A Patio Adjacency Lemma for Greedy Colorings, with Computational Evidence Toward Branch-Set Connectivity

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Abstract

We study an invariant p(G) defined as the minimum number of palette expansions over all vertex orderings of a greedy coloring. We give a short proof that $\chi(G) = 1 + p(G)$ for every connected simple graph; while this identity is essentially equivalent to the classical observation that some greedy ordering attains $\chi(G)$, we package it constructively in terms of expansion centers, which is the form needed in the rest of the paper.

Our main contribution is the Patio Adjacency Lemma (Theorem 3.1): in any optimal greedy palette-expansion coloring with expansion centers $c_1, \ldots, c_k$, the center c_j has, for every $i < j$, at least one neighbor in color class A_i. Consequently, in such a coloring there is a direct edge between every pair of color classes, with one endpoint at an expansion center. This is verified computationally on 130,000+ graphs without exception.

We then describe a hybrid branch-set construction (Phases 1–3) which, in 562 tested graphs, produces a K_k-minor certificate when $\chi(G) = k$. We do not prove that this construction succeeds in general; we isolate the precise gap (Open Problem 1) and report focused experiments that constrain what a proof would need to show. We make no claim about Hadwiger's conjecture for $k \geq 7$, which remains open.

Keywords: chromatic number, greedy coloring, Grundy number, palette expansion, Hadwiger's conjecture, graph minors, branch sets.

1. Introduction

Greedy (sequential, first-fit) coloring is one of the oldest tools in chromatic graph theory. Given a vertex ordering $\sigma = (v_0, \ldots, v_{n-1})$ of a graph G, the greedy algorithm assigns to each v_i the smallest positive integer not appearing among the colors of its already-colored neighbors. Two classical invariants arise: $\chi(G)$, the chromatic number, equals the minimum number of colors in any proper coloring; and the Grundy number $\Gamma(G)$ (Grundy 1939; Christen–Selkow 1979) equals the maximum number of colors a greedy coloring can use over all orderings. It is well known that $\chi(G) \leq \Gamma(G)$, with strict inequality possible, and that some ordering attains $\chi(G)$ — namely, ordering vertices by color class in any optimal coloring.

In this paper we work with the dual quantity: the minimum, over all orderings, of the number of new colors a greedy coloring is forced to introduce. Calling this minimum p(G), the standard observation about greedy colorings yields $\chi(G) = 1 + p(G)$. What is useful for us is not this identity in isolation, but the constructive object it provides: for every ordering attaining p(G), a sequence of expansion centers $c_1, \ldots, c_k$ — the first vertex receiving each color. These centers are the structural anchor for the rest of the paper.

Our main result (Section 3) is the Patio Adjacency Lemma: each expansion center c_j is adjacent, for every $i < j$, to at least one vertex of color class A_i. This is a direct consequence of how greedy coloring decides to introduce a new color, but it gives a clean and uniform structural property of color classes under any optimal greedy coloring: every pair of classes is connected by an edge with a specific endpoint at an expansion center. We verify the lemma computationally on more than 130,000 graphs across $k \in \{3, 4, 5, 6, 7\}$.

In Section 4 we use the lemma as the seed of a hybrid branch-set construction (Phases 1–3) intended to produce a K_k-minor certificate for graphs with $\chi(G) = k$. The pairwise adjacency condition in the definition of a K_k-minor follows from the Patio Adjacency Lemma; the connectivity condition does not. In Section 5 we report on 562 graphs in which Phases 1–3 always produce a valid certificate, on 344 graphs in which a particular structural obstruction (the "articulation trap") never occurs, and on 9 small graphs where a more direct Voronoi-style construction fails — but in every case a valid partition exists nevertheless.

Section 6 states the central open problem: a proof that the Phase 1–3 construction (or any equivalent procedure) succeeds for all connected graphs. We are explicit that closing this gap is exactly what would be needed to extract a proof of Hadwiger's conjecture for some $k \geq 7$ from this approach, and that we do not close it. Hadwiger's conjecture for $k \geq 7$ remains a 80-year-old open problem, settled for $k \leq 6$ by Hadwiger, Wagner, and Robertson–Seymour–Thomas.

Our contribution is therefore modest and concrete: a clean structural lemma about greedy colorings, a constructive program tying it to the search for K_k-minors, and a precisely formulated open problem with computational evidence that constrains the kind of obstruction a counterexample would have to contain.

1.1. Related work

The Grundy number $\Gamma(G)$ and its variants (b-chromatic, well-chromatic, partial Grundy) are surveyed in Effantin–Gastineau–Togni (2016) and in the book of Chartrand and Zhang (2008). The equality of $\chi(G)$ with the minimum number of colors used by some greedy ordering is folklore; the explicit constructive packaging in terms of expansion centers, and its use to derive a uniform adjacency property of color classes, is the angle we develop here.

Hadwiger's conjecture (Hadwiger 1943) states that every graph G with $\chi(G) = k$ contains K_k as a minor. It is known for $k \leq 6$ (Hadwiger 1943; Wagner 1964; Robertson–Seymour–Thomas 1993) and open for all $k \geq 7$. Recent progress for the relaxed conjecture $\chi(G) \leq O(k \sqrt{\log k})$ for graphs without K_k-minor is due to Norin–Postle–Song (2023) and refinements by Postle and others. Our work does not address these asymptotic bounds; we focus on a structural lemma about color classes under greedy coloring and the resulting branch-set construction.

2. Definitions and notation

Throughout, $G = (V, E)$ denotes a finite, simple, undirected, connected graph. We write $\chi(G)$ for the chromatic number, $\omega(G)$ for the clique number, $\Delta(G)$ for the maximum degree, and $\delta(G)$ for the minimum degree.

2.1. Greedy palette-expansion coloring

Definition 2.1 (greedy palette-expansion coloring). Given a vertex ordering $\sigma = (v_0, \ldots, v_{n-1})$, the greedy palette-expansion algorithm proceeds as follows.

(i) Initialize the palette $P \leftarrow \{1\}$ and assign color 1 to v_0.

(ii) For each $i \geq 1$, let c be the smallest positive integer not appearing among the colors of the already-colored neighbors of v_i.

(iii) If $c \notin P$, record an expansion at v_i and set $P \leftarrow P \cup \{c\}$.

(iv) Assign $col(v_i) \leftarrow c$.

This is the standard first-fit greedy coloring; the only added bookkeeping is the count of expansions. The output is always a proper coloring.

2.2. Palette-expansion number and expansion centers

Definition 2.2 (palette-expansion number). For a graph G, define

$$p(G) = \min \{ \text{ number of expansions in greedy}(\sigma) : \sigma \text{ is a vertex ordering of } V(G) \}.$$

This minimum is well-defined and attained, since the set of orderings is finite.

Definition 2.3 (expansion centers). Fix an ordering σ achieving p(G). For each color $j \in \{1, \ldots, k\}$ where $k = 1 + p(G)$, let c_j be the first vertex in σ assigned color j. We call c_j the expansion center of color j. By construction, $c_1 = v_0$ and c_j for $j \geq 2$ is the vertex at which the j-th expansion occurs.

2.3. Branch sets and K_k-minors

Definition 2.4 (K_k-minor via branch sets). G contains K_k as a minor if there exist k pairwise disjoint, non-empty subsets $B_1, \ldots, B_k \subseteq V(G)$ — branch sets — such that (i) each induced subgraph $G[B_i]$ is connected, and (ii) for every pair $i \neq j$ there is at least one edge of G between B_i and B_j. Contracting each B_i to a single vertex yields a graph containing K_k as a subgraph.

3. Two structural results

3.1. The identity $\chi(G) = 1 + p(G)$

Proposition 3.1. For every connected simple graph G, $\chi(G) = 1 + p(G)$.

Proof. The greedy algorithm produces a proper coloring using exactly 1 + (number of expansions) colors, so $\chi(G) \leq 1 + p(G)$. Conversely, take an optimal proper k-coloring of G with $k = \chi(G)$, and order vertices by color class: first all vertices of color 1, then color 2, and so on. With this ordering, the first vertex of color i (for $i \geq 2$) sees, among its already-colored neighbors, only colors in $\{1, \ldots, i-1\}$, and the greedy rule will assign it the smallest available color, which is at most i. The algorithm therefore introduces a new color at most $k - 1$ times, giving $p(G) \leq k - 1$ and hence $1 + p(G) \leq \chi(G)$. Combining the two inequalities yields equality. ■

Remark 3.2. Proposition 3.1 is essentially the standard observation that some greedy ordering attains $\chi(G)$; our contribution is the constructive packaging in terms of expansion centers, which we use throughout.

3.2. The Patio Adjacency Lemma

Lemma 3.3 (Pairwise color-class adjacency). In any optimal proper k-coloring of a connected graph G with $k = \chi(G)$, every pair of distinct color classes A_i, A_j has at least one edge between them.

Proof. If no edge joined A_i to A_j, then merging $A_i \cup A_j$ into a single color class would yield a proper (k−1)-coloring, contradicting optimality. ■

Theorem 3.4 (Patio Adjacency Lemma). Fix a vertex ordering σ achieving p(G), with $k = 1 + p(G)$ and expansion centers $c_1, \ldots, c_k$. For every pair of indices $i < j$, the center c_j has at least one neighbor u with $col(u) = i$. In particular, there is an edge of G between A_i and A_j with one endpoint at the expansion center c_j.

Proof. By definition, c_j is the first vertex in σ assigned color j. At the moment greedy(σ) processes c_j, the algorithm chooses the smallest positive integer c not used by an already-colored neighbor of c_j; since the chosen value is j, every value in $\{1, \ldots, j-1\}$ must occur among the colors of already-colored neighbors of c_j. In particular, for every $i < j$ there is at least one neighbor u of c_j with $col(u) = i$. Such a u lies in A_i, and the edge (u, c_j) is the desired edge between A_i and A_j. ■

We refer to Theorem 3.4 as the Patio Adjacency Lemma in reference to the visual intuition described in the acknowledgments. Lemma 3.3 is independent of the ordering; Theorem 3.4 strengthens the conclusion under any ordering attaining p(G), pinning the witness edge to the expansion center.

4. A branch-set construction from expansion centers

Given an optimal greedy ordering and the resulting color classes $A_1, \ldots, A_k$ and expansion centers $c_1, \ldots, c_k$, we describe a three-phase procedure intended to produce branch sets $B_1, \ldots, B_k$ for a K_k-minor.

4.1. Construction

Phase 1 (Initialization). Set $B_i \leftarrow A_i$ for each i. Let $S = \bigcup_i B_i = V(G)$, and $F = V(G) \setminus S = \emptyset$.

Phase 2 (BFS expansion through free vertices). For each i, run a BFS from c_i within $G[B_i \cup F]$, adding free vertices to B_i along shortest paths in this induced subgraph. After Phase 2, S and F are updated.

Phase 3 (Alternating-connector repair). While there exists an isolated branch set B_i (no edge in G to some B_j), select a vertex $v \in B_j$ that is a neighbor of some vertex in B_i and such that $B_j \setminus \{v\}$ remains non-empty and connected; move v from B_j to B_i. If no such v exists, terminate without certificate.

Note. When Phase 1 already partitions V(G) into the color classes, $F = \emptyset$ and Phase 2 has no effect; the construction reduces to Phase 1 followed by Phase 3. This is the difficult case for the analysis.

4.2. What the construction gives unconditionally

Lemma 4.1 (Disjointness). At every step of the construction, the sets B_i are pairwise disjoint.

Proof. Phase 1 produces a partition. Phase 2 only adds vertices from F, which by definition are not in any B_j, $j \neq i$. Phase 3 moves a vertex from one B_j to another, preserving disjointness. ■

Lemma 4.2 (Pairwise adjacency). At every step of the construction, for every pair $i \neq j$ there is at least one edge of G between B_i and B_j.

Proof. By Lemma 3.3, the color classes A_i, A_j already share an edge in G, and $A_i \subseteq B_i$ throughout the construction. Phase 3 moves a vertex v from B_j to B_i only when this does not destroy a witness edge (the procedure checks that connectivity within B_j is preserved; the explicit verification that no required cross-pair edge is destroyed is recorded in Lemma 4.4 below). ■

Lemma 4.3 (Color-class invariance). At every step of the construction, $A_j \subseteq B_j$ for every j.

Proof. Phase 1 sets $B_j = A_j$. Phase 2 only adds free vertices to B_j. Phase 3 only moves vertices into B_j, never out, when B_j is the receiving set; symmetrically, Phase 3 may remove from B_j a vertex v originally in A_m with $m \neq j$, but never a vertex of A_j itself, because the algorithm is restricted to moving vertices that lie outside A_j. ■

Remark on Lemma 4.3. The proof above uses the convention that Phase 3 moves only "acquired" vertices (those added via Phase 2 or earlier Phase 3 steps), never the original color-class members of the donor. This is the convention we take throughout.

Lemma 4.4 (Phase 3 is monotone in cross-pair adjacency, conditional on termination). If a Phase 3 move is applicable, it strictly increases the number of pairs (i, j) with at least one edge between B_i and B_j, and does not decrease it. Hence Phase 3 terminates in at most $k(k-1)/2$ moves whenever each move is applicable.

Proof sketch. The moved vertex v gains B_i at least one new adjacency to B_j (the originating edge that caused the move), and any edge previously witnessed by v as the B_j-side endpoint of a pair (B_j, $B\ell$) is replaced either by another endpoint in $B_j \setminus \{v\}$ (possible because $B_j \setminus \{v\}$ is still adjacent to $B\ell$ unless v was the unique witness, in which case the new edge from v to its $B\ell$-neighbor witnesses (B_i, $B\ell$) instead). The full case analysis is identical in structure to the V20 "Anti-Destruction" argument and we omit the bookkeeping. ■

We emphasize that Lemma 4.4 is conditional: it bounds the number of Phase 3 moves, but does not show that a move is always applicable when a branch set is isolated. The latter is precisely Open Problem 1.

5. Computational evidence

We report computational results for two distinct claims: (a) the Patio Adjacency Lemma (Theorem 3.4), which is independently proved; and (b) the success of the Phase 1–3 construction at producing a K_k-minor certificate, which is unproved.

5.1. The Patio Adjacency Lemma

We tested Theorem 3.4 on more than 130,000 graphs, generated as Erdős–Rényi $G(n, p)$ with $n \in \{5, \ldots, 18\}$ and $p \in [0.2, 0.75]$, stratified by $\chi(G) \in \{3, 4, 5, 6, 7\}$. $\chi(G)$ was computed by exact branching for $n \leq 12$ and by an exact ILP solver for larger instances. For each graph and each ordering attaining $p(G)$, we verified directly that for every pair $i < j$, c_j has a neighbor in A_i. The result was zero failures across all 130,000+ instances. This is consistent with Theorem 3.4 (the verification is essentially a sanity check on the implementation of the greedy and ordering routines) and we report it for transparency.

5.2. The Phase 1–3 construction

We tested the Phase 1–3 construction on a curated set of 562 graphs spanning small cycles C_5–C_{19}, complete graphs K_5–K_7, the Petersen graph, the Mycielski graph M_3, the Grötzsch graph, Kneser graphs K(5,2)–K(7,2) and K(10,2), and 500+ random graphs with $n \in \{5, \ldots, 20\}$. In every case, the construction produced k pairwise disjoint, internally connected, pairwise-adjacent branch sets — a valid K_k-minor certificate.

Total runtime on a single GPU (CUDA/CuPy) was approximately 130 minutes; CPU fallback runs were used for small graphs. Each subroutine was implemented in 10 independent scripts to cross-check intermediate results (pairwise adjacency, Phase 3 termination, certificate validation).

Table 1. Phase 1–3 construction results.

Graph family	χ(G)	p(G)	Certificates	Method
C_n, $n \in \{5,7,\ldots,19\}$	3	2	✓	Exhaustive
K_5, K_6, K_7	5–7	4–6	✓	GPU 3M
Petersen	3	2	✓	Exhaustive
Mycielski M_3, Grötzsch	4	3	✓	Mixed
Kneser K(5,2)–K(7,2)	3–5	2–4	✓	GPU 3M
Kneser K(10,2)	8	7	✓	GPU 3M
500+ random, $n \in \{5,\ldots,20\}$	2–10	1–9	✓ (562/562)	Mixed

These results are not a proof of correctness — the test set is finite and biased toward small graphs and standard families. They constrain the kind of obstruction a counterexample to the construction would have to exhibit.

5.3. The articulation trap

A natural obstruction to Phase 3 progress is the simultaneous configuration in which every frontier vertex of every donor branch set is an articulation point of that set. We call this an articulation trap. We tested 344 graphs (with $\chi(G) \in \{3,\ldots,9\}$) for the presence of such a trap during Phase 1–3 execution. None occurred. Whether the articulation trap can occur in any connected graph with $\chi(G) = k$ is open (and is closely related to Open Problem 1).

5.4. Voronoi-style alternative and small failure cases

A natural alternative to the color-class-seeded construction is a Voronoi-style construction in which each branch set B_i is grown by simultaneous BFS from the expansion center c_i alone, with no reliance on color classes. This guarantees connectivity by construction, but does not guarantee pairwise adjacency. On 92 small graphs we identified 9 graphs in which the Voronoi construction fails (a pair of branch sets has no edge). In each of these 9 cases, exhaustive search confirmed that a valid K_k-minor partition exists nevertheless; the failure is of the algorithm, not of the graph. The failing cases share a common shape: a small leaf-like branch set adjacent only to a large absorbing set, with at least one vertex in the large set whose reassignment would resolve the obstruction without disconnecting any set.

6. The central open problem

Open Problem 6.1. Prove or disprove: for every connected graph G with $\chi(G) = k$, the Phase 1–3 construction (or some equivalent procedure) produces k pairwise disjoint, internally connected, and pairwise-adjacent subsets of V(G).

Equivalently — and without reference to our construction — does every graph G with $\chi(G) = k$ admit a partition of V(G) into k connected, pairwise-adjacent sets? An affirmative answer is exactly Hadwiger's conjecture for that value of k.

We are therefore not proposing Open Problem 6.1 as easier than Hadwiger's conjecture; we are reformulating the unsolved part of our work as the standard formulation of the conjecture. The contribution of the present paper is upstream of this question: the Patio Adjacency Lemma supplies the pairwise-adjacency half of the certificate at no cost, and reduces the conjecture (within our framework) to the connectivity half. Whether this reformulation is technically useful is for future work to decide.

6.1. Subsidiary open problems

- Characterize the graphs for which an optimal palette-expansion ordering is unique up to a permutation of colors.
- Determine whether $p(G) \geq \omega(G) - 1$ in general; this is consistent with all our experiments but unproved.
- Find an ordering-free graph-theoretic characterization of expansion centers, or prove that none exists.
- Find tight bounds on the number of Phase 3 iterations required as a function of k.

7. On Hadwiger's conjecture

Hadwiger's conjecture (1943) asserts that every graph G with $\chi(G) = k$ contains K_k as a minor. The conjecture is settled for $k \in \{1, 2, 3\}$ (Hadwiger 1943, elementary), $k = 4$ (Hadwiger 1943; Wagner 1937), $k = 5$ (Wagner 1964 reduces it to the four-color theorem; Robertson–Seymour–Thomas 1993), and $k = 6$ (Robertson–Seymour–Thomas 1993, ≈ 100 pages of structural argument). For $k \geq 7$ it is open.

We make no claim toward Hadwiger's conjecture for $k \geq 7$. The Patio Adjacency Lemma supplies the adjacency half of a K_k-minor certificate; the connectivity half is exactly Open Problem 6.1 and remains open. The computational evidence in Section 5 is consistent with the conjecture (no counterexample appears among the 562 graphs tested) but has no force as a proof at the relevant scale.

8. Conclusion

We have given a clean proof of the Patio Adjacency Lemma (Theorem 3.4), a structural property of expansion centers under any optimal greedy palette-expansion coloring. The lemma supplies a uniform pairwise-adjacency property of color classes, and we have explored its use as the basis of a branch-set construction. Whether this construction can be turned into a proof of Hadwiger's conjecture for some $k \geq 7$ — by closing Open Problem 6.1 — is, we expect, a question about structural graph theory rather than about greedy coloring, and we leave it to specialists in that area.

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Acknowledgments

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