOH identity

Node OD3. Corrected O'Hara–Goodman statistics

Definition 0.5 (Rectangular partition sets and boundary states)**.** For nonnegative integers a, b, let

$$U(b, a) = \{p = (p_1, \dots, p_a) : 0 \le p_1 \le \cdots \le p_a \le b\}.$$

Use the boundary convention

$$p_j = 0 \quad (j \le 0), \qquad p_j = b \quad (j > a).$$

The rank is $|p| = p_1 + \cdots + p_a$. If $b < 0$, set $U(b, a) = \varnothing$. If $a = 0$ and $b \ge 0$, $U(b, 0)$ consists of one empty partition of rank 0. The auxiliary states $U(0, -1)$ and $U(0, -2)$ are singleton sets of rank 0; all other states with negative second coordinate are empty.

Definition 0.6 (Corrected spread, active set, and degree)**.** For $p \in U(b, a)$ define

$$\mathrm{spread}(p) = \max_{1 \le j \le a+2} (p_j - p_{j-2}), \qquad M(p) = \{j : 1 \le j \le a+2,\ p_j - p_{j-2} = \mathrm{spread}(p)\}.$$

Decompose $M(p)$ into maximal consecutive intervals D and put

$$\deg(p) = \sum_D \left\lfloor \frac{|D| + 1}{2} \right\rfloor.$$

For $a > 0$ define the internal active set

$$M_{\mathrm{act}}(p) = M(p) \cap \{2, 3, \dots, a+1\}.$$

Finally set

$$U(b, a, m) = \{p \in U(b, a) : \mathrm{spread}(p) = m\}, \quad U(b, a, m, d) = \{p \in U(b, a, m) : \deg(p) = d\},$$

$$U(b, a, \le m) = \bigcup_{j \le m} U(b, a, j).$$

For the empty object in $U(b, 0)$ the spread is b and the degree is 1 if $b > 0$; for $U(0, 0)$, $U(0, -1)$, and $U(0, -2)$ the spread is 0.

Lemma 0.7 (Endpoint regression and existence of the active index) *Let $a > 0$ and $p \in U(b, a)$ have positive spread m. Then $M_{\mathrm{act}}(p)$ is nonempty. If $1 \in M(p)$, then $2 \in M(p)$; if $a+2 \in M(p)$, then $a+1 \in M(p)$. Thus endpoints attach to active intervals but are never chosen as deletion indices.*

Proof. The endpoint candidates are

$$p_1 - p_{-1} = p_1, \qquad p_{a+2} - p_a = b - p_a.$$

They are dominated by internal candidates:

$$p_2 - p_0 = p_2 \geq p_1, \qquad p_{a+1} - p_{a-1} = b - p_{a-1} \geq b - p_a.$$

The same inequalities remain valid for $a = 1$, where $p_2 = b$ and $p_0 = 0$. Hence the maximum positive spread is attained in the internal range $2 \leq j \leq a + 1$.

If $1 \in M(p)$, then $p_1 = m$. Since $p_2 \geq p_1$ and $p_2 - p_0 \leq m$, one has $p_2 = m$, so $2 \in M(p)$. The reflected argument shows that $a + 2 \in M(p)$ implies $a + 1 \in M(p)$. □

Lemma 0.8 (Endpoint degree regression) *The corrected endpoint convention changes the internal degree only in the double-staircase case*

$$p = (m, m, 2m, 2m, \ldots, rm, rm) \in U((r + 1)m, 2r),$$

where the degree is increased by one. For $a = 0$, the empty element of $U(m, 0)$ has spread m and degree 1.

Proof. By the preceding lemma endpoints can only extend an already active interval. If the left endpoint interval continues, the equations force paired propagation:

$$p_{2i-1} = p_{2i} = im$$

as long as the interval continues. The right endpoint has the reflected propagation. Therefore a degree change is possible only when the full corrected interval is $[1, a + 2]$. Then $a = 2r$, the displayed double staircase is forced, and $b = (r + 1)m$. The internal interval has length $2r$ while the corrected interval has length $2r + 2$, and

$$\left\lfloor \frac{2r + 3}{2} \right\rfloor - \left\lfloor \frac{2r + 1}{2} \right\rfloor = 1.$$

The case $a = 0$ is exactly the stated boundary convention. □

Lemma 0.9 (Two-parity telescoping) *If $p \in U(b, a)$ has spread at most m, then for every $A \geq B$,*

$$p_A + p_{A+1} - p_B - p_{B+1} \leq m(A - B),$$

where boundary values are interpreted in the ambient rectangle.

Proof. Sum the inequalities $p_i - p_{i-2} \leq m$ along the parity class of A down to B, and along the parity class of $A + 1$ down to $B + 1$. If $A - B$ is even, the sums telescope exactly. If $A - B$ is odd, the same calculation leaves one adjacent comparison, and monotonicity of p closes the estimate. □

Node OD4. Internal-active product decomposition

Definition 0.10 (Internal-active deletion)**.** Let $p \in U(b, a, m, d)$ with $a > 0$ and $m > 0$, outside the endpoint singleton cases. Put

$$h = \max M_{\text{act}}(p), \qquad t = h - 1.$$

Thus $1 \le t \le a$ and $p_{t+1} - p_{t-1} = m$. Define the deletion $p' \in U(b - m, a - 2)$ by

$$p'_i = p_i \quad (1 \le i \le t - 2), \qquad p'_i = p_{i+2} - m \quad (t - 1 \le i \le a - 2).$$

The deleted charge is

$$r_1 = p_t + p_{t+1} + m(a - t + 1) - 2b.$$

Repeated deletion produces a tuple $r = (r_1, \dots, r_d)$ and a residual partition q.

Lemma 0.11 (Deletion surgery and degree drop) *One internal-active deletion is well-defined. It preserves the bound on spread by m and lowers the corrected degree by exactly one unless the residual spread has already dropped to at most $m - 1$.*

Proof. Since $p_{t+1} - p_{t-1} = m$,

$$p'_{t-1} = p_{t+1} - m = p_{t-1}.$$

Thus the splice is weakly increasing:

$$p'_{t-2} = p_{t-2} \le p_{t-1} = p'_{t-1}.$$

The tail entries satisfy $0 \le p_{i+2} - m \le b - m$, so $p' \in U(b - m, a - 2)$.

All two-step differences of p' are inherited from two-step differences of p, except at the two splice positions. For indices strictly to the left of the splice, the differences are unchanged. For indices strictly to the right, both terms are shifted by m, so the differences are again unchanged. At the splice the identities reduce to the original inequalities $p_j - p_{j-2} \le m$ and to the exact active relation $p_{t+1} - p_{t-1} = m$. Hence no two-step difference in p' is greater than m.

It remains to prove the degree drop. Let $D = [u, v]$ be the last corrected active interval of $M(p)$ which contains $h = t + 1 = \max M_{\text{act}}(p)$. Only this final interval can change. Every earlier active interval lies strictly to the left of the splice and is transported identically; the inactive gap before D remains inactive, because otherwise the transported two-step inequality would create an active bridge in p before deletion.

The transport of the final interval is explicit:

case	$D \subset M(p)$	$D' \subset M(p')$	degree change
singleton	$\{h\}$	$\varnothing$	$1 \to 0$
internal final block	$[u, h]$ $(u < h,\ h \le a + 1)$	$[u, h - 2]$ or $\varnothing$	-1
right endpoint attached	$[u, a + 2]$ $(h = a + 1)$	$[u, a]$ or $\varnothing$	-1

Here D' is read in the corrected endpoint range of the smaller rectangle. In the singleton case the contribution is $\lfloor (1+1)/2 \rfloor = 1$ and then disappears. In the two non-singleton cases the image interval is shorter by exactly two. If $L = |D| \geq 2$, then

$$\left\lfloor \frac{L+1}{2} \right\rfloor - \left\lfloor \frac{(L-2)+1}{2} \right\rfloor = 1.$$

For the right-endpoint attached case, $a+2 \in M(p)$ implies $a+1 \in M(p)$, so $h = a+1$ and $t = a$; after deletion the corrected endpoint range of $U(b-m, a-2)$ ends at a, giving precisely the interval $[u, a]$ above.

No two active intervals merge across the splice. A merger would require the inactive gap immediately before D to become active in p', but its two-step difference is one of the transported two-step differences of p or is bounded by the strict splice inequality; either alternative contradicts maximality of the final interval in p.

Thus one deletion lowers the corrected degree by exactly one whenever spread m remains. If no active interval remains, the residual spread is at most $m-1$. □

Lemma 0.12 (Deleted charge tuple) *The tuple produced by repeated internal-active deletion satisfies*

$$0 \leq r_1 \leq r_2 \leq \cdots \leq r_d \leq am + 2m - 2b.$$

Hence $r \in U(am + 2m - 2b, d)$.

Proof. For the first deletion, the lower bound follows from two-parity telescoping with $(A, B) = (a+1, t)$:

$$2b - p_t - p_{t+1} \leq m(a+1-t),$$

which is equivalent to $r_1 \geq 0$. The upper bound follows from telescoping with $(A, B) = (t, -1)$:

$$p_t + p_{t+1} \leq m(t+1),$$

and therefore $r_1 \leq am + 2m - 2b$.

The same upper bound remains unchanged after one deletion because

$$(a-2)m + 2m - 2(b-m) = am + 2m - 2b.$$

Let the next internal active index, expressed in the original coordinates, be $u+1$. Since $t+1$ was maximal internal active, $u \leq t-2$. Rewriting the next charge in original coordinates gives

$$r_2 = p_u + p_{u+1} + m(a-u+1) - 2b.$$

Thus

$$r_2 - r_1 = m(t-u) - \big((p_t + p_{t+1}) - (p_u + p_{u+1})\big).$$

Two-parity telescoping with $(A, B) = (t, u)$ gives

$$(p_t + p_{t+1}) - (p_u + p_{u+1}) \leq m(t-u),$$

so $r_2 \geq r_1$. Iteration proves the full monotone chain. □

Lemma 0.13 (Insertion intervals) *Let* $p' \in U(b-m, a-2)$ *have spread at most* m. *For* $1 \leq t \leq a$ *put*

$$L_t = p'_{t-1} + p'_t - mt, \qquad R_t = p'_{t-2} + p'_{t-1} - m(t-1),$$

using the boundary convention in the smaller rectangle. The half-open intervals

$$I_t = (L_t, R_t] \quad (1 \leq t < a), \qquad I_a = [L_a, R_a]$$

are disjoint and cover the full range $[2b - m(a+2), 0]$.

Proof. The spread bound gives $p'_t - p'_{t-2} \le m$, hence $L_t \le R_t$. Moreover

$$R_{t+1} = p'_{t-1} + p'_t - mt = L_t,$$

so the intervals are adjacent and disjoint with the stated endpoint convention. The top endpoint is $R_1 = p'_{-1} + p'_0 = 0$. The bottom endpoint is

$$L_a = p'_{a-1} + p'_a - ma = (b-m) + (b-m) - ma = 2b - m(a+2).$$

□

Lemma 0.14 (Charge order and tail maximality) *In the recursive insertion process, after the tail $r_2, \ldots, r_d$ has been inserted to form p', the insertion determined by r_1 creates the maximal internal active index of the reconstructed partition.*

Proof. Let t_1 be the insertion position selected for r_1. If the tail is empty, there is nothing to prove beyond the local verification in the insertion lemma. Otherwise let

$$h' = u + 1 = \max M_{\mathrm{act}}(p')$$

be the maximal internal active index of the already constructed tail partition p'. The first tail charge, rewritten in the parameters of the larger rectangle, is

$$r_2 = p'_u + p'_{u+1} + m(a-u+1) - 2b.$$

The insertion value for r_1 is

$$X_1 = r_1 - m(a+2) + 2b.$$

Since the charge tuple is weakly increasing, $r_1 \le r_2$, and hence

$$X_1 \le p'_u + p'_{u+1} - m(u+1) = L_{u+1}.$$

The insertion intervals are ordered downward and satisfy $R_{j+1} = L_j$. Therefore $X_1 \le L_{u+1}$ forces the selected insertion position to satisfy

$$t_1 \ge u + 2.$$

Thus every internal active index of p' is at most $u + 1 \le t_1 - 1$. After insertion, these old active indices remain to the left of the new exact active relation

$$p_{t_1+1} - p_{t_1-1} = m.$$

The local inequality $X_1 > L_{t_1}$ gives, for $t_1 < a$,

$$p_{t_1+2} - p_{t_1} < m,$$

so the immediate right index $t_1 + 2$ is not active. Every internal index $j \ge t_1 + 3$ in the reconstructed partition corresponds to the index $j - 2 \ge t_1 + 1$ in p', and no such index is active by $t_1 \ge u + 2$. Hence no internal active index of p exceeds $t_1 + 1$.

If $t_1 = a$, the corrected endpoint $a + 2$ may attach to the final corrected interval, but it is not an internal deletion index. Thus the new active pair is exactly the maximal internal active pair. □

Lemma 0.15 (Insertion inverse) *Given*

$$(q, r) \in U(b - md, a - 2d, \le m - 1) \times U(am + 2m - 2b, d),$$

there is a unique recursive insertion inverse to internal-active deletion.

Proof. Assume the inverse has already built p' from q and the tail $r_2, \dots, r_d$. Put

$$X = r_1 - m(a+2) + 2b.$$

Since $0 \le r_1 \le am + 2m - 2b$, one has $X \in [2b - m(a+2), 0]$. Choose the unique t with $X \in I_t$ and set

$$p_i = p'_i \quad (1 \le i \le t-1),$$
$$p_t = X + mt - p'_{t-1},$$
$$p_i = p'_{i-2} + m \quad (t+1 \le i \le a).$$

The inequality $X \le R_t$ gives

$$p_t \le p'_{t-2} + m,$$

so $p_t - p_{t-2} \le m$. The inequality $X > L_t$ gives, for $t < a$,

$$p_t > p'_t,$$

so

$$p_{t+2} - p_t = (p'_t + m) - p_t < m.$$

For $t = a$ there is no interior index to the right of the splice; only the corrected right endpoint may attach.

Adjacent order is separate. Since $p'_{t-1} \le p'_t$ and $p'_t < p_t$ for $t < a$, while the endpoint case is read with the same boundary convention, we have

$$p_{t-1} = p'_{t-1} \le p_t.$$

Also

$$p_t \le p'_{t-2} + m \le p'_{t-1} + m = p_{t+1}.$$

Thus p is weakly increasing and has spread at most m away from the inserted exact relation

$$p_{t+1} - p_{t-1} = m.$$

Therefore $t+1 \in M_{\text{act}}(p)$. By the tail-maximality lemma, no larger internal active index is created. Hence internal-active deletion removes exactly the inserted pair and recovers p' and r_1. Induction on d proves uniqueness and the inverse identities. □

Lemma 0.16 (Rank shift) *For one deletion step,*

$$|p| = |p'| + r_1 + 2(b - m).$$

After d recursive steps,

$$|p| = |q| + |r| + 2bd - md(d+1).$$

Proof. From the deletion formula,

$$|p'| = \sum_{i=1}^{t-2} p_i + \sum_{i=t-1}^{a-2} (p_{i+2} - m) = \sum_{i=1}^{t-2} p_i + \sum_{j=t+1}^{a} p_j - m(a-t).$$

Therefore

$$|p| - |p'| = p_{t-1} + p_t + m(a-t).$$

Since $p_{t+1} - p_{t-1} = m$,

$$|p| - |p'| = p_t + p_{t+1} + m(a-t-1).$$

But

$$r_1 + 2(b-m) = p_t + p_{t+1} + m(a-t+1) - 2b + 2b - 2m = p_t + p_{t+1} + m(a-t-1).$$

This proves the one-step shift. At the jth deletion the current first rectangle parameter is $b-(j-1)m$, so the accumulated constant is

$$\sum_{j=1}^{d} 2(b-jm) = 2bd - md(d+1).$$

Adding the charge sum $|r|$ and the final residual rank $|q|$ gives the claimed formula. □

Theorem 0.17 (Corrected product decomposition) *For $a, b, m, d > 0$ there is a rank-shifting bijection*

$$\Phi : U(b, a, m, d) \longrightarrow U(b - md, a - 2d, \leq m-1) \times U(am + 2m - 2b, d),$$

such that, if $\Phi(p) = (q, r)$, then

$$|p| = |q| + |r| + 2bd - md(d+1).$$

The statement is compatible with the boundary singleton conventions.

Proof. For nonexceptional strata, repeated internal-active deletion gives (q, r). The deletion lemma proves that after exactly d deletions the first component has spread at most $m-1$. The charge lemma proves that r lies in the required rectangle. The insertion lemma proves bijectivity, and the rank-shift lemma proves the displayed identity.

It remains to check endpoint and auxiliary singleton states.

For the double-staircase endpoint correction,

$$p_\star = (m, m, 2m, 2m, \ldots, rm, rm) \in U((r+1)m, 2r, m, r+1),$$

the target is

$$U(0, -2, \leq m-1) \times U(0, r+1),$$

a singleton product. Its rank shift is

$$2(r+1)^2 m - m(r+1)(r+2) = mr(r+1) = |p_\star|.$$

For the empty element of $U(m, 0, m, 1)$, the target is

$$U(0, -2, \leq m-1) \times U(0, 1),$$

again a singleton product, and the rank shift is $2m - 2m = 0$.

The odd auxiliary state $U(0, -1)$ occurs when the repeated deletion leaves height -1 and zero width. The basic visible case is $a = 1$, $b = m$, and $d = 1$: for $p = (x) \in U(m, 1, m, 1)$, the target is

$$U(0, -1, \leq m-1) \times U(m, 1).$$

Here the deleted charge is $r_1 = x$, so the second component is the one-part partition (x) in $U(m, 1)$, the first component is the singleton $U(0, -1)$, and the rank shift is

$$2bd - md(d+1) = 2m - 2m = 0,$$

so $|p| = |r_1|$.

In general, suppose $a - 2d = -1$ and a recursive branch survives to the odd terminal height. Just before the final deletion the current height is 1 and the current width is

$$b_{\mathrm{cur}} = b - (d-1)m.$$

A one-row element $p = (x) \in U(b_{\mathrm{cur}}, 1)$ has corrected spread

$$\max\{x,\ b_{\mathrm{cur}},\ b_{\mathrm{cur}} - x\} = b_{\mathrm{cur}}.$$

Since the final deletion belongs to the exact-spread-m layer, $b_{\mathrm{cur}} = m$. Hence the residual width after the final deletion is

$$b_{\mathrm{cur}} - m = 0,$$

equivalently $b - md = 0$. Thus the residual component is precisely the singleton $U(0, -1)$ and contributes rank zero. The same rank formula and inverse construction remain valid.

All other negative-height states are empty and contribute nothing. □

State	**Value**	**Role**
$U(0, A),\ A \geq 0$	singleton	ordinary terminal branch
$U(0, -1)$	singleton	auxiliary odd terminal state
$U(0, -2)$	singleton	endpoint correction state
$U(B, A),\ B \neq 0$, level 0	empty	terminal branch dies
$U(B, A),\ A < 0$ except $(0, -1), (0, -2)$	empty	illegal negative-height state

Node OD5. q-enumeration recurrence

Definition 0.18 (Cumulative weight)**.** For $a, b \geq 0$ and $m \geq 0$ define

$$v(b, a; m) = \sum_{p \in U(b,a,\leq m)} q^{|p|}.$$

The same notation is used for the auxiliary singleton states. If the relevant set is empty, the value is 0. If $m \geq b$, then $v(b, a; m) = G(b, a)$.

Theorem 0.19 (Corrected q-enumeration recurrence) *Let $a, b \geq 0$ and $m \geq 1$. With the boundary conventions above,*

$$v(b, a; m) = v(b, a; m-1) + \sum_{d \geq 1} q^{2bd - md(d+1)} G(am + 2m - 2b, d)\, v(b - md, a - 2d; m - 1).$$

Equivalently, one may write a sum over $d \geq 0$ only if the $d = 0$ term is interpreted separately as the lower-layer branch $v(b, a; m-1)$.

Proof. The layer of exact spread m decomposes by degree:

$$v(b, a; m) - v(b, a; m-1) = \sum_{d \geq 1} \sum_{p \in U(b,a,m,d)} q^{|p|}.$$

The product decomposition and its rank shift give

$$\sum_{p \in U(b,a,m,d)} q^{|p|} = q^{2bd - md(d+1)} \left(\sum_{r \in U(am+2m-2b,d)} q^{|r|} \right) \left(\sum_{q \in U(b-md,a-2d,\leq m-1)} q^{|q|} \right).$$

For $d > 0$ the first parenthesis is $G(am + 2m - 2b, d)$, with the convention that it is zero if the first argument is negative. The second parenthesis is $v(b - md, a - 2d; m - 1)$. Adding the lower layer gives the recurrence. The recurrence is not applied at $m = 0$. $\square$

Node OD6. Iteration to the KOH identity

Lemma 0.20 (Terminal collapse) *After iterating the recurrence from $v(k,n;k)$ down to level $m=0$, a branch indexed by choices $d_k, d_{k-1}, \dots, d_1$ can survive only if*

$$B_1 = k - \sum_{s=1}^{k} s d_s = 0.$$

When $B_1 = 0$, the terminal factor is 1 for the ordinary state $A_1 \geq 0$ and for the auxiliary states $A_1 = -1, -2$. All other terminal states vanish. If $B_1 = 0$ but $A_1 < -2$, the branch has already vanished through a negative Gaussian factor.

Proof. At level $m=0$, the only ordinary rectangle with spread at most 0 is the zero-width rectangle $U(0, A_1)$ with $A_1 \geq 0$, and it contributes 1. The auxiliary states $U(0,-1)$ and $U(0,-2)$ are singletons. All states with $B_1 \neq 0$ or with illegal negative height are empty.

Assume $B_1 = 0$ and $A_1 < -2$. Let s_0 be the smallest index with $d_{s_0} > 0$. The Gaussian first argument created at step s_0 is

$$s_0 n - 2(k - s_0) + 2 \sum_{t > s_0} (t - s_0) d_t = s_0(A_1 + 2),$$

which is negative. Hence that Gaussian factor is zero, and the branch is killed before the terminal layer. □

Theorem 0.21 (KOH identity derived from the recurrence) *For $n, k \geq 0$,*

$$G(n,k) = \sum_{\lambda \vdash k} q^{2n(\lambda)} \prod_{s=1}^{k} G\left(sn - 2(k-s) + 2\sum_{t>s}(t-s)d_t,\ d_s \right),$$

where $\lambda = (1^{d_1} 2^{d_2} \cdots k^{d_k})$ and

$$2n(\lambda) = \sum_{r=1}^{k} (D_r^2 - D_r), \qquad D_r = \sum_{s \geq r} d_s.$$

Proof. Use $G(n,k) = G(k,n)$ and start from

$$G(n,k) = G(k,n) = v(k,n;k).$$

At step $s = k, k-1, \dots, 1$, choose $d_s \geq 0$ and set

$$B_{k+1} = k, \qquad A_{k+1} = n,$$

$$B_s = B_{s+1} - s d_s, \qquad A_s = A_{s+1} - 2 d_s.$$

After the previous choices $d_k, d_{k-1}, \dots, d_{s+1}$,

$$B_{s+1} = k - \sum_{t>s} t d_t, \qquad A_{s+1} = n - 2\sum_{t>s} d_t.$$

The Gaussian factor produced at step s is

$$G(sA_{s+1} + 2s - 2B_{s+1}, d_s).$$

Substitution gives

$$sA_{s+1} + 2s - 2B_{s+1} = sn - 2(k-s) + 2\sum_{t>s}(t-s)d_t.$$

By terminal collapse, surviving branches are exactly multiplicity vectors satisfying $\sum_s s d_s = k$, hence partitions $\lambda \vdash k$.

The accumulated exponent is computed from the step contributions

$$e_s = 2B_{s+1}d_s - s d_s(d_s+1).$$

Using $B_{s+1} = k - \sum_{t>s} t d_t$ and $k = \sum_u u d_u$ on surviving branches,

$$\sum_s e_s = \sum_s s d_s^2 + 2\sum_{s<t} s d_s d_t - \sum_s s d_s.$$

For $D_r = \sum_{s \geq r} d_s$,

$$\sum_{r\geq 1} D_r^2 = \sum_s s d_s^2 + 2\sum_{s<t} s d_s d_t, \qquad \sum_{r\geq 1} D_r = \sum_s s d_s.$$

Therefore $\sum_s e_s = \sum_r (D_r^2 - D_r) = 2n(\lambda)$. Combining factors over all surviving branches gives the displayed identity. $\square$

Remark 0.22 (Equivalent Stanton–Zeilberger indexing)**.** The same product can be written as

$$\prod_{i=0}^{k-1} G\left((k-i)n - 2i + \sum_{j=0}^{i-1} 2(i-j)d_{k-j},\ d_{k-i}\right),$$

by putting $s = k - i$.

PART III. The sign bridge from KOH

Node OD7. Alpha estimates and exceptional refund

Throughout this part let $k \geq 0$, let $r \geq 0$, and put

$$c = \left\lceil \frac{k}{2} \right\rceil, \qquad m = \min(c, r).$$

For a partition $\lambda \vdash k$ define

$$\alpha(\lambda) = m - \sum_s \left\lceil \frac{d_s}{2} \right\rceil.$$

Lemma 0.23 (Large-r estimate) *If $r \geq \lceil k/2 \rceil$, then $\alpha(\lambda) \geq 0$ for every partition $\lambda \vdash k$.*

Proof. Here $m = \lceil k/2 \rceil$. Let $S = \sum_s \lceil d_s/2 \rceil$. For each part size s, group its d_s copies into $\lceil d_s/2 \rceil$ packets, each of size one or two. Every packet with $s \geq 2$ has total weight at least 2. For $s = 1$, every packet has total weight 2 except possibly one singleton of weight 1. Hence

$$k \geq 2S - 1.$$

Since S is an integer, $S \leq \lceil k/2 \rceil = m$. □

Lemma 0.24 (Small-r estimate and exceptional factor) *Assume $r < \lceil k/2 \rceil$ and let the KOH summand indexed by $\lambda \vdash k$ be nonzero after the specialization $n = 2r$. Then either $\alpha(\lambda) \geq 0$, or λ has exactly $r + 1$ distinct parts. In the latter case one KOH factor is $G(0, 1) = 1$, and the corresponding summand still lies in $\mathcal{A}$ in the sign-bridge proof.*

Proof. Here $m = r$. Since the summand is nonzero, the factor with part size $s = 1$ has nonnegative first argument:

$$r - (k - 1) + \sum_{t>1} (t - 1) d_t \geq 0.$$

Let $\ell = \sum_s d_s$ be the number of parts. Since

$$k = \ell + \sum_{t>1} (t - 1) d_t,$$

we get $\ell \leq r + 1$.

If $\ell \leq r$, then $\sum_s \lceil d_s/2 \rceil \leq \ell \leq r = m$. If $\ell = r + 1$ and some multiplicity is at least 2, then $\sum_s \lceil d_s/2 \rceil \leq \ell - 1 = r$. Thus only the case of $r + 1$ distinct parts remains.

Let $s = \min \lambda$. Then $d_s = 1$, $\sum_{t>s} d_t = r$, and $k = s + \sum_{t>s} t d_t$. The corresponding KOH factor has

$$\frac{A_s}{2} = rs - (k - s) + \sum_{t>s} (t - s) d_t = rs - s \sum_{t>s} d_t = 0.$$

Thus this factor is $G(0, 1) = 1$. It is left as 1 rather than replaced by $(1 - q)G(0, 1)/(1 - q)$; the unreplaced factor refunds one power of $(1 - q)$ and repairs the apparent exponent $\alpha = -1$. □

Node OD8. Finite sign theorem

Theorem 0.25 (Finite sign theorem) *If $N, k \geq 0$ and Nk is even, then*

$$(1-q)^m G(N,k) \in \mathcal{A}, \qquad m = \min\left(\left\lceil \frac{k}{2} \right\rceil, \left\lfloor \frac{N+1}{2} \right\rfloor\right).$$

Proof. By symmetry $G(N,k) = G(k,N)$, it is enough to prove the theorem for $N = 2r$. The cases $k = 0$ and $r = 0$ are immediate. The case $k = 1$ follows from

$$G(2r,1) = 1 + q + \cdots + q^{2r}, \qquad (1-q)G(2r,1) = 1 - q^{2r+1} \in \mathcal{A}.$$

Assume $k \geq 2$ and argue by strong induction on the pair $(k, 2r)$, ordered lexicographically by k and then by the first argument.

Apply KOH to $G(2r,k)$. Discard any summand with a negative Gaussian first argument. For each remaining factor $G(A_s, d_s)$, the first argument A_s is nonnegative and even. The induction hypothesis gives

$$(1-q)^{u_s} G(A_s, d_s) \in \mathcal{A}, \quad u_s = \min\left(\left\lceil \frac{d_s}{2} \right\rceil, \left\lfloor \frac{A_s+1}{2} \right\rfloor\right),$$

where the induction is on the smaller pair (d_s, A_s); the only possible factor with $d_s = k$ occurs for $\lambda = 1^k$, and then its first argument is $2r - 2(k-1) < 2r$ unless the summand has already vanished. Since $u_s \leq \lceil d_s/2 \rceil$, closure under extra multiplication by powers of $(1-q)$ gives

$$F(A_s, d_s) = (1-q)^{\lceil d_s/2 \rceil} G(A_s, d_s) \in \mathcal{A}.$$

For $d_s = 0$ this means $G(A_s, 0) = 1$.

A nonzero summand of $(1-q)^m G(2r,k)$ becomes

$$(1-q)^{\alpha(\lambda)} q^{2n(\lambda)} \prod_s F(A_s, d_s),$$

except in the exceptional case of the previous lemma, where the $G(0,1)$ factor is left unreplaced and one power of $(1-q)$ is refunded. The product lies in $\mathcal{A}$, the shift is even, and the alpha lemmas give a nonnegative remaining power of $(1-q)$. Hence every nonzero summand lies in $\mathcal{A}$, and finite summation preserves membership.

If N is even, this is the direct case. If N is odd and Nk is even, then k is even; by symmetry $G(N,k) = G(k,N)$, and the already proved even-first-argument case applies to $G(k,N)$. $\square$

PART IV. The limiting exit

Node OD9. Coefficientwise limit and Odlyzko's theorem

Lemma 0.26 (Gaussian limit) *For every fixed k,*

$$G(N,k) \longrightarrow \frac{1}{(q)_k}$$

coefficientwise as $N \to \infty$. The convergence may be taken along even values of N.

Proof. Fix a degree d. If $N > d$, then the numerator

$$\prod_{j=1}^{k}(1-q^{N+j})$$

has no nonconstant term of degree at most d. Thus the coefficient of q^d in $G(N,k)$ equals the coefficient of q^d in $1/(q)_k$ for every $N > d$. $\square$

Theorem 0.27 (Odlyzko's sign-alternation theorem) *For every integer $k \geq 0$,*

$$\frac{(1-q)^k}{(q)_k} \in \mathcal{A}.$$

Equivalently, the reciprocal product in Odlyzko's conjecture has alternating coefficients.

Proof. Fix k and let $N \to \infty$ through even integers. By the finite sign theorem,

$$(1-q)^{\lceil k/2 \rceil} G(N,k) \in \mathcal{A}.$$

The Gaussian limit and coefficientwise closure give

$$\frac{(1-q)^{\lceil k/2 \rceil}}{(q)_k} \in \mathcal{A}.$$

Since $(1-q)^{k-\lceil k/2 \rceil} \in \mathcal{A}$ and $\mathcal{A}$ is closed under products,

$$\frac{(1-q)^k}{(q)_k} \in \mathcal{A}.$$

The equivalent form from OD1 identifies this series with $R_k(q)$ and completes the proof of the sign-alternation theorem. $\square$

APPENDIX. Sharpness

Node OD10. Sharp exponent threshold

Theorem 0.28 (Sharp exponent threshold) *For $k \geq 1$ and $e \in \mathbb{Z}_{\geq 0}$,*

$$\frac{(1-q)^e}{(q)_k} \in \mathcal{A} \quad \Longleftrightarrow \quad e \geq \left\lceil \frac{k}{2} \right\rceil.$$

Proof. Sufficiency at $e = \lceil k/2 \rceil$ follows from the finite sign theorem and the Gaussian limit; larger e follows by multiplying by extra powers of $(1-q)$.

For necessity, assume $e < \lceil k/2 \rceil$. At $q = 1$,

$$(q)_k = k!(1-q)^k + O((1-q)^{k+1}),$$

so $(1-q)^e/(q)_k$ has a pole of order $k - e$ at $q = 1$. At any other root of unity of order $t \geq 2$, the pole order is at most the number of $j \leq k$ divisible by t, hence at most $\lfloor k/2 \rfloor$. Since $e < \lceil k/2 \rceil$,

$$k - e > \left\lfloor \frac{k}{2} \right\rfloor.$$

Thus $q = 1$ is the unique dominant pole. Its principal part gives

$$[q^n]\frac{(1-q)^e}{(q)_k} \sim \frac{n^{k-e-1}}{k!(k-e-1)!} > 0.$$

For all sufficiently large odd n, the coefficient is positive, contradicting membership in $\mathcal{A}$. $\square$

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