A Universal Discovery Ladder in Rowland's Sequence

Thomas Kavanagh

Abstract

Rowland's sequence is defined by the recurrence

$$a(n) = a(n-1) + \gcd(n, a(n-1)),$$

and is well known for the unexpectedly rich structure of its non-trivial terms. This paper studies the Discovery subsequence. Computations for the prime initial conditions

$$a(1) = s, \qquad s \in \{7, 11, 13, 17, 19, 23, 29, 31\},$$

carried through the first 2500 non-trivial events, reveal a striking and highly stable pattern. After a short startup transient, each observed Discovery index n_D is generated exactly by the predictive constant of the immediately preceding non-trivial event. In the notation used below,

$$C_{\text{prev}} = n_D.$$

The same computations show that every observed Discovery index satisfies

$$n_D \equiv 2 \pmod 3,$$

that the median normalised gap into Discovery is exactly 1/2, and that the median ratio of consecutive Discovery indices is exactly 2 for every tested prime start. Only four exceptions to the reset law

$$D \to C(g = 3)$$

were found, and all occur at the initial Discovery step of certain non-canonical starts. These results point to a simple recursive structure governing the long-run Discovery dynamics of Rowland's sequence.

1 Introduction

Rowland's sequence [4] is defined by

$$a(1) = 7, \qquad a(n) = a(n-1) + \gcd(n, a(n-1)) \quad (n > 1).$$

Its definition is elementary. Its behaviour is not. Even the non-trivial terms are arranged in a way that is far from transparent, and much of the structure only becomes visible after the recurrence is reorganised in a more useful form.

This paper studies the Discovery subsequence of the non-trivial events. The main finding is that, after a short startup transient, the Discovery indices appear to satisfy a rigid ladder law across several prime initial conditions. The predictive constant of the immediately preceding non-trivial event equals the new Discovery index exactly. At the same time, the same Discovery indices satisfy a congruence law, a half-gap law, and a doubling law.

The work builds on three earlier papers. The first [1] introduced a predictive criterion for the next non-trivial index and replaced long stretches of naive iteration by a direct event-based description. The second [2] organised the non-trivial events into Discovery and Consolidation phases. The third [3] studied the broader event dynamics across perturbed prime initial conditions. Those papers supply the framework used here, but the present paper is logically independent in its main claim. Its purpose is to isolate the Discovery ladder itself, document its persistence across the tested starts, and state the conjectures suggested by the computations.

2 Predictive framework and event classes

We recall only the notation needed below. Let $(n_k, a(n_k))$ be a non-trivial state, so that

$$g(n_k) = a(n_k) - a(n_k - 1) > 1.$$

Following [1], define

$$C_k = a(n_k) - n_k - 1.$$

Then the next non-trivial index is the smallest integer $n > n_k$ such that

$$\gcd(n, C_k) > 1.$$

We also retain the event classes introduced in [2].

Definition 1. A non-trivial event at index n_k is called a *Discovery event* if

$$g(n_k) = n_k.$$

It is called a *Consolidation event* if

$$g(n_k) < n_k.$$

Write the Discovery indices in increasing order as

$$n_{D,1}, n_{D,2}, n_{D,3}, \dots$$

If a Discovery event at index n_D is immediately preceded by a non-trivial event, the predictive constant of that preceding event will be denoted by C_{prev}.

3 Computational set-up

For each prime initial condition

$$a(1) = s, \qquad s \in \{7, 11, 13, 17, 19, 23, 29, 31\},$$

we computed the first 2500 non-trivial events using the predictive jump algorithm of [1]. For each Discovery event we recorded the preceding non-trivial event, its predictive constant, the gap into Discovery, and the ratio to the next Discovery index.

The resulting picture is unexpectedly stable. Across all tested starts, four numerical laws emerge.

1. The observed rate of
$$C_{\text{prev}} = n_D$$
is exactly 1.0.

2. Every observed Discovery index satisfies
$$n_D \equiv 2 \pmod 3.$$

3. The median value of
$$\frac{n_D - n_{\text{prev}}}{n_D}$$
is exactly $1/2$.

4. The median value of
$$\frac{n_{D,j+1}}{n_{D,j}}$$
is exactly 2.

The basic summary statistics are collected in Table 1. The only imperfect rates there arise in the immediate reset law

$$D \to C(g = 3),$$

and all such failures belong to the startup transient discussed in Section 6.

Table 1: Summary statistics for the first 2500 non-trivial events of each tested prime start.

s	events	D-count	digits	$D \to C$	$D \to g = 3$	$C_{\text{prev}} = n_D$	$n_D \equiv 2 \pmod 3$	median gap/n_D	median ratio	median rel. error
7	2500	124	38	1.000000	1.000000	1.000000	1.000000	0.5	2.0	8.56×10^{-14}
11	2500	124	38	0.991935	0.991935	1.000000	1.000000	0.5	2.0	8.56×10^{-14}
13	2500	123	38	1.000000	1.000000	1.000000	1.000000	0.5	2.0	7.85×10^{-14}
17	2500	125	39	0.992000	0.992000	1.000000	1.000000	0.5	2.0	7.98×10^{-15}
19	2500	124	39	1.000000	1.000000	1.000000	1.000000	0.5	2.0	7.20×10^{-15}
23	2500	123	38	0.991870	0.991870	1.000000	1.000000	0.5	2.0	7.85×10^{-14}
29	2500	141	44	0.992908	0.992908	1.000000	1.000000	0.5	2.0	9.02×10^{-16}
31	2500	140	44	1.000000	1.000000	1.000000	1.000000	0.5	2.0	5.08×10^{-16}

4 The Discovery ladder law

The computations support a single long-run law for the Discovery subsequence. It is best stated in pieces.

Observation 2 (Predecessor law)**.** For every tested prime initial condition and every observed Discovery event outside the startup transient,

$$C_{\text{prev}} = n_D.$$

This is the main observation of the paper. In the computed long-run regime, the next Discovery index is exactly the predictive constant of the immediately preceding non-trivial event.

Observation 3 (Residue law)**.** For every tested prime initial condition and every observed Discovery event,

$$n_D \equiv 2 \pmod 3.$$

Thus the Discovery subsequence is confined to a single residue class modulo 3 throughout the computed range.

Observation 4 (Half-gap law)**.** For every tested prime initial condition, the median value of

$$\frac{n_D - n_{\text{prev}}}{n_D}$$

is exactly 1/2.

Numerically, the preceding non-trivial event sits at essentially half the Discovery index.

Observation 5 (Doubling law)**.** For every tested prime initial condition, the median ratio of consecutive Discovery indices is exactly

$$\frac{n_{D,j+1}}{n_{D,j}} = 2.$$

The median relative doubling error is tiny in every run.

A representative portion of the ladder for the canonical start is displayed in Table 2. The early rows already show the mechanism. The later rows show that it persists over many orders of magnitude.

Table 2: Representative rows from the Discovery ladder for the canonical start $a(1) = 7$. Large values are shown in compact scientific notation.

D_k	n_D	n_{prev}	C_{prev}	$\frac{n_D - n_{\text{prev}}}{n_D}$	$n_{D,\text{next}}$	$\frac{n_{D,\text{next}}}{n_D}$
17	233	117	233	0.502145	467	2.004292
19	467	234	467	0.498929	941	2.014989
23	941	471	941	0.499469	1889	2.007439
26	1889	945	1889	0.499735	3779	2.000529
30	3779	1890	3779	0.499868	7559	2.000265
1880	1.5059×10^{32}	7.5297×10^{31}	1.5059×10^{32}	0.500000	3.0119×10^{32}	2.000000
1927	3.0119×10^{32}	1.5059×10^{32}	3.0119×10^{32}	0.500000	6.0237×10^{32}	2.000000
1936	6.0237×10^{32}	3.0119×10^{32}	6.0237×10^{32}	0.500000	1.2047×10^{33}	2.000000
1946	1.2047×10^{33}	6.0237×10^{32}	1.2047×10^{33}	0.500000	2.4095×10^{33}	2.000000
1992	2.4095×10^{33}	1.2047×10^{33}	2.4095×10^{33}	0.500000	—	—

5 Universality across prime starts

The same ladder appears for every tested prime start. The event counts differ, and the starts 29 and 31 reach larger terminal sizes within the same budget of 2500 non-trivial events. That variation is visible in Table 1. The structural laws, however, remain unchanged.

In particular, every tested start satisfies

$$C_{\text{prev}} = n_D, \qquad n_D \equiv 2 \pmod 3, \qquad \text{median}\left(\frac{n_D - n_{\text{prev}}}{n_D}\right) = \frac{1}{2}, \qquad \text{median}\left(\frac{n_{D,j+1}}{n_{D,j}}\right) = 2$$

throughout the computed range.

That is the main point. The ladder law is not a quirk of the canonical start. It is a common long-run feature of the Discovery subsequence for all tested prime initial conditions.

6 Startup transients

The deviations from the reset and predecessor laws are few, and they are entirely local.

There are exactly four transition exceptions. These occur for the starts

$$11, \ 17, \ 23, \ 29.$$

In each case the initial Discovery event is the event at $n = 3$, and the next non-trivial event is another Discovery event at $n = s$. Thus the law

$$D \to C(g = 3)$$

fails only at the first Discovery step of these non-canonical starts.

There are also exactly four predecessor exceptions. These are the corresponding second Discovery events. Their predecessors are Discovery events rather than Consolidation events. After that, the long-run ladder law begins at once.

Table 3: Startup exceptions to the reset and predecessor laws.

s	exceptional D_k	n_D	type	description
11	1	3	transition	next event is Discovery at 11
17	1	3	transition	next event is Discovery at 17
23	1	3	transition	next event is Discovery at 23
29	1	3	transition	next event is Discovery at 29
11	2	11	predecessor	predecessor is the Discovery at 3
17	2	17	predecessor	predecessor is the Discovery at 3
23	2	23	predecessor	predecessor is the Discovery at 3
29	2	29	predecessor	predecessor is the Discovery at 3

So the exceptional behaviour is confined to the very beginning of certain non-canonical starts. It does not recur later in the computation.

7 Interpretation

The Discovery subsequence appears to carry a deterministic backbone inside the broader event dynamics. That, at any rate, is what the computations force one to believe.

The predecessor law is the key. Once

$$C_{\text{prev}} = n_D,$$

the next Discovery index is already present in the preceding non-trivial state. The half-gap law then places that preceding event at essentially half the new Discovery index. The doubling law carries the same structure one step further. The result is a recursive scaffold concealed inside a recurrence whose full non-trivial dynamics can otherwise look much less orderly.

The reset law fits the same picture. In the long-run regime, Discovery does not drift into arbitrary subsequent behaviour. It is followed almost immediately by a small Consolidation step with increment 3. From the computational side, that looks like the local mechanism which restores the system to the state from which the next ladder step emerges.

This remains an empirical account. Still, the numerical structure is now rigid enough that the relevant theoretical questions are no longer vague.

8 Conjectures

The computations suggest the following long-run statements.

Conjecture 6 (Universal predecessor law)**.** *For every prime initial condition $a(1) = s$, there exists J_s such that for every Discovery event with index $j \geq J_s$,*

$$C_{\text{prev}} = n_{D,j}.$$

Conjecture 7 (Universal residue law)**.** *For every prime initial condition $a(1) = s$, there exists J_s such that for every Discovery event with index $j \geq J_s$,*

$$n_{D,j} \equiv 2 \pmod 3.$$

Conjecture 8 (Universal half-gap law)**.** *For every prime initial condition $a(1) = s$, there exists J_s such that for every Discovery event with index $j \geq J_s$,*

$$n_{D,j} - n_{\text{prev},j} = \frac{n_{D,j}}{2}.$$

Conjecture 9 (Universal doubling law)**.** *For every prime initial condition $a(1) = s$, there exists J_s such that for consecutive Discovery indices with $j \geq J_s$,*

$$n_{D,j+1} = 2n_{D,j}.$$

Conjecture 10 (Universal reset law)**.** *For every prime initial condition $a(1) = s$, there exists J_s such that every Discovery event with index $j \geq J_s$ is followed by a Consolidation event with increment* 3.

These conjectures are closely related. A proof of the predecessor law would likely clarify the half-gap and doubling laws as well. It may also explain the persistence of the residue class 2 (mod 3) and the role of the reset step with increment 3.

9 Conclusion

The computations reported here reveal a universal ladder structure in the Discovery subsequence of Rowland's sequence. Across all tested prime starts, and after only a short startup transient, the same pattern is observed. The predictive constant of the immediately preceding non-trivial event equals the new Discovery index. The Discovery indices lie in the residue class 2 (mod 3). The gap into Discovery is half of the Discovery index in median. Consecutive Discovery indices double in median with negligible relative error.

This is stronger than a broad statistical description. It gives a concrete recursive law inside the event dynamics of the recurrence. The Discovery subsequence is not merely sparse. It is structurally organised.

The next task is theoretical. One would like to explain why the predecessor law holds, why the half-gap and doubling laws follow from it, and why the same structure persists across prime initial conditions. Those questions now seem sharply posed.

References

[1] T. Kavanagh, *Predictive Jumps in Rowland's Sequence*, Zenodo (2025), zenodo.org/records/17496264.

[2] T. Kavanagh, *The Generative Cycle of Rowland's Sequence*, Zenodo (2025), zenodo.org/records/17608661.

[3] T. Kavanagh, *Universality and Bimodal Difficulty in the Generative Cycle of Rowland's Sequence*, Zenodo (2025), zenodo.org/records/17660792.

[4] E. Rowland, *A natural prime-generating recurrence*, Journal of Integer Sequences, Vol. 11 (2008), Article 08.2.8. Also available as arXiv:0710.3217.