

On Green's Function for a System of non-Intersecting Spheres.

By Prof. W. BURNSIDE. Received January 8th, 1894. Read January 11th, 1894.

1. *Introduction.*

In the application of the method of images to the problem of two electrified spheres, series are obtained of the form

$$\frac{1}{OP} - \frac{c_1}{O_1P} + \frac{c_2}{O_2P} - \dots,$$

where $O_1, O_2, \&c.$, are the images of the original point O , backwards and forwards, in the spheres, and c_1, c_2, \dots are certain constants depending on the radii of the spheres and the position of the points O . The point P is the variable point, whose coordinates are the independent variables of which the above series is a function.

Green's function for two spheres and the potential at any external point due to the charged spheres are both expressible by one or more series of the above form. Now, though for purposes of actual calculation these series are expressed in a sufficiently convenient form, they give, in the form written, no information as to the nature of the analytical function of x, y, z represented by them. A very simple modification of the separate terms of the series brings this latter point into a clear light. It may, in fact, be shown by elementary geometry that, if P_n is the point derived from P by taking in the inverse order the set of inversions through which O_n proceeds from O , then

$$\frac{c_n}{O_nP} = \frac{\mu_n^{\frac{1}{2}}}{OP_n},$$

where μ_n is the ratio of the infinitesimal linear elements at P_n and P . The series is thus expressible in the form

$$\sum_n (-1)^n \frac{\mu_n^{\frac{1}{2}}}{OP_n},$$

and the quasi-automorphic character of the function represented, for the group of point-transformations generated by inversions at the two spheres, is almost immediately evident.

It is the object of the present paper to lead up to this result, start-

ing directly from a consideration of the group arising from any finite number of non-intersecting spheres. The convergency of the chief series involved, namely, $\sum \mu_i^{-1}$, is not proved unconditionally; but it is shown that, whatever be the number of spheres, this series will certainly be convergent when certain inequalities between their radii and distances apart are satisfied. The groups of transformations dealt with are the analogues for three dimensions of those treated in the author's paper "On a Class of Automorphic Functions" (*Proc. Lond. Math. Soc.*, Vol. XXIII.); and many details in the treatment of the two are practically identical. The attempt has, however, been made to ensure that this paper shall be complete in itself.

2. Notation.

From n given spheres, and a given point, a series of points may be formed by successive inversions. The points thus proceeding from a given point P may be conveniently denoted by a suffix notation $P_{123\dots}$, this symbol denoting the point derived from P by inverting first in sphere 1, then in sphere 2, and so on. It may be assumed that in the suffix no symbol occurs twice running, since two successive inversions in a sphere produce no change.

Where there is no risk of confusion, any point of the series will be represented by P_i , the suffix i being an abbreviation for any possible combination of the symbols 1, 2, 3, ... n .

The coordinates of each point in the series are rational functions of the coordinates of P ; but, since when two points are inverses of each other the coordinates of either are rational functions of those of the other, it follows that the coordinates of all the points in the series are rational functions of those of any one of them, such as P_i .

The n rational reversible operations, by means of which the coordinates of the successive points in the series are expressed in terms of those of P , namely, the inversions at the n spheres, may be replaced by an inversion at a single sphere, say sphere 1, and $n-1$ pairs of inversions taken at the spheres 1 and 2, 1 and 3, ... 1 and n .

The new operations thus introduced, each consisting of a pair of inversions, are evidently also rational and reversible; and they also form a group, in the sense that the result of performing successively two operations of the series is always equivalent to some other operation of the same series. This latter statement is also clearly true of

the original series of inversions, but it is not true of the series of operations each consisting of an odd number of inversions. If P_i be now used to denote any point proceeding from P by the operations of the group, which arises from the $n-1$ pairs of inversions, the original series of points will be denoted by the symbols P_i and P_{ii} , where i is replaced in succession by every possible combination of an even number of the original suffixes.

Since the corresponding infinitesimal figures each of which is the inverse of the other with respect to a sphere are similar, the same is true of the two corresponding infinitesimal figures one of which is derived from the other by any operation of the group now introduced. The symbol μ_i will be used to denote the ratio of an infinitesimal element of length in the neighbourhood of the point P_i to the corresponding element in the neighbourhood of P , or, in other words, the linear magnification at P_i . For a single inversion the linear magnification is evidently a rational function of the coordinates of either of the two inverse points considered; and therefore μ_i is a rational function of x_i, y_i, z_i the coordinates of P_i .

3. *Quasi-Automorphic Functions.*

Consider now the series

$$\sum_i \mu_i^m f(x_i, y_i, z_i) = F(x, y, z),$$

where the summation is extended to all the operations of the group, suffix 0 corresponding to the identical operation, so that x_0, y_0, z_0 are x, y, z , and μ_0 is unity. When the group is finite, and therefore also the number of terms in the series, the latter will be a one-valued function of the position of P , so long as $f(x, y, z)$ is a one-valued function of x, y, z , and m is integral. The latter condition may be dispensed with, if it is agreed that μ_i^m shall represent the real positive m^{th} power of μ_i (which is itself necessarily positive) whatever m may be. If the group is of infinite order, the convergency of the series must be considered. This will be dealt with later, and is for the present assumed.

If S_k is that operation of the group that changes P into P_k , so that

$$S_k x = x_k, \quad S_k y = y_k, \quad S_k z = z_k,$$

then

$$S_k F(x, y, z) = F(x_k, y_k, z_k).$$

Now the totality of the points P_i are unaltered by the operation S_k ; and therefore the particular point P_i must by this operation be changed into some other one of the series. Moreover, no two can be changed into the same point, for from

$$S_k P_i = S_k P_r,$$

in consequence of the reversibility of the operation S_k ,

$$P_i = P_r,$$

would follow.

$$\text{Hence} \quad S_k P_i = P_j,$$

where, when i takes every possible value once, so also does j .

$$\text{Now} \quad \mu_i^2 = \frac{dx_i^2 + dy_i^2 + dz_i^2}{dx^2 + dy^2 + dz^2},$$

$$\text{and hence} \quad (S_k \mu_i)^2 = \frac{dx_j^2 + dy_j^2 + dz_j^2}{dx_k^2 + dy_k^2 + dz_k^2} = \frac{\mu_j^2}{\mu_k^2},$$

$$\text{or} \quad S_k \mu_i = \frac{\mu_j}{\mu_k};$$

$$\text{therefore} \quad S_k \sum_i \mu_i^m f(x_i, y_i, z_i) = \sum_j \frac{\mu_j^m}{\mu_k^m} f(x_j, y_j, z_j),$$

where, on the right-hand, j takes every possible value once.

$$\text{Hence, finally,} \quad F(x_k, y_k, z_k) = \mu_k^{-m} F(x, y, z),$$

so that the function $F(x, y, z)$ is quasi-automorphic with regard to the group of operations considered.

Finite groups of operations of the kind here considered are exhaustively dealt with in a memoir by Goursat (*Ann. de l'Ecole Nor. Sup.*, t. 6). When two or more of the spheres intersect at angles which are incommensurable with two right angles, an infinite number of points P_i lie in the neighbourhood of any given one, and the series above considered cannot be convergent. The case in which each of the spheres is external to all the others is the one which it is proposed here to deal with.

In this case, if P is external to all the spheres, the points P_i are all within the spheres; and by a process which need not here be given in detail, as it is precisely analogous to the first convergency proof

given in Poincaré's memoir on Fuchsian functions (*Acta Mathematica*, t. 1), it is easily shown that the series

$$\sum_i \mu_i^m$$

is absolutely convergent if m is not less than 3.

When the distances apart of the centres of the spheres are sufficiently great in comparison with their radii, the second convergence proof given in the author's paper "On a Class of Automorphic Functions" (*Proc. Lond. Math. Soc.*, Vol. XXIII.) may be modified as follows, to show that the above series is convergent so long as m is positive.

Let a and b be the radii of two spheres, and d the distance apart of their centres. Then it may easily be shown by elementary geometry that the linear magnification resulting from an inversion first at the sphere whose radius is a , and then at the sphere whose radius is b , is always less than

$$\left(\frac{b}{d-a}\right)^2.$$

If, now, λ is the greatest value of this fraction for any pair of the n given circles, the linear magnification after any operation compounded of r of the original operations is not greater than λ^r . Also corresponding to each operation compounded of r of the original operations there are $2(n-1)-1$, i.e., $2n-3$, compounded of $r+1$. Hence the series

$$\sum_i \mu_i^m$$

is certainly convergent if $\sum [(2n-3)\lambda^m]^r$

is convergent. For instance, with three equal spheres whose centres form an equilateral triangle, the series for $m = \frac{1}{2}$ is certainly convergent if the diameters of the spheres are less than half the distances between the centres.

The proof just given shows that

$$\sum_i \mu_i$$

is certainly convergent for the group arising from a system of spheres each of which is external to all the others, provided that certain inequalities hold between the radii and the distances apart of the centres. It is series of this form, viz., $m = \frac{1}{2}$, that occur in the physical problems referred to in the title of the present paper; and,

with regard to the limitations imposed by the inequalities mentioned above, it is to be noticed that these inequalities have been shown to be sufficient to ensure the convergence of the series, and therefore to justify the application of the method to the corresponding cases. But they are clearly not necessary conditions, as a consideration of the proof itself will show.

The convergency of the series $\sum_i \mu_i^m$ carries with it that of the series

$$\sum_i \mu_i^m f(x_i, y_i, z_i),$$

so long as $f(x, y, z)$ does not become infinite at any one of the series of points P_i . For, if this is the case, $f(x_i, y_i, z_i)$ will have a maximum value M independent of i , and then

$$\sum_i \mu_i^m f(x_i, y_i, z_i) < M \sum_i \mu_i^m.$$

4. Physical Application.

If (x', y', z') is the inverse point of (x, y, z) in a given sphere, and if $f(x, y, z)$ is a solution of Laplace's equation, so also is $\mu'^{\frac{1}{2}} f(x', y', z')$, and the two solutions are numerically equal at the surface of the sphere. This is a known theorem; a proof of it is given in a paper by Mr. W. D. Niven (*Proc. Lond. Math. Soc.*, Vol. VIII., p. 66). If, now, (x'', y'', z'') is the inverse point of (x', y', z') in another sphere, the linear magnification in passing from (x', y', z') to (x'', y'', z'') is μ''/μ' . Hence, since $f(x', y', z')$ is a solution of Laplace's equation when x', y', z' are the variables, so also by the above theorem is

$$\frac{\mu''^{\frac{1}{2}}}{\mu'^{\frac{1}{2}}} f(x'', y'', z'').$$

But, if $F(x', y', z')$ is a solution when x', y', z' are the variables, $\mu'^{\frac{1}{2}} F(x', y', z')$ is a solution with x, y, z as variables. Hence, finally, $\mu''^{\frac{1}{2}} f(x'', y'', z'')$ is a solution when x, y, z are the variables. It follows that $f(x, y, z)$ and $\mu'^{\frac{1}{2}} f(x', y', z')$ are simultaneously solutions of Laplace's equation when x', y', z' are derived from x, y, z by any number of inversions, even or odd.

Let, now, accented symbols be used to denote points derived from (x, y, z) by an odd number of inversions, and suppose the set of spheres such that $\sum_i \mu_i^{\frac{1}{2}}$ is a uniformly convergent series. Then, if $f(x, y, z)$ is a solution of Laplace's equation, whose only infinities are

external to all the spheres, the function $F(x, y, z)$ defined by the series

$$\sum_i \mu_i^{\frac{1}{2}} f(x_i, y_i, z_i) - \sum_i \mu_i^{\frac{1}{2}} f(x'_i, y'_i, z'_i)$$

has the following properties :—

- (i.) It satisfies Laplace's equation.
- (ii.) Its only infinities in the space external to all the spheres coincide with those of $f(x, y, z)$.
- (iii.) It vanishes at the surface of each sphere.

That the function satisfies Laplace's equation is evident, since it is defined by a uniformly convergent series, each of whose terms satisfies the equation.

If $f(x, y, z)$ becomes infinite at the point A , $f(x_i, y_i, z_i)$ becomes infinite at the point derived from A by the inverse of the operation which leads from P to P_i ; and since, by supposition, A is external to all the spheres, $f(x_i, y_i, z_i)$ can therefore only be infinite at points within some one of the spheres.

The quantities μ_i have a superior limit in the space external to the spheres. It follows that the only infinities of the series outside the spheres are those of $f(x, y, z)$.

Finally, corresponding to each term $\mu_i^{\frac{1}{2}} f(x_i, y_i, z_i)$ of the first sum, there is a single term $\mu_i^{\frac{1}{2}} f(x'_i, y'_i, z'_i)$ of the second, such that x'_i, y'_i, z'_i is the inverse of x_i, y_i, z_i at any specified one of the system of spheres, and at the surface of this sphere the two terms are numerically equal. Hence at the surface of this sphere the terms of the two sums can be taken in pairs which destroy each other.

If, now, $\mu_i^{\frac{1}{2}} f(x'_i, y'_i, z'_i) = f_1(x, y, z)$,

and if the operation which changes (x, y, z) into (x_i, y_i, z_i) transforms (x', y', z') into (x'_i, y'_i, z'_i) , then

$$\frac{\mu'_i}{\mu_i^{\frac{1}{2}}} f(x'_i, y'_i, z'_i) = f_1(x_i, y_i, z_i);$$

and therefore $\sum \mu_i^{\frac{1}{2}} f(x'_i, y'_i, z'_i) = \sum \mu_i^{\frac{1}{2}} f_1(x_i, y_i, z_i)$.

Hence the function $F(x, y, z)$ can be expressed in the form

$$\sum_i \mu_i \{f(x_i, y_i, z_i) - f_1(x_i, y_i, z_i)\},$$

and it is therefore such that it is reproduced, except as to a factor depending on the transformation, when (x, y, z) is transformed by any one of the operations of the group.

For the case of two spheres, which has already been treated from the point of view of calculation in a great variety of ways, the necessary formulæ for representing explicitly the result just obtained may be conveniently written as follows.

$$\begin{aligned}\text{Taking} \quad x^2 + y^2 + z^2 + 2x \coth \alpha + 1 &= 0, \\ x^2 + y^2 + z^2 - 2x \coth \beta + 1 &= 0\end{aligned}$$

as the equations to the two spheres, the system of points proceeding from (x, y, z) by an even number of inversions all lie in a plane through the axis of x , and, when ρ is used for $\sqrt{y^2 + z^2}$, are given by

$$\begin{aligned}x_n &= \frac{(x^2 + \rho^2 + 1) \sinh n\theta \cosh n\theta + x \cosh 2n\theta}{\sinh^2 n\theta (\{x + \coth n\theta\}^2 + \rho^2)}, \\ \rho_n &= \frac{\rho}{\sinh^2 n\theta (\{x + \coth n\theta\}^2 + \rho^2)}, \\ \mu_n &= \frac{\rho_n}{\rho},\end{aligned}$$

where

$$\theta = \alpha + \beta.$$

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It has been usual to assume that the velocity potential of the fluid motion which consists of a train of progressive waves propagated along a canal of uniform cross-section can be represented by an expression of the form $f(y, z) \cos(mx - nt)$, the notation being the same as in Basset's *Hydrodynamics*, Vol. II., Art. 392. The wave motion which has a velocity potential of this form must be such that the crests of the waves are always in planes perpendicular to the length of the canal, the particles of fluid describing ellipses whose planes are perpendicular to the cross-section. In what follows it is proposed to investigate in what cases it is possible to propagate a train of such waves of any given wave length along a canal whose sides are planes equally inclined to the vertical.