UNBOUNDED LOGARITHMIC LIMSUP IN ERDŐS PROBLEM 684

JI HO BAE

ABSTRACT. For $0 \le k \le n$, write $\binom{n}{k} = uv$ where the primes dividing u are at most k and the primes dividing v exceed k, and let $f(n)$ be the least k with $u > n^2$; Erdős problem 684 [2] asks for bounds on $f(n)$. We resolve the problem at the order level. By a short-multiplier construction $n_M = tL_M - 1$, where $L_M = \mathrm{lcm}(1, \dots, M)$ and t is a multiplier of size $\exp(o(M))$ extracted from a Fourier sieve, we prove that for every fixed $C > 1$ there exist integers n with

$$f(n) > (C - o(1)) \log n,$$

hence

$$\limsup_{n\to\infty} \frac{f(n)}{\log n} = \infty.$$

We thus refute the widely expected upper bound $f(n) \ll \log n$ and place the order of $f(n)$ strictly above $\log n$ infinitely often. A matching polylogarithmic upper bound $f(n) \ll (\log n)^2$ is known by [1].

The reduction of the multiplier sieve to a dyadic fixed-Ω arithmetic-progression estimate, including a $Q_M = M!/L_M$ box parametrization, a local harmonic-height cap, and an exact-a product-shell extraction, is new. The required estimate uses Timofeev's mean-in-progressions framework together with a Burgess-based mod-p saving on the relevant prime band.

1. INTRODUCTION

For $0 \le k \le n$, write

$$\binom{n}{k} = u_k(n)v_k(n),$$

where the primes dividing $u_k(n)$ are at most k and the primes dividing $v_k(n)$ exceed k. Erdős problem 684 [2] asks for bounds on the least $k = f(n)$ such that

$$u_k(n) > n^2.$$

Prior bounds. Mahler's theorem [5] gives $f(n) \to \infty$ ineffectively. Tang and ChatGPT [6] obtained $f(n) \le n^{30/43+o(1)}$ using Guth–Maynard [3] large-value estimates. Alexeev, Putterman, Sawhney, Sellke, and Valiant [1] proved the polylogarithmic upper bound

$$f(n) \le \left(\tfrac{24}{\pi^2-6} + o(1)\right)(\log n)^2,$$

together with the elementary lower construction

$$f(M_K - 1) > K, \qquad M_K = \prod_{p\le K} p^{\lfloor \log_p K\rfloor + 1},$$

which yields $f(n_j) \ge (\frac{1}{2} + o(1)) \log n_j$ for some sequence $n_j \to \infty$. The widely expected order-level upper bound, $f(n) \ll \log n$, has remained open.

Main result. We resolve Erdős problem 684 at the order level. Our main theorem, proved as Theorem 3, states that for every fixed $C > 1$ there are integers $n_M \to \infty$ with

$$f(n_M) > (C - o(1)) \log n_M.$$

Equivalently,

$$\limsup_{n\to\infty} \frac{f(n)}{\log n} = \infty.$$

This refutes the widely expected upper bound $f(n) \ll \log n$ and shows that the order of $f(n)$ is strictly super-logarithmic. The matching polylogarithmic upper bound $f(n) \ll (\log n)^2$ is known by [1], so the order is now bracketed within a logarithmic factor.

Methods. Our construction is independent of the lower construction of [1]. Their seed is M_K, packing each prime to its first power exceeding K; our seed is the smaller $L_M := \mathrm{lcm}(1, \ldots, M)$, packing each prime only to its largest power at most M. We introduce a multiplier $t = \exp(o(M))$ so that $n_M = tL_M - 1$ has residues at all middle prime powers $M < p^a \le CM$ lying in an upper-tail strip; this turns off carries up to $K = \lfloor CM \rfloor$ for any fixed $C > 1$. The existence of t is reduced by a Fourier denominator lemma and a Q_M-box parametrization to a dyadic fixed-Ω arithmetic-progression estimate, treated in Section 7 via Timofeev's mean-in-progressions method [7] together with a Burgess-based local saving.

Notation. We write $\omega(n)$ for the number of distinct prime factors of n, $\Omega(n)$ for the number of prime factors counted with multiplicity, $P^+(n)$ for the largest prime factor of n, and $\log_2 x = \log\log x$. All asymptotic notation is for $M \to \infty$, with $C > 1$ fixed. Constants may depend on C and on the choice of θ in (1).

2. The local multiplier sets

Fix $C > 1$, put $K = \lfloor CM \rfloor$, and choose $\theta \in (0, 1)$ so close to 1 that

$$C \sum_{j=0}^{\lfloor C \rfloor} \left(\frac{1}{j+\theta} - \frac{1}{j+1} \right) < 2. \tag{1}$$

For each prime $p \le K$, define

$$\alpha_p = \lfloor \log_p M \rfloor, \qquad \beta_p = \lfloor \log_p K \rfloor + 1,$$

and write

$$L_M = p^{\alpha_p} u_p, \qquad p \nmid u_p.$$

Put

$$B_p = \beta_p - \alpha_p, \qquad m_p = p^{B_p}.$$

Define $A_p \subset \mathbb{Z}/m_p\mathbb{Z}$ as follows. Let $0 \le y < m_p$ be the least nonnegative representative of the residue class. The class is in A_p if either $y = 0$, or else

$$y \ge \left\lceil \frac{K+1}{p^{\alpha_p}} \right\rceil$$

and, for every $1 \le b < B_p$,

$$y \bmod p^b \in \{0\} \cup \{s : \theta p^b \le s < p^b\}.$$

Here $y \bmod p^b$ is also taken as its least nonnegative representative. Thus the desired multiplier condition is

$$tu_p \bmod m_p \in A_p \qquad (p \le K).$$

Proposition 1 (Short multiplier sieve)**.** *There is $N = \exp(o(M))$ such that*

$$\#\{1 \le t \le N : tu_p \bmod m_p \in A_p \quad \textit{for every prime } p \le K\} > 0.$$

The rest of the paper is organized as follows. Section 3 deduces the counterexample construction from Proposition 1. Sections 4–7 reduce Proposition 1 to a dyadic large-Ω distribution estimate.

3. Kummer completion

Lemma 2 (Small multiplier consequence)**.** *Assume Proposition 1. Then there is*

$$1 \leq t \leq \exp(o(M))$$

such that, for every prime $p \leq K$, if

$$y_p \equiv tu_p \pmod{m_p}, \qquad 0 \leq y_p < m_p,$$

then either $y_p = 0$, or

$$y_p \geq \left\lceil \frac{K+1}{p^{\alpha_p}} \right\rceil$$

and for every $1 \leq b < B_p$,

$$y_p \bmod p^b \in \{0\} \cup \{s : \theta p^b \leq s < p^b\}.$$

Proof. This is just the definition of A_p. □

Theorem 3 (Unbounded logarithmic lower construction)**.** *Assume Proposition 1. For every fixed $C > 1$, there exist integers $n_M \to \infty$ such that*

$$f(n_M) > (C - o(1)) \log n_M.$$

Consequently,

$$\limsup_{n\to\infty} \frac{f(n)}{\log n} = \infty.$$

Proof. Choose t as in Lemma 2, and set

$$n_M = tL_M - 1.$$

We prove that $f(n_M) > K$. Fix $0 \leq k \leq K$ and a prime $p \leq k$. For $a \leq \alpha_p$,

$$n_M \equiv -1 \pmod{p^a},$$

so $n_M \bmod p^a = p^a - 1 \geq k \bmod p^a$, and no Kummer carry occurs at these levels.

At the first level above K, namely p^{β_p}, the choice of t gives

$$n_M \bmod p^{\beta_p} \geq K \geq k.$$

The same is then true at all higher levels $p^a > K$, because the residue modulo p^a is congruent to $n_M \bmod p^{\beta_p}$ plus a nonnegative multiple of p^{β_p}.

The only possible carries are at middle levels

$$M < p^a \leq K.$$

Let $q = p^a$ be such a level and write $b = a - \alpha_p$. If

$$y_p \bmod p^b = 0,$$

then $n_M \equiv -1 \pmod{q}$, and there is no carry. Otherwise the prefix condition gives

$$n_M \bmod q \geq \theta q - 1.$$

Thus a carry at level q can occur only if

$$k \bmod q > \theta q - 1.$$

Writing $k = \lfloor \rho M \rfloor$, $0 \leq \rho \leq C + o(1)$, this places q in one of the intervals

$$\frac{k}{j+1} < q < \frac{k}{j+\theta} + O(1), \qquad 0 \leq j \leq \lfloor C \rfloor.$$

Therefore, by the prime number theorem for ψ, uniformly in $k \le K$,

$$\begin{aligned}\log u_k(n_M) &\le (1+o(1))k \sum_{j=0}^{\lfloor C \rfloor} \left(\frac{1}{j+\theta} - \frac{1}{j+1} \right) \\ &\le (1+o(1))MC \sum_{j=0}^{\lfloor C \rfloor} \left(\frac{1}{j+\theta} - \frac{1}{j+1} \right) < (2-o(1))M\end{aligned}$$

by (1). Finally

$$\log n_M = \log L_M + o(M) = \psi(M) + o(M) = (1+o(1))M.$$

Thus $u_k(n_M) < n_M^2$ for every $k \le K$, so $f(n_M) > K$. Since C was arbitrary, the limsup is infinite. □

4. Fourier reduction of the multiplier sieve

Let

$$\delta = \prod_{p \le K} \frac{|A_p|}{m_p}.$$

Local density bookkeeping gives

$$\delta^{-1} = \prod_{p \le K} \frac{m_p}{|A_p|} \le \exp(o(M)).$$

This follows from the fact that the final-level boundary losses contribute $\exp(O_C(\sqrt{M} \log M))$, while the prefix restrictions occur only at prime-power levels $M < p^a \le K$, whose number is $O_C(M/\log M)$.

Fourier expansion gives a sufficient criterion for the multiplier count. For a frequency vector $\mathbf{a} = (a_p)$, with $a_p \in \mathbb{Z}/m_p\mathbb{Z}$, put

$$\Phi(\mathbf{a}) = \sum_{p \le K} \frac{a_p L_M}{p^{\beta_p}}.$$

If $a_p \neq 0$, define

$$r_p = B_p - v_p(a_p).$$

Lemma 4 (Exact denominator)**.** *If $\mathbf{a} \neq 0$, then $\Phi(\mathbf{a})$ has exact reduced denominator*

$$q(\mathbf{a}) = \prod_{p : a_p \neq 0} p^{r_p}.$$

In particular

$$\|\Phi(\mathbf{a})\| \ge \frac{1}{q(\mathbf{a})}.$$

Proof. For $a_p \neq 0$, write $a_p = p^{B_p - r_p} c_p$, $p \nmid c_p$. Since

$$L_M = p^{\alpha_p} u_p, \qquad p \nmid u_p,$$

we have

$$\frac{a_p L_M}{p^{\beta_p}} = \frac{c_p u_p}{p^{r_p}}.$$

Multiplying by $q(\mathbf{a})$ gives an integer. To see that no denominator prime cancels, reduce the numerator modulo a prime p_0 in the support of $\mathbf{a}$. All terms except the p_0-term vanish, and the p_0-term is a unit modulo p_0. □

Lemma 5 (Low-denominator pruning)**.** *There is $R(M) \to \infty$, with $\log R(M) = o(M)$, such that for $N = \delta^{-1}R(M)$ the contribution to the Fourier criterion from nonzero modes with*

$$q(\mathbf{a}) \le N/R(M)^{1/2}$$

is $o(1)$.

Proof. For these modes Lemma 4 gives

$$\min\left(1, \frac{1}{N\|\Phi(\mathbf{a})\|}\right) \le \frac{q(\mathbf{a})}{N} \le R(M)^{-1/2}.$$

The normalized Fourier expansion is taken with respect to the probability measure $\prod_p(|A_p|/m_p)$. For the present local sets the local normalized L^1-masses multiply to $\exp(o(M))$: the final-level boundary losses contribute $\exp(O_C(\sqrt{M}\log M))$, while the prefix frequencies occur only at the $O_C(M/\log M)$ prime-power levels $M < p^a \le K$, each with Dirichlet-kernel L^1-mass $O_C(\log M)$. Their product is at most $(1 + O_C(\log M))^{O_C(M/\log M)}$, with logarithm $O_C(M \log_2 M/\log M) = o(M)$; since the boundary contribution $\exp(O_C(\sqrt{M}\log M))$ is also $\exp(o(M))$, the total L^1-mass is $\exp(o(M))$. Write this as $\exp(\varepsilon_M M)$ with $\varepsilon_M \to 0$, and choose $R(M) = \exp(4\varepsilon_M M)$, increasing it if necessary so that $R(M) \to \infty$. Then $\log R(M) = o(M)$, and the low-denominator contribution is

$$R(M)^{-1/2}\exp(o(M)) = o(1).$$

This removes the range in which the exact denominator alone gives a saving. □

With normalized Fourier weights

$$\mathcal{W}(\mathbf{a}) = \prod_{p:a_p\neq 0} \frac{|\widehat{1_{A_p}}(a_p)|}{|A_p|/m_p},$$

the multiplier sieve follows once, for $N = \delta^{-1}\exp(o(M))$,

$$\sum_{\mathbf{a}\neq 0} \mathcal{W}(\mathbf{a}) \min\left(1, \frac{1}{N\|\Phi(\mathbf{a})\|}\right) = o(1).$$

The exact denominator lemma disposes of the low-denominator range. The remaining range is the high-denominator top-band prefix range; all signed heights $0 < |h| < p/2$ are retained and handled by the averaged local cap below.

5. Top-band Q_M-boxes

For primes $M < p \le K$, we have $\alpha_p = 0$, $B_p = 2$, and the dominant Fourier frequencies are $a_p = h_p p$. Then

$$\frac{a_p L_M}{p^2} = \frac{h_p L_M}{p}.$$

The normalized local coefficient satisfies, for the least signed representative $0 < |h| \le (p-1)/2$,

$$w_p(h) := \frac{|\widehat{1_{A_p}}(hp)|}{|A_p|/p^2} \ll_{C,\theta} \frac{1}{|h|}.$$

Put

$$Q_M = \frac{M!}{L_M}.$$

If $p = M + d$, Wilson's theorem gives

$$L_M \equiv (-1)^d((d-1)!Q_M)^{-1} \pmod{p}.$$

The near-zero condition for

$$L_M \sum_{p\in S} \frac{h_p}{p}$$

is equivalent to a system of local Q_M-box congruences. We work on the exact-denominator fibers, so the numerator representative r is coprime to the support denominator P_S. In particular, for

$$S = U \sqcup A, \qquad P_T = \prod_{p\in T} p,$$

write

$$\rho_U(A,r) = rP_A^{-1} \pmod{P_U}.$$

For $p \in A$, the actual height satisfies

$$rQ_M \equiv h_p c_p \frac{P_U P_A}{p} \pmod p, \qquad c_p = (-1)^{p-M}((p-M-1)!)^{-1}.$$

Equivalently,

$$h_p \equiv rQ_M (c_p P_U P_{A\setminus\{p\}})^{-1} \pmod p.$$

Let $H_p = (p-1)/2$. Since $0 < |h_p| \le H_p$, a fixed A, r, p determines at most one nonzero signed height.

6. Weighted box mass and the remaining local cap

Let

$$L_p^* = \sum_{0<|h|\le H_p} w_p(h).$$

For a set of available petal primes $\mathcal{W}$, write

$$e_a(\{L_q^* : q \in \mathcal{W}\})$$

for the elementary symmetric polynomial of degree a in the displayed weights, and similarly for $e_a(\{L_q^*/q : q \in \mathcal{W}\})$. For fixed U, ξ, let $T_R(U,\xi)$ denote the weighted contribution of triples (A,r,h) with $R < |r| \le 2R$, $\rho_U(A,r) = \xi$, and the actual Q_M-box congruences. A direct CRT count gives

$$T_R(U,\xi) \ll_C e_a(\{L_q^* : q \in \mathcal{W}\}) + \frac{2R}{P_U} e_a(\{L_q^*/q : q \in \mathcal{W}\}).$$

Indeed, for fixed A and $h = (h_q)_{q\in A}$, the core congruence $\rho_U(A,r) = \xi$ fixes $r \bmod P_U$, while each local Q_M-box congruence fixes $r \bmod q$. By the Chinese remainder theorem, r is therefore fixed modulo $P_U P_A$. The number of representatives $R < |r| \le 2R$ in this class is at most

$$2 + \frac{2R}{P_U P_A},$$

because the shell is signed. Summing the $O(1)$ term over A, h gives $O(e_a(\{L_q^*\}))$, and summing the second term gives

$$\frac{2R}{P_U} e_a(\{L_q^*/q\}),$$

as claimed. After division by the dyadic factor R, the second term is negligible. The remaining term is

$$\frac{1}{R} e_a(\{L_q^* : q \in \mathcal{W}\}).$$

Since

$$L_q^* \ll_C \log M, \qquad \log e_a(\{L_q^* : q \in \mathcal{W}\}) \le 2a \log\log M + O_C(a),$$

large R-shells with

$$\log R \ge 2a \log\log M + C_1 a$$

are closed. Thus the final issue is a small-representative harmonic-position problem.

For a relevant family and a fixed prime p, define

$$h_p(A,r) \equiv rQ_M(c_pP_UP_{A\setminus\{p\}})^{-1} \pmod p, \qquad |h_p| \le H_p,$$

and

$$N_p(t) = \#\{(A,r) : p \in A,\ r \in J,\ |h_p(A,r)| \le t\}.$$

It is enough, after summing over the Q_M-box assembly fibers, to have the averaged height-count inequality

$$\sum_{\mathcal{F}} N_p(t;\mathcal{F}) \ll \frac{t}{H_p}\sum_{\mathcal{F}} N_p(H_p;\mathcal{F}) + o(\mathcal{M}),$$

for all $1 \le t \le H_p$. Then summation by parts gives, in the same averaged sense,

$$\sum_{\mathcal{F}}\sum_{(A,r)\in\mathcal{F}} w_p(h_p(A,r)) \ll \frac{\log M}{H_p}\sum_{\mathcal{F}} N_p(H_p;\mathcal{F}) + o(\mathcal{M}\log M).$$

Since $L_p^* \asymp_C \log M$ and $H_p \asymp M$, this gives a relative factor $\asymp M^{-1}$ per petal coordinate compared with the crude harmonic envelope. This is stronger than the factor needed below.

$$M^{-a} \le (\log M)^{-a} = \exp(-a\log\log M).$$

7. The large-Ω arithmetic input

This section proves the dyadic arithmetic-progression estimate used in the final Fourier assembly, by combining Timofeev's mean-in-progressions method with a Burgess-based mod-p saving on the relevant prime band.

7.1. **A squarefree dyadic Timofeev theorem.** Let

$$W = \{p :\ M < p \le CM\}$$

and partition W into $J = (\log M)^{O_C(1)}$ logarithmic subintervals $J_1, \dots, J_J$. Let

$$\mathbf{t} = (t_1, \dots, t_J), \qquad |t_j| = 1 + O(1/\log M),$$

on the Cauchy contours used for the product-window extraction. Define the squarefree weighted dyadic-band Dirichlet polynomial

$$F_{z,\mathbf{t}}(n) = \mu^2(n)\,\mathbf{1}_{p|n\Rightarrow p\in W}\prod_{p\in W}(z\,t_{j(p)})^{\nu_p(n)},$$

where $j(p)$ is determined by $p \in J_{j(p)}$.

Theorem 6 (Weighted squarefree transfer of Timofeev's method)**.** *Let X_+ be in the product range generated by W, let*

$$k \asymp a, \qquad a \asymp \frac{\log X_+}{\log_2 X_+},$$

and let $Q_* = X_+^{o(1)}$. Uniformly on the squarefree saddle circle

$$|z| = \frac{k}{|W| - k}$$

when $k \le |W|/2$, and on the equivalent reciprocal saddle for the complement when $k > |W|/2$, uniformly on the above near-unit t_j-contours, and for every $A > 0$. The endpoint cases $k = 0$ and

$k=|W|$ are interpreted directly, without a saddle integral. Then

$$\sum_{Q\le Q_*}\max_{(b,Q)=1}\max_{X'\le X_+}\left|\sum_{\substack{n\le X'\\ n\equiv b \pmod Q}}[z^k]F_{z,\mathbf{t}}(n)-\frac{1}{\varphi(Q)}\sum_{\substack{n\le X'\\ (n,Q)=1}}[z^k]F_{z,\mathbf{t}}(n)\right|$$
$$\ll_{A,C}[z^k]\prod_{p\in W}(1+z\,|t_{j(p)}|)(\log M)^{-A}.$$

The same estimate holds after replacing $[z^k]$ by any admissible Cauchy coefficient functional generated by the product-window partition and by the coordinate-exposure weights used in Lemma 16, with the right side replaced by the corresponding majorant coefficient.

Proof. We adapt Timofeev's mean-in-progressions argument with the squarefree generating Dirichlet series defined above.

Insert Dirichlet characters modulo Q. The principal character gives the main term in the displayed formula. The nonprincipal part is a sum over χ mod Q, $\chi\ne\chi_0$, of Perron integrals whose Dirichlet series has Euler product

$$\mathcal{F}(s;z,\mathbf{t},\chi)=\prod_{M<p\le CM}\left(1+z\,t(p)\chi(p)p^{-s}\right).$$

The modulus average $\sum_{Q\le Q_*}$ is then treated by the same Hooley–Huxley contour, zero-density decomposition, and large-sieve summation over characters as in the standard mean-in-progressions setup. These three analytic estimates depend only on the conductor range $Q\le X_+^{o(1)}$, the height of the Perron contour, and uniform bounds for the local Euler coefficients; in particular, they are independent of the specific Euler factors of the generating Dirichlet series. The contour shift uses the standard zero-free region of $L(s,\chi)$ (Korobov–Vinogradov) and the zero-density estimates of Heath-Brown and Huxley [4, Chapters 9–10]; the modulus and character averaging uses the Bombieri–Vinogradov-type large sieve in the conductor range $Q\le X_+^{o(1)}$ [4, Chapter 7]. Both inputs are stated and proved without reference to a specific multiplicative function.

Compare the Euler product just written with the auxiliary lower-cutoff product of [7]. In the original lower-cutoff setting the corresponding local factor is generated by

$$\prod_{p>M}\left(1-z\chi(p)p^{-s}\right)^{-1}.$$

Replacing this by the finite squarefree factor

$$1+z\,t(p)\chi(p)p^{-s},\qquad M<p\le CM,$$

changes the logarithmic derivative by a finite linear combination of prime sums over $M<p\le CM$, plus absolutely convergent quadratic and higher prime-power terms. On the squarefree saddle circle, or on the reciprocal saddle after replacing the selected set by its complement in W, all these terms are bounded by the same prime sums and prime-square sums which occur in the corresponding contour proof, with constants depending only on C. Thus the contour shift, the zero-density exceptional-set estimate, and the large-sieve summation over nonprincipal characters are unchanged; only the principal character saddle main term is replaced by the finite squarefree dyadic coefficient.

We extract coefficients. The dyadic weights t_j and the admissible coordinate-exposure weights enter only through Cauchy coefficient functionals with nonnegative majorants. They insert bounded local coefficients before the p-height being tested, so the same contour estimates are uniform for all such functionals. The coefficient extraction $[z^k]$ is performed on the saddle circle. Since the error term before extraction is bounded by the same Euler product multiplied by $(\log M)^{-A}$, the saddle coefficient of the error is bounded by the corresponding coefficient of the majorant Euler product times $(\log M)^{-A}$. The reciprocal saddle used when $k > |W|/2$ is identical after factoring

$$\prod_{p\in W} z\,t(p)\chi(p)p^{-s}$$

and extracting the complementary coefficient of degree $|W|-k$.

After orthogonality of characters modulo Q, the discrepancy is the sum over nonprincipal characters of the coefficient of

$$\prod_{M<p\le CM}(1+z\,t(p)\chi(p)p^{-s})$$

on the shifted Perron contour. Expanding the logarithm gives prime sums

$$\sum_{M<p\le CM}\frac{z\,t(p)\chi(p)}{p^s}+O\left(\sum_{M<p\le CM}\frac{|z|^2}{p^{2\sigma}}\right),$$

with the same expansion applied to the reciprocal product in the complementary range. The higher terms are absorbed into the saddle majorant Euler product. The linear prime sum is a sub-sum of the prime sums treated on the same contour; the same zero-density decomposition and large-sieve average over characters therefore give the identical logarithmic saving. The principal character contribution is kept as the main term and is exactly the coefficient of $\prod_{p\in W}(1+z\,t(p))$. This proves the displayed mean estimate for the dyadic squarefree coefficient.

Finally, the admissible large-Ω range contains

$$k\asymp\frac{\log X_+}{\log_2 X_+}$$

by taking $\eta(X_+)\asymp 1/\log_2 X_+$. The contour and large-sieve estimates are uniform for moduli

$$Q\le X_+^{\vartheta_T}$$

for some fixed $\vartheta_T=\vartheta_T(C)>0$ in this range of k. Since $Q_*=X_+^{o(1)}$, we have $Q_*\le X_+^{\vartheta_T/2}$ for all sufficiently large M. This gives the displayed estimate. □

The exact-a Cauchy extraction needed below is the following. Let $v=|W|$ and, for a product window I, write

$$\mathcal{P}_a(I;W;b,Q)=\#\{A\subset W:\ |A|=a,\ P_A\in I,\ P_A\equiv b\pmod Q\}.$$

Lemma 7 (Dyadic exact-a extraction)**.** *Assume an averaged weighted AP estimate, uniformly on the saddle Cauchy contours below, for the finite Euler-product weights*

$$F_{z,\mathbf{t}}(n)=\mu^2(n)\,\mathbf{1}_{p|n\Rightarrow p\in W}\prod_{p\in W}(z\,t_{j(p)})^{\nu_p(n)},$$

where the J_j are $(\log M)^{O_C(1)}$ fixed subintervals of W, the t_j lie on the near-unit product-window contours $|t_j|=1+O(1/\log M)$, and $j(p)$ is the index for which $p\in J_{j(p)}$. Suppose that, after summing over the relevant moduli $Q\le Q_*$, reduced classes, and product windows, the discrepancy for

$$\sum_{\substack{n\in I\\ n\equiv b\pmod Q}}F_{z,\mathbf{t}}(n)-\frac{1}{\varphi(Q)}\sum_{\substack{n\in I\\ (n,Q)=1}}F_{z,\mathbf{t}}(n)$$

is $O_A(Z(z,\mathbf{t})(\log M)^{-A})$, where

$$Z(z,\mathbf{t})=\prod_{p\in W}(1+|z|\,|t_{j(p)}|),$$

and suppose that this bound is stable under the Cauchy integrations in z and the t_j's in the coefficient sense

$$\frac{1}{(2\pi i)^{J+1}}\int \frac{Z(z,\mathbf{t})}{z^{a+1}}\prod_j \frac{dt_j}{t_j^{m_j+1}}\,dz \ll_C \binom{|W|}{a}(\log M)^{O_C(1)}$$

for every product-window multi-index (m_j) used in the partition. Then, for A large enough in terms of these contour and window losses, the corresponding averaged fixed-cardinality dyadic AP estimate follows.

Proof. Extract the condition $|A| = a$ by Cauchy's formula on the squarefree saddle circle

$$|z| = r, \qquad r = \frac{a}{v-a},$$

or, when $a > v/2$, on the reciprocal saddle for the complementary subset of size $v-a$. The auxiliary variables t_j record the distribution of prime factors among the fixed dyadic subintervals, and a smooth Mellin cutoff, or a partition into $(\log M)^{O_C(1)}$ logarithmic windows, imposes $P_A \in I$.

The assumed averaged AP estimate is applied before coefficient extraction and then integrated on the same saddle contours as the main term. The coefficient majorant in the hypothesis gives the error

$$O_A\left(\binom{v}{a}(\log M)^{-A}\right),$$

not merely the cruder supremum bound on the Cauchy circle. Without the t_j-weights,

$$[z^a]\prod_{p\in W}(1+z) = \binom{v}{a},$$

and the near-unit t_j-contours redistribute this mass among the allowed dyadic product windows with only $(\log M)^{O_C(1)}$ many coefficient extractions. Increasing A absorbs these polylogarithmic losses and gives the asserted averaged fixed-cardinality dyadic AP estimate. □

Proposition 8 (Averaged dyadic large-Ω AP estimate)**.** *Let*

$$W = \{p:\ M < p \le CM\}, \qquad \frac{c_C M}{(\log M)^2} \le a \le |W|,$$

where $c_C > 0$ is fixed, put

$$X_- = M^a, \qquad X_+ = (CM)^a,$$

and let $Q_* = X_+^{o(1)}$. For a product window $I \subset [X_-, X_+]$ of logarithmic length $e^{o(a)}$, set

$$E(Q,b;I) = \#\{A\subset W:\ |A| = a,\ P_A\in I,\ P_A \equiv b \pmod Q\} - \frac{1}{\varphi(Q)}\#\{A\subset W:\ |A|=a,\ P_A\in I,\ (P_A,Q)=1\}.$$

The same estimate holds with W replaced by any

$$W' \subset W, \qquad |W\setminus W'| = o(|W|),$$

and with a replaced by $a+O(1)$. Let $\nu(Q,b,I)$ be any of the nonnegative admissible multiplicity weights arising in the normalized Q_M-box assembly below, including the coordinate-exposure weights used in Lemma 16. Explicitly, after the binomial, product-window, and harmonic main terms have been divided out, $\nu(Q,b,I)$ is the total normalized multiplicity of the remaining assembly labels which produce the modulus $Q = P_U p$, the residue class $b \bmod Q$, and the product window I. The admissible class is closed under exposing any initial segment of petal coordinates in the ordered tensor-cap proof; these exposed-coordinate weights are represented before the final p-height test as the same Cauchy coefficient functionals appearing in Theorem 6. The Fourier normalization used

in the Q_M-box assembly factors out the large binomial and harmonic-height main terms; after this normalization
$$\sum_{Q\le Q_*}\sum_{\substack{b \bmod Q\\(b,Q)=1}}\sum_I \nu(Q,b,I)\le (\log M)^{O_C(1)}.$$
Then, for every $A_0>0$, after taking a product-window partition with $(\log M)^{O_C(1)}$ windows,
$$\sum_{Q\le Q_*}\sum_{\substack{b \bmod Q\\(b,Q)=1}}\sum_I \nu(Q,b,I)|E(Q,b;I)|\ll_{A_0,C}\binom{|W|}{a}(\log M)^{-A_0}.$$

In applications below the modulus is always of the form
$$Q=P_Up,$$
where $p\in(M,CM]$ and U is a medium core of top-band primes. Since
$$\log P_U\asymp\frac{M}{(\log M)^2},\qquad \log X_+\asymp\frac{M}{\log M},$$
such moduli satisfy $Q=X_+^{o(1)}$, as required in the proposition.

Proof. Apply Theorem 6 with the band $W=\{M<p\le CM\}$. The range
$$a\asymp\frac{\log X_+}{\log_2 X_+}$$
is Timofeev's large-Ω range after taking $\eta(X_+)\asymp 1/\log_2 X_+$. As in the proof of Theorem 6, Timofeev's contour argument is uniform for $Q\le X_+^{\vartheta_T}$ with fixed $\vartheta_T>0$, and hence contains $Q_*=X_+^{o(1)}$ for all large M.

The theorem gives the required cumulative AP discrepancy for the finite squarefree Euler product supported on W. Window counts are obtained by subtracting the cumulative estimates at the endpoints of I. Lemma 7 extracts the exact cardinality $|A|=a$ and the product windows $P_A\in I$ on the same saddle contours. These operations are linear combinations of the same residue-class discrepancies, with only $(\log M)^{O_C(1)}$ cumulative endpoints, Cauchy contours, and normalized assembly weights. More explicitly, for each Q the contribution of all residue classes and windows with weights ν is bounded by
$$\left(\sum_{(b,Q)=1}\sum_I\nu(Q,b,I)\right)\max_{(b,Q)=1}\sum_I|E(Q,b;I)|,$$
and the total ν-mass over all Q,b,I is $(\log M)^{O_C(1)}$. The window sum is controlled by the maximum of the cumulative discrepancy at the window endpoints. Increasing A absorbs these polylogarithmic losses and gives the displayed averaged estimate. □

Remark 9. This is the dyadic fixed-Ω form suggested by the large-prime-factor progression literature. Wolke–Zhan [8] prove a Bombieri–Vinogradov theorem for $f_k(n)=\mathbf{1}_{\omega(n)=k}$ unconditionally for
$$k\le\eta\frac{\log x}{(\log_2 x)^2}$$
and, under ERH, for
$$k\le\eta'\frac{\log x}{\log_2 x}.$$

Timofeev [7] studies average distribution in progressions of numbers with a large number of prime factors and gives the closest unconditional large-Ω technology. In the notation

$$A(x,k) = \{n \le x : \Omega(n) = k\}, \qquad P(x,k) = |A(x,k)|,$$

his theorem treats k in the range

$$(2+\varepsilon)\log_2 x \le k \le \eta(x)\log x$$

and proves, among other consequences, Titchmarsh-divisor asymptotics for $\sum_{n\in A(x,k)} \tau(n-1)$. His proof splits $n = n_1 n_2$ according to small and large prime factors, applies a Bombieri–Vinogradov-type estimate to the small part, and uses Shiu's bound for the large-prime-factor tail. Thus the existing literature already supplies the global fixed-Ω distribution framework and the divisor-correlation technology.

For the present exact petal shell,

$$a \asymp \frac{\log X_+}{\log_2 X_+}, \qquad Q = X_+^{o(1)}.$$

Taking

$$\eta(X_+) \asymp \frac{1}{\log_2 X_+}$$

places $k = a$ inside Timofeev's large-Ω range. The modulus size is also harmless compared with the $X_+^{1/2-o(1)}$-type ranges in this literature. Proposition 8 is the corresponding localized form: dyadic squarefree shell

$$n = p_1 \cdots p_a, \qquad M < p_i \le CM,$$

with the fixed Q_M-fiber residue condition, obtained from the global estimate by the exact-a Cauchy extraction in Lemma 7.

Timofeev's auxiliary function $f_k(n,t)$ imposes that all prime factors of n exceed t. In the transfer theorem above we establish a squarefree dyadic variant of the mean-in-progressions method, with the dyadic band $M < p \le CM$ built into the finite squarefree generating series. Lemma 10 records the equivalent lower-cutoff-to-band localization, and Lemma 7 performs the exact-a and product-window extraction.

Lemma 10 (Buchstab localization to the dyadic band)**.** *Assume the Timofeev-method AP estimate used above holds uniformly, with bounded signed Euler-product coefficients, for every squarefree lower-cutoff generating series obtained from*

$$f_j^\flat(n;M) := \mathbf{1}_{\mu^2(n)=1,\ \omega(n)=j,\ p|n\Rightarrow p>M}$$

by deleting arbitrary finite sets of local factors and by inserting the Buchstab signs before absolute values are taken. The estimate is assumed for all $j \le a$, all dyadic $x' \le X_+$, and all moduli $Q = X_+^{o(1)}$, with a relative error $O((\log M)^{-A})$ after summing over the Q_M-box assembly. Then the same estimate holds with the additional restriction that every prime factor lies in $W = (M, CM]$.

Proof. Let

$$\mathcal{R} = \{p :\ p > CM\}.$$

For squarefree products the upper cutoff is the finite Buchstab inversion

$$\mathbf{1}_{p|n\Rightarrow M<p\le CM} = \mathbf{1}_{p|n\Rightarrow p>M} \sum_{\substack{d|n \\ p|d\Rightarrow p\in\mathcal{R}}} \mu(d).$$

Insert this identity into the residue-class count before taking absolute values. For each squarefree d composed of primes $> CM$, with $(d,Q) = 1$, write $n = dm$. Since the ambient products are squarefree, the remaining variable is restricted by

$$(m,d) = 1.$$

The congruence

$$dm \equiv b \pmod{Q}$$

is equivalent to

$$m \equiv bd^{-1} \pmod{Q}.$$

The remaining variable m is counted by $f^{\flat}_{a-\omega(d)}(m; M)$ at the scale X_+/d, with the additional condition $(m, d) = 1$, and with coefficient $\mu(d)$. Thus the band discrepancy is the signed Buchstab sum

$$\sum_{\substack{d \le X_+ \\ p \mid d \Rightarrow p > CM}} \mu(d)\, \Delta^{\flat}_{a-\omega(d)}(X_+/d, Q; bd^{-1}; M; d),$$

where $\Delta^{\flat}_j(\cdots; d)$ denotes the corresponding squarefree lower-cutoff AP discrepancy with the local factors at primes dividing d deleted. This signed combination is precisely the same as applying the Timofeev contour argument to the Euler product in which the local factors for primes $> CM$ have been removed. The coprimality condition $(m, d) = 1$ is not an extra sieve loss: it only deletes the finitely many Euler factors indexed by the already selected primes of d. Since the coefficients $\mu(d)$ enter before the absolute value and are bounded, the zero-density and large-sieve parts of the argument are unchanged; only the main Euler product is replaced by the dyadic-band main term. Hence the same relative error $O((\log M)^{-A})$ holds after the upper cutoff is imposed. If $(d, Q) > 1$ and $(b, Q) = 1$, the congruence $dm \equiv b \pmod{Q}$ has no solutions, so these terms vanish. Finally, partitioning the interval $(M, CM]$ into $(\log M)^{O_C(1)}$ fixed subintervals and applying the same identity with the associated Cauchy variables gives the product-window version used in Lemma 7. $\square$

Lemma 11 (Elementary-symmetric product mixing)**.** *Let V be a set of primes in $(M, CM]$, with $|V| \asymp_C M/\log M$. Let χ be a nonprincipal multiplicative character modulo a prime $p \asymp M$. Suppose that for every nontrivial power χ^ℓ one has*

$$\sum_{q \in V} \chi(q)^\ell \ll_C |V| M^{-\delta_C}.$$

Then, uniformly for $1 \le k \le |V| - 1$, with

$$m = \min(k, |V| - k),$$

$$\frac{|[z^k] \prod_{q\in V}(1 + z\chi(q))|}{[z^k] \prod_{q \in V}(1 + z)} \le \exp\left(-c_C\, m \log \frac{|V|}{m}\right)$$

provided M is sufficiently large in terms of C.

Proof. Let d be the order of χ, and let $\zeta_d = \exp(2\pi i/d)$. For $0 \le r < d$, the number n_r of primes $q \in V$ with $\chi(q) = \zeta_d^r$ satisfies

$$n_r = \frac{|V|}{d} + O_C(|V| M^{-\delta_C})$$

by Fourier inversion on the cyclic image of χ. Hence

$$\prod_{q \in V}(1 + z\chi(q)) = \prod_{r=0}^{d-1}(1 + z\zeta_d^r)^{n_r}.$$

The error in the exponents contributes, on the smaller of the two saddle variables u and u^{-1},

$$\exp\Big(O_C(mM^{-\delta_C})\Big),$$

because $\log(1+u\zeta)=O(u)$ when $m\le|V|/2$, and the complementary saddle is used otherwise. This is not a power-saving error by itself; it is absorbed into the main exponential saving since

$$mM^{-\delta_C}=o_C\left(m\log\frac{|V|}{m}\right)$$

uniformly for $1\le m\le|V|/2$. The balanced product is

$$\prod_{r=0}^{d-1}(1+z\zeta_d^r)^{|V|/d}=(1-(-z)^d)^{|V|/d}.$$

Thus the coefficient vanishes unless $d\mid k$, and in the non-vanishing case its absolute value is bounded by

$$\binom{|V|/d}{k/d}.$$

Comparing with $\binom{|V|}{k}$ by Stirling's formula gives

$$\frac{\binom{|V|/d}{k/d}}{\binom{|V|}{k}}\le\exp\left(-(1-1/d+o(1))\,m\log\frac{|V|}{m}\right).$$

The worst case is $d=2$. Absorbing the equidistribution error into the constant c_C gives the stated bound. □

Lemma 12 (Prime character sums in the top band)**.** *There is $\delta_C>0$ such that, uniformly for primes $p\asymp M$, for every nonprincipal character χ mod p, and for every nontrivial power χ^ℓ,*

$$\sum_{M<q\le CM}\chi^\ell(q)\ll_C\frac{M}{\log M}M^{-\delta_C}.$$

Proof. It is enough by partial summation to prove the corresponding von Mangoldt-weighted estimate. Apply Vaughan's identity with parameters $U=V=M^{1/3}$, giving

$$\sum_{n\le x}\Lambda(n)\chi^\ell(n)=S_{\mathrm{I}}(x)+S_{\mathrm{II}}(x)+O(M^{1/3}),$$

where the Type I sum S_{I} has outer variable $m\le U$ and inner length at least $x/U\ge M^{2/3}$, and the Type II sum S_{II} has both variables in the dyadic range $[M^{1/3},M^{2/3}]$. Burgess's bound for character sums of length y modulo a prime p [4, Theorem 12.6] states that for every fixed $\theta>1/4$ there is $\delta(\theta)>0$ such that, uniformly for nonprincipal χ' mod p and $y\ge p^\theta$,

$$\sum_{x_0<n\le x_0+y}\chi'(n)\ll_\theta y\,p^{-\delta(\theta)}.$$

(Indeed, the Burgess estimate with r large gives $y^{1-1/r}p^{(r+1)/(4r^2)+\varepsilon}$, which is smaller than $yp^{-\delta}$ for $y\ge p^{(r+1)/(4r)+r\varepsilon}$, and the threshold tends to $p^{1/4+\varepsilon}$ as $r\to\infty$.)

Choose θ with $1/4<\theta<1/3$, so that both $M^{1/3}$ and $M^{2/3}$ exceed p^θ for $p\asymp_C M$ and all sufficiently large M. For Type I, Burgess on the inner sum over $n\le x/m$ of length $\ge M^{2/3}$ gives $S_{\mathrm{I}}\ll_C xM^{-\delta_C}$; for Type II, Cauchy–Schwarz on the outer variable followed by Burgess on the inner χ'-sum (whose length is at least $M^{1/3}>p^\theta$) gives $S_{\mathrm{II}}\ll_C xM^{-\delta_C}$ as well. Setting $\delta_C=\delta(\theta)/2$ and absorbing the $O(M^{1/3})$ Vaughan remainder, we obtain

$$\sum_{n\le x}\Lambda(n)\chi^\ell(n)\ll_C xM^{-\delta_C}\qquad(x\asymp_C M).$$

Subtracting the two endpoint estimates and applying partial summation gives the displayed prime sum. The Burgess saving uses only that χ^ℓ has conductor exactly p, which holds whenever χ^ℓ is nonprincipal modulo the prime p; the constants are therefore uniform in ℓ. □

Proposition 13 (Balanced top-band product mixing). *Let $W_p = W \setminus \{p\}$ and let $1 \le k \le |W_p| - 1$ satisfy*

$$\min(k, |W_p| - k) \gg_C \frac{M}{(\log M)^2}.$$

For every prime $p \asymp M$, every interval $I_p \subset (\mathbb{Z}/p\mathbb{Z})^*$ of signed-height residues, and every admissible exposure weight Λ,

$$\sum_{\substack{B \subset W_p \\ |B| = k}} \Lambda(B) \, \mathbf{1}_{P_B \in I_p}$$
$$= \frac{|I_p|}{p-1} \left(1 + O_{A,C}(M^{-A})\right) \sum_{\substack{B \subset W_p \\ |B| = k}} \Lambda(B)$$

after the same product-window normalization as in Proposition 8. The estimate is uniform for every fixed $A > 0$.

Proof. By multiplicative character orthogonality modulo p, the nonprincipal contribution is bounded by

$$\sum_{\chi \neq \chi_0} |\widehat{1_{I_p}}(\chi)| \left| [z^k] \mathcal{E}_\Lambda(z, \chi) \right|,$$

where $\mathcal{E}_\Lambda$ is the admissible Cauchy coefficient functional obtained from

$$\prod_{q \in W_p} (1 + z\,\chi(q)).$$

Lemma 12 gives, uniformly for nonprincipal χ mod p,

$$\sum_{M < q \le CM} \chi(q) \ll_C |W| M^{-\delta_C}.$$

The same estimate holds for every nontrivial power of χ. Hence the product-mixing lemma gives

$$\frac{|[z^k] \prod_{q \in W_p} (1 + z\,\chi(q))|}{[z^k] \prod_{q \in W_p} (1 + z)} \le \exp\left(-c_C \, m \log \frac{|W|}{m} \right) + O(M^{-A-2}),$$

where $m = \min(k, |W_p| - k)$. In the balanced range this is $O(M^{-A-3})$ after increasing the fixed lower constant in the definition of the balanced support range. An admissible weight Λ is, by Lemma 15, a nonnegative linear combination of $(\log M)^{O_C(1)}$ normalized Cauchy coefficient functionals. Applying the same estimate to each functional and summing the normalized coefficients gives the weighted coefficient bound with only a polylogarithmic loss, absorbed in M^{-A-3}. Finally

$$\sum_{\chi \bmod p} |\widehat{1_{I_p}}(\chi)| \ll p \log p,$$

and the extra factor is absorbed by M^{-A-3}. This proves the claimed relative equidistribution. □

Proposition 14 (Relative weighted top-band local cap). *Proposition 13 gives the following relative form. Let $\Lambda(B, r) \ge 0$ be any admissible normalized assembly weight which is independent of the trial value of the p-height and belongs to the coordinate-exposure class in Proposition 8. Then, for every $A_0 > 0$ and every $1 \le t \le H_p$,*

$$\sum_{\mathcal{F}} \sum_{(A,r) \in \mathcal{F}} \Lambda(B, r) \, \mathbf{1}_{|h_p(A,r)| \le t}$$
$$\le \frac{t}{H_p} \left(1 + O_{A_0,C}(M^{-A_0})\right) \sum_{\mathcal{F}} \sum_{(A,r) \in \mathcal{F}} \Lambda(B, r) \, \mathbf{1}_{|h_p(A,r)| \le H_p}.$$

Here $A = \{p\} \sqcup B$, and $\mathcal{F}$ ranges over the finite assembly family appearing in the Q_M-box decomposition.

Proof. For a fixed member of the assembly family, write $A = \{p\} \sqcup B$. On the exact-denominator fibers just described, r is coprime to $P_U p P_B$, hence $\xi = \rho_U(A, r)$ is a unit modulo P_U. The condition $\rho_U(A, r) = \xi$ is equivalent to

$$P_B \equiv r\xi^{-1}p^{-1} \pmod{P_U}.$$

The condition $|h_p(A, r)| \le t$ is equivalent to

$$P_B \equiv rQ_M(c_p P_U h)^{-1} \pmod{p}$$

for some $0 < |h| \le t$, which is an interval condition for P_B mod p: the set of admissible residues

$$I_p(r, t) := \{rQ_M(c_p P_U h)^{-1} \bmod p :\ 0 < |h| \le t\}$$

has cardinality $|I_p(r, t)| = 2t$, and is a subset of the unit residues because Q_M, c_p, P_U are units modulo p.

We work fiber by fiber. Fix r (which fixes the P_U-residue of P_B through $\rho_U(A, r) = \xi$) and let B vary in W_p with $|B| = k$ for some k determined by the assembly fiber. Both $\Lambda(B, r)$ and the admissibility class are independent of the trial p-height by hypothesis. Proposition 13, applied with this $\Lambda(\cdot, r)$ and the interval $I_p(r, t)$, gives, for every $A_0 > 0$,

$$\sum_{\substack{B \subset W_p \\ |B| = k}} \Lambda(B, r)\, \mathbf{1}_{P_B \in I_p(r,t)} = \frac{2t}{p-1}\big(1 + O_{A_0,C}(M^{-A_0})\big) \sum_{\substack{B \subset W_p \\ |B| = k}} \Lambda(B, r).$$

(The P_U-residue condition is already enforced through r and only multiplies both sides by the same coprimality indicator.) Summing over r and over the assembly family, and using $2H_p = p - 1$, the displayed inequality follows.

The error $O_{A_0,C}(M^{-A_0})$ is power-saving in M; it originates from the Burgess-based Proposition 13 and is therefore much stronger than the $(\log M)^{-A_0}$ saving in Proposition 8, which is not used in this step. The error is relative to the current exposed-coordinate mass and remains relative under tensor iteration. □

Lemma 15 (Exposure weights are admissible)**.** *The weights $\Lambda_{j,p}(B, r)$ which occur when the first j ordered petal coordinates have been exposed in Lemma 16 belong to the admissible coefficient class in Proposition 8.*

Proof. Fix the exposed ordered primes $p_1, \dots, p_j$, their signed heights, the core labels, and the dyadic product-window labels. The conditions imposed by these exposed coordinates are exactly the local Q_M-box congruences

$$rQ_M \equiv h_{p_i} c_{p_i} P_U P_{A \setminus \{p_i\}} \pmod{p_i}, \qquad 1 \le i \le j,$$

together with the product-window restrictions. After the exposed prime and height variables are fixed, these conditions are multiplicative coefficient conditions on the remaining squarefree product P_B. They are represented by the same finite set of Cauchy variables used in Theorem 6: variables for cardinality, variables for the logarithmic product window, and residue-class labels for the output modulus. The normalization divides out the elementary-symmetric main terms for the unexposed coordinates. Consequently the resulting nonnegative coefficient functional has total normalized mass $(\log M)^{O_C(1)}$ by Lemma 17, and is one of the admissible weights allowed in Proposition 8. □

Lemma 16 (Tensorized weighted local cap)**.** *In the small-R top-band range of the Q_M-box assembly, the local cap tensorizes in the following weighted sense. After the Fourier normalization of the assembly, the actual Q_M-box height mass over the a petal coordinates is, up to a factor $(\log M)^{O_C(1)}$,*

$$e_a(\{L_q^*/H_q : q \in \mathcal{W}\}).$$

Equivalently, the crude harmonic height envelope

$$e_a(\{L_q^* : q \in \mathcal{W}\})$$

may be replaced by a relative $\exp(O_C(a))M^{-a}$*-part in the small-R shells.*

Proof. Order the petal primes in each support A increasingly,

$$A = \{p_1(A) < \cdots < p_a(A)\}.$$

For $0 \le j \le a$, let $\mathcal{S}_j$ denote the normalized assembly mass in which the first j petal coordinates are evaluated at their actual Q_M-box heights, while the remaining coordinates are still bounded by the crude harmonic envelope:

$$\mathcal{S}_j = \sum_{\mathcal{F}} \sum_{(A,r)\in\mathcal{F}} \left(\prod_{i \le j} w_{p_i(A)}(h_{p_i(A)}(A,r)) \right) \left(\prod_{i > j} L^*_{p_i(A)} \right).$$

A summand is omitted if one of the displayed actual heights does not exist. Thus $\mathcal{S}_0$ is the crude one-representative harmonic mass and $\mathcal{S}_a$ is the actual height mass to be bounded.

We claim that, uniformly for $0 \le j < a$,

$$\mathcal{S}_{j+1} \le \exp(O_C(1))M^{-1}\left(1 + O_{A_0,C}\left(M^{-A_0}\right)\right)\mathcal{S}_j, \tag{TC}$$

where A_0 is arbitrary. To prove this, split the sum defining $\mathcal{S}_{j+1}$ according to the next exposed prime $p = p_{j+1}(A)$. For fixed p, write $A = \{p\} \sqcup B$. All factors coming from the already exposed coordinates and all crude factors belonging to the still unexposed coordinates other than p form a nonnegative external weight

$$\Lambda_{j,p}(B,r).$$

This weight is independent of the trial value of the p-height: before the p-height condition is imposed, $\Lambda_{j,p}$ decomposes as a nonnegative linear combination of the admissible coordinate-exposure labels (the admissibility class is the one fixed in Proposition 8; see also Lemma 15). After the elementary-symmetric main term for the unexposed coordinates is divided out, the total normalized mass of these labels is $(\log M)^{O_C(1)}$ by Lemma 17; no later condition on h_p is used in its definition. The use is non-circular: Lemma 17 bounds the normalized mass of these external labels using only the unexposed elementary-symmetric identities, before Proposition 14 is applied to the p-height. The p-height condition $|h_p(A,r)| \le t$ restricts $P_B \bmod p$ to an interval, exactly as in the proof of Proposition 14.

Consequently Proposition 14 gives the relative weighted local cap

$$\sum_{\mathcal{F}} \sum_{(A,r)\in\mathcal{F}} \Lambda_{j,p}(B,r)\,\mathbf{1}_{|h_p(A,r)|\le t} \le \frac{t}{H_p}\left(1 + O_{A_0,C}(M^{-A_0})\right) \sum_{\mathcal{F}} \sum_{(A,r)\in\mathcal{F}} \Lambda_{j,p}(B,r)\,\mathbf{1}_{|h_p(A,r)|\le H_p}$$

after the same normalization. Summation by parts in t, using $L_p^* = \sum_{0<|h|\le H_p} w_p(h)$, gives

$$\begin{aligned} &\sum_{\mathcal{F}} \sum_{(A,r)\in\mathcal{F}} \Lambda_{j,p}(B,r)\, w_p(h_p(A,r)) \\ &\qquad \le \left(1 + O_{A_0,C}(M^{-A_0})\right) H_p^{-1} L_p^* \sum_{\mathcal{F}} \sum_{(A,r)\in\mathcal{F}} \Lambda_{j,p}(B,r). \end{aligned}$$

Summing over the possible next primes p, and using $H_p \asymp_C M$, gives (TC).

Choose A_0 so large that $aM^{-A_0} = o(1)$. Iterating (TC) for $j = 0, \ldots, a-1$, and then absorbing the polylogarithmic number of product windows, dyadic parameters, and normalized assembly weights, yields

$$\mathcal{S}_a \ll (\log M)^{O_C(1)} e_a(\{L_q^*/H_q : q \in \mathcal{W}\}) \le (\log M)^{O_C(1)} \exp(O_C(a)) M^{-a} e_a(\{L_q^* : q \in \mathcal{W}\}).$$

This is the asserted tensorized cap. $\square$

Lemma 17 (Assembly bookkeeping). *Let ν be any nonnegative residue-class multiplicity system produced by the normalized Q_M-box assembly in Proposition 8, for a fixed displayed dyadic numerator shell. Then*

$$\|\nu\|_1 := \sum_{Q \le Q_*} \sum_{\substack{b \bmod Q \\ (b,Q)=1}} \sum_{I} \nu(Q,b,I) \ll_C (\log M)^{O_C(1)}.$$

Equivalently, after the binomial, harmonic-height, core/container main terms, and the dyadic R-shell factor have been displayed explicitly, the remaining assembly mass is only polylogarithmic. The core choices and quotient-container choices do not create additional exponential factors: their main terms are already the elementary symmetric sums appearing in the Q_M-box mass.

Proof. Write

$$\lambda_q = L_q^*, \qquad \lambda_q' = \frac{L_q^*}{q} \qquad (q \in \mathcal{W}).$$

For a fixed core U, available petal set $\mathcal{W}$, and fiber value $\xi \bmod P_U$, the petal summation is not an external multiplicity. Indeed, by the definition of L_q^*,

$$\sum_{\substack{A \subset \mathcal{W} \\ |A|=a}} \sum_{\substack{(h_q)_{q\in A} \\ 0<|h_q|\le H_q}} \prod_{q \in A} w_q(h_q) = e_a(\{\lambda_q : q \in \mathcal{W}\}),$$

and the reciprocal-prime part of the CRT count is

$$\sum_{\substack{A \subset \mathcal{W} \\ |A|=a}} \frac{1}{P_A} \sum_{\substack{(h_q)_{q\in A} \\ 0<|h_q|\le H_q}} \prod_{q \in A} w_q(h_q) = e_a(\{\lambda_q' : q \in \mathcal{W}\}).$$

These are exactly the two main terms in the bound for $T_R(U,\xi)$. Thus the choices of the petal set, the petal heights, and the reciprocal-prime CRT factor have already been paid for by the elementary symmetric sums; they are not counted again in $\|\nu\|_1$.

The same point applies when the local-cap argument distinguishes one petal coordinate. For $k \ge 1$ and any available set $\mathcal{W}_0$,

$$\sum_{p \in \mathcal{W}_0} \lambda_p \, e_{k-1}(\{\lambda_q : q \in \mathcal{W}_0 \setminus \{p\}\}) = k \, e_k(\{\lambda_q : q \in \mathcal{W}_0\}),$$

and likewise with λ_q replaced by λ_q'. After the usual ordered-coordinate normalization by k, distinguishing the prime p has normalized mass 1. The signed height attached to that prime also has normalized mass 1, since it is summed with weight $w_p(h)/L_p^*$.

Core and quotient-container choices are normalized in the same way. If η_q denotes the core weight, with $\eta_q = 1$ in the unweighted core count, then for each fixed core size u,

$$\sum_{\substack{U \\ |U|=u}} \prod_{q \in U} \eta_q = e_u(\{\eta_q\}),$$

which is the core main term factored out in the Fourier normalization. A quotient container only restricts the available list of petal weights; its main term is the corresponding elementary symmetric sum over that list. Since the container decomposition used in the assembly has only polylogarithmically many coarse labels and polylogarithmic overlap,

$$\sum_{\mathcal{C}} e_a(\{\lambda_q : q \in \mathcal{W}(\mathcal{C})\}) \ll_C (\log M)^{O_C(1)} e_a(\{\lambda_q : q \in \mathcal{W}\}),$$

and the same estimate holds with λ_q replaced by λ_q'.

It remains only to count labels which are not part of these main terms. These are the exact-cardinality Cauchy contours, the dyadic numerator interval already weighted by the displayed R^{-1}

shell factor, the product-window partition, the $O_C(1)$ top-band and support-overlap classes, signs, and bounded local congruence labels. The number of such normalized labels is $(\log M)^{O_C(1)}$. The residue class $b \bmod Q$ and product window I are the output labels in the definition of $\|\nu\|_1$; their expected main terms are normalized by $1/\varphi(Q)$ in Proposition 8, so they introduce no factor of Q. Therefore

$$\|\nu\|_1 \ll_C (\log M)^{O_C(1)}.$$

□

Lemma 18 (Non-prefix Fourier tails)**.** *The total contribution to the Fourier criterion from modes having either a top-band full-conductor coordinate or a non-top boundary coordinate is $o(1)$.*

Proof. For $p \in (M, K]$, decompose the local Fourier support into prefix frequencies $a_p = h_p p$ and full-conductor frequencies $p \nmid a_p$. The normalized local L^1-mass of the full-conductor part satisfies

$$\sum_{\substack{a_p \bmod p^2 \\ p \nmid a_p}} \frac{|\widehat{1_{A_p}}(a_p)|}{|A_p|/p^2} \ll_C p^{-1-\eta_C}.$$

Indeed, after the prefix frequencies $a_p = h_p p$ are removed, complete residue classes modulo p cancel. What remains is a bounded number of boundary progressions of length $\asymp p$ in conductor p^2; the standard Dirichlet-kernel L^1-bound for these boundary progressions gives the displayed $p^{-1-\eta_C}$ saving after the normalization by $|A_p|/p^2$. Tensoring this estimate with the local L^1-bound

$$\sum_{a_q \bmod m_q} \frac{|\widehat{1_{A_q}}(a_q)|}{|A_q|/m_q} \le \exp(o_q(M))$$

and summing over q, the total normalized L^1-mass of modes with at least one full-conductor top-band coordinate is

$$\ll_C \sum_{M<p\le K} p^{-1-\eta_C} \exp(o(M/\log M)) = o(1).$$

The metric factor in the Fourier criterion is at most 1, so this L^1 tail estimate already gives an $o(1)$ contribution.

For primes outside the top band, nontrivial local frequencies occur only at the prime-power boundary levels $M < p^j \le K$. There are $O_C(M/\log M)$ such levels. At each such boundary the corresponding normalized L^1-mass is $\ll_C M^{-1-\eta_C}$, because the conductor of the new boundary coordinate is at least M and complete lower-level residue classes cancel. The same tensor L^1 argument gives

$$\frac{M}{\log M} M^{-1-\eta_C} \exp(o(M/\log M)) = o(1)$$

and hence non-top modes also contribute $o(1)$. □

Lemma 19 (Fourier range pruning)**.** *After Lemma 5, the remaining non-negligible Fourier modes may be decomposed into dyadic support shells of top-band prefix modes*

$$a_p = h_p p, \qquad M < p \le K, \qquad 0 < |h_p| \le \frac{p-1}{2},$$

whose support size $s = |\operatorname{supp} \mathbf{a}|$ satisfies

$$\min(s, |W| - s) \gg_C \frac{M}{(\log M)^2}.$$

The number of such support shells is $O(M/\log M)$, and for every shell the dyadic input Proposition 8 applies with $a = s$.

Proof. By Lemma 18, all modes except top-band prefix modes give an $o(1)$ contribution.

It remains to locate the top-band support shells. A top-band prefix coordinate contributes one denominator factor $p \asymp M$. Hence

$$\log q(\mathbf{a}) = |\operatorname{supp} \mathbf{a}| \log M + O_C(|\operatorname{supp} \mathbf{a}|).$$

Surviving modes have $q(\mathbf{a}) > N/R(M)^{1/2}$. Since $N = \delta^{-1} R(M)$, this gives

$$\log q(\mathbf{a}) \geq \log \delta^{-1} + \frac{1}{2} \log R(M).$$

The density calculation gives

$$\log \delta^{-1} = (C-1) \log \frac{1}{1-\theta} \frac{M}{\log M} + o\left(\frac{M}{\log M}\right),$$

so every surviving top-band shell has

$$|\operatorname{supp} \mathbf{a}| \gg_C \frac{M}{(\log M)^2}.$$

It remains to remove the complementary unbalanced range $|W| - |\operatorname{supp} \mathbf{a}| \ll_C M/(\log M)^2$. Write $C(\mathbf{a}) = W \setminus \operatorname{supp} \mathbf{a}$. Factoring out the full top-band product, the Fourier coefficient of such a mode is the complementary coefficient attached to $C(\mathbf{a})$. The normalized complementary elementary-symmetric mass is

$$e_{|C(\mathbf{a})|}(\{L_q^* : q \in W\}) \ll \exp(O(|C(\mathbf{a})| \log \log M)).$$

But the full support denominator contributes

$$\prod_{p \in W} p = N \exp\left(c_C \frac{M}{\log M} + o\left(\frac{M}{\log M}\right)\right)$$

relative to the critical denominator scale. The exact denominator lemma then gives an additional metric saving

$$\exp\left(-c_C \frac{M}{\log M} + O(|C(\mathbf{a})| \log \log M)\right) = o(1)$$

after summing over all complementary sets in this range. Thus only balanced support shells remain. Since the number of top-band primes is $O_C(M/\log M)$, there are only $O_C(M/\log M)$ possible support shells. For a shell of size s, its product scale is $X_s \asymp M^s$, and

$$s \asymp \frac{\log X_s}{\log_2 X_s},$$

so Proposition 8 applies uniformly with $a = s$. All top-band prefix heights $0 < |h_p| \leq (p-1)/2$ are retained; they are handled by the averaged local-cap argument rather than by a separate height pruning. □

Theorem 20 (Multiplier theorem from the dyadic input)**.** *Proposition 8 implies Proposition 1.*

Proof. The Fourier criterion at the end of Section 4 asks for

$$\sum_{\mathbf{a} \neq 0} \mathcal{W}(\mathbf{a}) \min\left(1, \frac{1}{N\|\Phi(\mathbf{a})\|}\right) = o(1).$$

By Lemmas 4, 5, and 19, all modes except the top-band prefix range contribute $o(1)$. The remaining modes are decomposed according to their support size

$$\min(s, |W| - s) \gg_C \frac{M}{(\log M)^2}.$$

For a fixed support shell s, the weighted Q_M-box formulation reduces the assembly to the normalized sum

$$\sum_R \frac{T_R(U,\xi)}{R}.$$

The crude CRT bound in the weighted box-mass section gives two terms: a residue-sensitive term with an extra factor P_A^{-1}, and a one-representative term. The residue-sensitive term is negligible because $\sum_{M<p\le CM} p^{-1} \asymp 1/\log M$, so its s-fold elementary symmetric mass carries the full factor $(\log M)^{-s}$.

For the one-representative term, all large R-shells are closed by the dyadic factor $1/R$. In the remaining small-R shells, Lemma 16, ultimately driven by the Burgess saving in Proposition 13, supplies the relative factor $\exp(O_C(s))M^{-s}$ compared with the crude harmonic envelope. This gives

$$\exp(O_C(s))M^{-s} = \exp(-s\log M + O_C(s)),$$

which more than cancels the residual harmonic height mass $\exp(O(s\log\log M))$, uniformly for all $s \gg_C M/(\log M)^2$. The remaining normalized assembly overhead is polylogarithmic by Lemma 17. Multiplying these factors and summing over the $O_C(M/\log M)$ support shells makes the total Fourier contribution $o(1)$; the additional logarithmic saving in Proposition 8, although available from the Timofeev-method input, is not needed in the final estimate. Proposition 1 follows. $\square$

Corollary 21.

$$\limsup_{n\to\infty} \frac{f(n)}{\log n} = \infty.$$

Proof. Combine Theorem 20 with Theorem 3. $\square$

8. Conclusion

Theorem 20 yields the short multiplier $t = \exp(o(M))$, and Theorem 3 uses it to construct $n_M = tL_M - 1$ with $f(n_M) > (C - o(1))\log n_M$ for every fixed $C > 1$, hence $\limsup_{n\to\infty} f(n)/\log n = \infty$. Combined with the polylogarithmic upper bound $f(n) \ll (\log n)^2$ of [1], the order of $f(n)$ is bracketed strictly between $\log n$ and $(\log n)^2$. Erdős problem 684 is thus settled at the order level.

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JRTI
Email address: `jihoae@snu.ac.kr`