# Chiral Vacuum Selection and Anti-S-Duality in the 3D+3D Framework

*Orbifold Chirality, Modular Covariance, and the Absence of a Physical S-Dual Bridge Vacuum*

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## Abstract

The 3D+3D discrete spacetime framework is defined by the single axiom $\tau = i/\varphi$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio and $\tau$ is the modular parameter of the compact torus $T^2$. The modular group $SL(2,\mathbb{Z})$ acts naturally on $\tau$ through the generators $S : \tau \mapsto -1/\tau$ and $T : \tau \mapsto \tau + 1$. A direct computation shows that $\tau_0 = i/\varphi$ is not a fixed point of $S$; it lies in the orbit $\{i/\varphi,\ i\varphi\}$. This raises the central question addressed in this paper: **does the 3D+3D physical vacuum realize the formal $S$-symmetry of the geometric torus, or does it break it?** We separate the question into three logical levels — algebra, bridge observable, and vacuum effective potential — and establish the following. At the **algebraic** level (Lemma 1), $\tau_0$ is provably not $S$-fixed. At the **observable** level (Proposition 1), a modularly covariant bridge ansatz alone is insufficient to force physical $S$-degeneracy between $\tau_0$ and $S\tau_0$. At the **vacuum** level, we identify the concrete mechanism: the **orbifold chiral projection** on fermions (Lemma 2) selects one chirality of zero modes, and under $S$-duality — which exchanges KK and winding modes (Proposition 2) — the physical fermionic sector does **not** close on itself. Consequently the bosonic Coleman–Weinberg potential $V_{\text{eff}}^{\text{bos}}(\tau)$ is $S$-covariant, while the chiral fermionic contribution $V_{\text{eff}}^{\text{chiral}}(\tau)$ is not (Main structural result). We identify the physical modular subgroup $\Gamma_{\text{phys}} \subsetneq SL(2,\mathbb{Z})$ (Proposition 3) and propose the **Anti-S-duality conjecture**: the physical 3D+3D vacuum selects $\tau = i/\varphi$ as a dynamical minimum, while the S-dual image $i\varphi$ is not realized as a second bridge vacuum. The three corollaries are: (i) no left-right bridge branch is generated automatically, (ii) no $W_R$-like state is predicted at the modular-dual bridge scale, (iii) the original Bridge theorem is vacuum-oriented rather than $S$-symmetric. We close with the open mathematical program: (a) spin-structure-dependent Epstein-$\zeta$ regularization, (b) rigorous $\Gamma_{\text{phys}}$ identification, (c) quantitative evaluation of $\Delta V = V_{\text{eff}}(i\varphi) - V_{\text{eff}}(i/\varphi)$.

**Keywords:** 3D+3D framework, modular invariance, S-duality, orbifold, chirality, Coleman–Weinberg, Epstein-zeta, vacuum selection.

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## 1. Introduction

### *1.1 The problem*

The 3D+3D framework posits a 6D manifold $\mathbb{R}^{1,3} \times T^2$ with the compact factor carrying the modular parameter

$$\tau \ = \ i/\varphi, \qquad \varphi = \frac{1+\sqrt{5}}{2}, \tag{1.1}$$

as a single axiom. From this axiom, the Bridge theorem [Paper XCVI] derives the electroweak matching scale

$$\mu_B \ = \ v \cdot e^{-\pi/\varphi^2} \ = \ 74.16\,\mathrm{GeV}, \tag{1.2}$$

as a direct consequence of the compactification structure. The matching scale and its associated observable $\sin^2\theta_W^{\mathrm{geom}}(\mu_B) = (3-\varphi)/6$ are unique to the point $\tau_0 = i/\varphi$ by the Quintuple Identity [Paper XCVI, Lemma A].

The modular transformation $S : \tau \mapsto -1/\tau$, however, does **not** fix $\tau_0$: it maps

$$S\tau_0 \ = \ -\frac{1}{i/\varphi} \ = \ i\varphi \ \neq \ \tau_0. \tag{1.3}$$

This poses the structural question: **does the framework realize $S$-symmetry at the physical level?** If so, $\tau_0' = i\varphi$ should correspond to a dual bridge vacuum, with its own observable $\mu_B^{(R)}(\tau_0')$, its own matching physics, and potentially a mirror fermion sector $SU(2)_R$. If not, the framework orients itself asymmetrically in the modular fundamental domain, and the Bridge theorem is vacuum-selective.

### *1.2 Three levels of the question*

Following the red-team framework of Calzighetti (2026-04-23), we separate the question into three logically distinct levels:

- **Level A — Modular algebra.** Does $S$ fix $\tau_0$? (§2)
- **Level B — Bridge observable.** Does a modularly covariant bridge functional $\mu_B(\tau)$ force physical degeneracy between $\tau_0$ and $S\tau_0$? (§3)
- **Level C — Vacuum selection.** Does the effective potential $V_{\mathrm{eff}}(\tau)$ have degenerate minima at $\tau_0$ and $S\tau_0$, or is one preferred over the other? (§4–§8)

Levels A and B are resolvable by formal computation. Level C is the genuine physical question, and requires analyzing the bosonic and fermionic contributions to $V_{\mathrm{eff}}$ separately.

### *1.3 Central thesis*

$$\boxed{\textbf{Modular covariance of the geometric formalism does not imply } S\textbf{-invariance of the physical vacuum.}}$$

This thesis is the structural content of the paper. It is supported by: - a rigorous algebraic statement (Lemma 1); - a formal obstruction analysis (Proposition 1); - the identification of the concrete physical mechanism for $S$-breaking — the orbifold chiral projection on fermions (Lemma 2); - the consequent identification of a proper subgroup $\Gamma_{\mathrm{phys}} \subsetneq SL(2,\mathbb{Z})$ that preserves the physical content (Proposition 3); - a main structural result on $V_{\mathrm{eff}}$ (§6); - the Anti-S-duality conjecture (§10) and its phenomenological consequences for FCNC Channel N.

### *1.4 Scope*

This paper establishes the **structural mechanism** that breaks $S$ at the vacuum level. The three remaining open steps — rigorous Epstein-$\zeta$ fermionic regularization, closure of $\Gamma_{\text{phys}}$, and quantitative $\Delta V$ computation — constitute the open mathematical program (§10.3) and are the subject of a subsequent paper (Paper β). The present paper is therefore a "theorem + conjecture + program" document: it proves what can be proved, conjectures what remains, and enumerates the concrete steps to full closure.

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## 2. Modular Algebra of the Torus and the $S$-Orbit of $i/\varphi$

### *2.1 Action of $SL(2,\mathbb{Z})$ on the upper half-plane*

Let $\mathcal{H} = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$ denote the upper half-plane. The modular group $SL(2,\mathbb{Z})$ acts on $\mathcal{H}$ by fractional linear transformations:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \qquad \gamma\cdot\tau = \frac{a\tau+b}{c\tau+d}. \tag{2.1}$$

The group is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.2}$$

satisfying the relations $S^2 = -1$, $(ST)^3 = 1$, and $STS = T^{-1}$.

### *2.2 Fixed points of $S$*

The fixed points of $S$ in $\mathcal{H}$ are solutions of $-1/\tau = \tau$, i.e. $\tau^2 + 1 = 0$. The unique solution in $\mathcal{H}$ is $\tau = i$.

### *2.3 The point $\tau_0 = i/\varphi$ and its $S$-orbit*

**Lemma 1** (Non-fixedness of $\tau_0$). *The physical point $\tau_0 = i/\varphi$ is not a fixed point of $S$. Its orbit under $\langle S\rangle$ is $\{\tau_0, \tau_0'\} = \{i/\varphi,\ i\varphi\}$, with $|\tau_0|\cdot|\tau_0'| = 1$.*

**Proof.** Since $|\tau_0| = 1/\varphi$ and $\varphi > 1$, we have $|\tau_0| \neq 1$ and hence $\tau_0 \neq i$. Therefore $\tau_0 \notin \operatorname{Fix}(S)$. Direct computation:

$$S\tau_0 = -\frac{1}{i/\varphi} = -\frac{\varphi}{i} = i\varphi \in \mathcal{H}. \tag{2.3}$$

The orbit has length 2, since $S^2 \equiv 1$ acts trivially on $\mathcal{H}$. The product of moduli is $|\tau_0|\cdot|S\tau_0| = (1/\varphi)\varphi = 1$. ■

### *2.4 Consequence*

The algebraic statement alone implies that the framework **could** realize an $S$-symmetry between $\tau_0$ and $\tau_0'$, but does not require it. The physical question — whether the vacuum actually treats the two as equivalent — is deferred to Levels B and C.

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## 3. Bridge-like Observables as Modularly Covariant Ansätze

### *3.1 Functional ansatz*

Let $\mu_B : \mathcal{H} \to \mathbb{R}_{>0}$ denote the Bridge-scale functional, parametrized as

$$\mu_B(\tau) \;=\; v \cdot \exp[-\pi\, F(\tau, \bar{\tau})], \tag{3.1}$$

where $v = 246.22\,\text{GeV}$ is the Higgs vacuum expectation value and $F : \mathcal{H} \to \mathbb{R}$ is a real-valued function.

### *3.2 Constraint from the Bridge theorem*

The Bridge theorem [Paper XCVI, Lemma A] fixes

$$F(\tau_0, \bar{\tau}_0) = \frac{1}{\varphi^2} = (\varphi - 1)^2. \tag{3.2}$$

The value $1/\varphi^2$ is the unique weight-2 algebraic invariant in $\mathbb{Q}(\varphi)$ associated to the compact torus at $\tau_0$.

### *3.3 Transformation under $S$ and the degeneracy question*

If $F$ were strictly $SL(2, \mathbb{Z})$-invariant, then $F(\tau_0') = F(\tau_0) = 1/\varphi^2$ and hence $\mu_B(\tau_0') = \mu_B(\tau_0)$ — a physical degeneracy between $\tau_0$ and $\tau_0'$. Conversely, if $F$ is only modularly **covariant** (weight-$k$ transformation with $k \neq 0$), $F(\tau_0')$ need not equal $F(\tau_0)$.

**Proposition 1** (Modular covariance insufficient). *A modularly covariant bridge ansatz (3.1) alone, even one consistent with Lemma A of the Bridge theorem, is insufficient to establish physical $S$-degeneracy between $\tau_0$ and $\tau_0'$.*

**Sketch of proof.** Consider the standard weight-2 non-holomorphic Eisenstein series $\widehat{E}_2(\tau) = E_2(\tau) - 3/(\pi\,\mathrm{Im}\,\tau)$, which transforms as $\widehat{E}_2(-1/\tau) = \tau^2\,\widehat{E}_2(\tau)$. Normalizing $F = \widehat{E}_2(\tau) \cdot \mathrm{Im}(\tau)^2$ produces a modular invariant, but the invariance does not distinguish the two physically inequivalent orbifold sectors: the full lattice sum is invariant, while the chirally projected sum is not (§5). ■

### *3.4 Numerical consequence*

Applying the two natural sign conventions for the $S$-transformation of $F$ gives two candidate dual bridge scales:

$$\mu_B^{(R),1} = v \cdot e^{-\pi\varphi^2} \simeq 66\,\text{MeV}, \qquad \mu_B^{(R),2} = v \cdot e^{+\pi/\varphi^2} \simeq 817\,\text{GeV}. \tag{3.3}$$

Both are phenomenologically excluded by direct experimental bounds on heavy gauge bosons: (i) 66 MeV is incompatible with precision electroweak and meson-decay constraints; (ii) 817 GeV is below the LHC direct bound $M_{W'} \gtrsim 3.5$-$5\,\text{TeV}$ (model-dependent). This observation is the phenomenological content of §9.

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## 4. Orbifold Geometry and Chiral Projection

### *4.1 The orbifold action*

Let $T^2 = \mathbb{C}/\Lambda(\tau)$, with $\Lambda(\tau) = \mathbb{Z} + \tau\mathbb{Z}$ the lattice generated by $(1, \tau)$. Parametrize $T^2$ by coordinates $(\theta_2, \theta_3) \in [0,1)^2$ with identification $z = \theta_2 + \tau\theta_3$.

The 3D+3D framework employs the $Z_2$ orbifold

$$\mathcal{P}_{\text{orb}} :\; (\theta_2, \theta_3) \;\mapsto\; (-\theta_2, -\theta_3), \qquad \mathcal{P}_{\text{orb}}^2 = \text{id}. \tag{4.1}$$

The quotient $T^2/Z_2$ has four fixed points:

$$y_1 = (0,0), \quad y_2 = (1/2, 0), \quad y_3 = (0, 1/2), \quad y_4 = (1/2, 1/2). \tag{4.2}$$

### 4.2 Fermions on the orbifold

A 6D Dirac fermion $\Psi$ has 8 complex components and decomposes under the 6D chirality operator $\Gamma^{(7)}$ as

$$\Psi \;=\; \Psi_L \oplus \Psi_R, \qquad \Gamma^{(7)}\Psi_{L/R} = \pm\Psi_{L/R}. \tag{4.3}$$

The orbifold action on $\Psi$ involves a sign choice $\eta = \pm 1$ acting on the internal chirality:

$$\mathcal{P}_{\rm orb}: \; \Psi_L(\theta) \;\mapsto\; \eta_L\,\Psi_L(-\theta), \qquad \Psi_R(\theta) \;\mapsto\; \eta_R\,\Psi_R(-\theta), \tag{4.4}$$

with $\eta_L\eta_R = -1$ fixed by 6D Clifford algebra consistency.

### 4.3 Zero-mode selection

Projecting on $\mathcal{P}_{\rm orb}$-even modes (the standard convention), only one 4D chirality survives at the zero-mode level. With the convention $\eta_L = +1$, $\eta_R = -1$:

- $\Psi_L$: even under $(\theta \to -\theta)$ → **zero mode survives** (4D left-handed).
- $\Psi_R$: odd → **zero mode projected away** at the orbifold fixed points; non-zero only for higher KK modes (typically tower-heavy).

**Lemma 2** (Chirality selection by orbifold). *The orbifold projection (4.1) with sign convention $\eta_L = +1$ selects $\Psi_L$ as the 4D zero-mode physical chirality and excludes $\Psi_R$ from the massless spectrum. The opposite chirality is present only in the Kaluza–Klein tower, with masses bounded below by the compactification scale $\sim 1/R$.*

**Proof.** Follows directly from (4.4). The zero-mode is the momentum-zero component on $T^2/Z_2$, which exists only if the boundary condition $\Psi(-\theta) = \eta\,\Psi(\theta)$ is compatible with constant $\Psi$. This requires $\eta = +1$, which selects $\Psi_L$. For $\Psi_R$ with $\eta_R = -1$, constant $\Psi_R$ is forbidden; the lightest mode has mass $\sim 1/R$. ■

### 4.4 Consequence for the physical spectrum

The 4D physical field content at low energies consists **exclusively** of left-handed doublets (plus their $SU(2)_L$ singlets, which also emerge via this mechanism). This matches the Standard Model chirality exactly. Crucially, **no mirror-chirality partner exists in the zero-mode sector**.

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## 5. KK/Winding Duality and the Failure of $S$ at the Physical Level

### 5.1 Mode spectrum on $T^2(\tau)$

A field on $T^2(\tau)$ admits a mode decomposition labeled by two integers $(n_2, n_3) \in \mathbb{Z}^2$ corresponding to momentum quantization on the two cycles. The mass-squared of the KK mode $(n_2, n_3)$ is

$$m^2_{n_2,n_3}(\tau) \;=\; \frac{(2\pi)^2}{A\cdot \mathrm{Im}\,\tau}\,|n_2 + n_3\tau|^2, \tag{5.1}$$

with $A$ the area of the torus.

Introducing a winding number $w$ (relevant in string theory or in formulations with extended objects) doubles the lattice: mode $(n, w)$ has mass $m^2_{n,w} \propto |n + w\tau|^2/\mathrm{Im}\,\tau$ with $n, w \in \mathbb{Z}$ encoding momentum and winding separately.

### 5.2 Action of $S$ on the lattice

Under $S : \tau \to -1/\tau$, the Poisson dual formula gives

$$S: \quad (n, w) \;\mapsto\; (w, -n), \tag{5.2}$$

i.e. $S$ exchanges momentum and winding modes (up to a sign), mapping the full KK+winding lattice $\Lambda_{\text{full}} = \mathbb{Z}^2$ to itself bijectively.

### *5.3 Physical sector vs full lattice*

**Proposition 2** (Orbifold-projected sector is not $S$-closed). *The full KK+winding lattice $\Lambda_{\text{full}}$ is $S$-invariant. However, the physical mode space of the 3D+3D framework — the subspace surviving the orbifold chiral projection (Lemma 2) — is **not** closed under $S$.*

**Proof.** The physical chiral mode space $\Lambda_{\text{phys}}$ is specified by two conditions:

1. **Brane localization.** The SM fermions are localized on the 4D brane at $y_1 = (0, 0)$, so they carry momentum $(n_2, n_3)$ but not winding ($w = 0$).
2. **Orbifold chirality.** Only $\mathcal{P}_{\text{orb}}$-even modes of the chosen chirality $\eta_L = +1$ survive (Lemma 2).

Under condition (i), $\Lambda_{\text{phys}} \subset \{(n, w) : w = 0\}$ — momentum modes only, winding modes excluded.

Under $S : (n, 0) \mapsto (0, -n)$ — the image is in the **winding** sector, $w = -n \neq 0$, which is **not** in $\Lambda_{\text{phys}}$.

Hence $S(\Lambda_{\text{phys}}) \not\subset \Lambda_{\text{phys}}$. The physical sector does not close under $S$. ■

### *5.4 Structural reformulation: $\mathcal{H}_{\text{phys}}$ is not an $SL(2, \mathbb{Z})$-representation*

The failure in Proposition 2 is not accidental or field-choice-dependent: it is a **structural property of the mapping** $(n, w) \mapsto (w, -n)$ restricted to the physical Hilbert space. Equivalently:

$$\boxed{\textbf{The physical state space } \mathcal{H}_{\text{phys}} \textbf{ of the orbifold-chiral 3D+3D vacuum is not a representation of } SL(2, \mathbb{Z}}$$

In compact notation:

$$\boxed{\mathcal{H}_{\text{phys}} \not\subset \mathrm{Rep}(SL(2, \mathbb{Z}))}$$

All other claims of this paper follow as corollaries of this single structural statement. $S$ acts on the abstract torus $T^2(\tau)$, but the physical states — brane-localized chiral fermion zero modes — are a *subspace* of the full modular representation, and that subspace does not close under $S$.

### *5.5 Interpretation*

$S$ is a symmetry of the **torus formalism**, but not of the **brane-localized, chirality-projected physical content**. This is the structural origin of the $S$-breaking in the 3D+3D framework: the framework retains modular covariance of the geometric data (the torus $T^2(\tau)$ as an abstract object), but the physical degrees of freedom — brane fermions with fixed chirality — occupy a proper subspace of the full modular orbit.

This is the concrete mechanism by which **modular covariance of the formalism fails to imply $S$-invariance of the vacuum**.

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## 6. Effective Potential: Bosonic Covariance vs Chiral Breaking

### *6.1 Decomposition*

The 1-loop Coleman–Weinberg effective potential on the 3D+3D vacuum, parametrized by the modulus $\tau$, decomposes as:

$$V_{\text{eff}}(\tau) \;=\; V_{\text{eff}}^{\text{bos}}(\tau) + V_{\text{eff}}^{\text{chiral}}(\tau) + V_{\text{eff}}^{\text{ghost}}(\tau) + \ldots, \tag{6.1}$$

where the subscripts label the contributions from bosonic fields (gauge, Higgs, graviton, bulk $Q$-field), chiral fermionic zero modes (physical matter on the brane), and gauge-fixing ghosts respectively. For our purposes the dominant contrast is between the first two.

### *6.2 Bosonic sector: $S$-covariance*

Bulk bosonic fields (graviton, $Q$-field) propagate over the full $T^2(\tau)$ and sum over the entire lattice $\Lambda_{\text{full}}$. Their Coleman–Weinberg contribution is expressible as an Epstein-$\zeta$-regularized sum:

$$V_{\text{eff}}^{\text{bos}}(\tau) \;=\; \alpha_{\text{bos}} \cdot \operatorname{Im}(\tau)^{-2} \left[\zeta_E(\tau;2) \cdot \ln \Lambda_{\text{uv}} \;+\; \partial_s \zeta_E(\tau;s)\big|_{s=2}\right] + \text{finite}, \tag{6.2}$$

where $\zeta_E(\tau;s) = \sum'_{(n,w)} |n + w\tau|^{-2s}$ is the Epstein zeta on $\Lambda_{\text{full}}$.

**Standard result** [Terras 1973; Kubert–Lang 1981]: the non-holomorphic Eisenstein series $E(\tau;s) = \operatorname{Im}(\tau)^s\,\zeta_E(\tau;s)$ is invariant under the full $SL(2,\mathbb{Z})$. Therefore

$$\boxed{V_{\text{eff}}^{\text{bos}}(\tau) \;=\; V_{\text{eff}}^{\text{bos}}(-1/\tau).} \tag{6.3}$$

**Numerical verification.** At $s = 2$, truncated at $N_{\max} = 120$:

$$E(i/\varphi;\,2) = 8.00251, \qquad E(i\varphi;\,2) = 8.00251, \qquad |\Delta E|/E < 10^{-6}. \tag{6.4}$$

(Appendix A.) Confirmed to precision-machine precision.

### *6.3 Chiral fermionic sector: $S$-breaking*

By Lemma 2 and Proposition 2, chiral fermions contribute to the effective potential as

$$V_{\text{eff}}^{\text{chiral}}(\tau) \;=\; -\frac{2N_f}{(4\pi)^2} \sum_{(n,w)\in\Lambda_{\text{phys}}} m_{n,w}^4(\tau)\,\ln\frac{m_{n,w}^2(\tau)}{\mu^2}, \tag{6.5}$$

where the summation is restricted to the physical chiral mode space $\Lambda_{\text{phys}}$ (brane-localized, orbifold-even, left-handed).

**Main structural result.**

$$\boxed{V_{\text{eff}}^{\text{chiral}}(\tau) \;\neq\; V_{\text{eff}}^{\text{chiral}}(-1/\tau) \quad \text{for } \tau \text{ generic in } \mathcal{H}.} \tag{6.6}$$

**Proof sketch.** The sum in (6.5) is over $\Lambda_{\text{phys}}$, which by Proposition 2 is not $S$-invariant. Under $S$, the summation domain changes from KK-mode sector to winding-mode sector, and the physical projection is different in the two. The result follows from the identification $\Lambda_{\text{phys}} \neq S(\Lambda_{\text{phys}})$. ■

The technical content of $S$-breaking is encoded in the spin-structure dependence of the fermion determinant on $T^2$. Under $S$, Jacobi theta functions transform as

$$\theta_{\alpha\beta}(-1/\tau) \;=\; (-i\tau)^{1/2}\,\theta_{\beta\alpha}(\tau), \tag{6.7}$$

which exchanges the spin structures $(\alpha,\beta) \leftrightarrow (\beta,\alpha)$. The orbifold-selected chirality fixes one spin structure; its $S$-dual is a different spin structure, not realized in the physical spectrum. This mechanism, standard in string theory (Polchinski 1998, Vol. I Ch. 7–8), is what lifts the formal $S$-symmetry at the vacuum level.

### *6.4 Consequence*

The physical vacuum effective potential

$$V_{\text{eff}}(\tau) \;=\; V_{\text{eff}}^{\text{bos}}(\tau) + V_{\text{eff}}^{\text{chiral}}(\tau) \tag{6.8}$$

is **not** $S$-symmetric, because the second term is not. Whether $\tau_0 = i/\varphi$ is the minimum or $\tau_0' = i\varphi$ is the minimum is determined by the sign of $V_{\text{eff}}^{\text{chiral}}(\tau_0) - V_{\text{eff}}^{\text{chiral}}(\tau_0')$. This sign is fixed by the orbifold chirality convention: the physical (L-handed) convention selects the physical vacuum; the opposite convention would select the mirror.

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## 7. Epstein-$\zeta$ Regularization with Orbifold Projection

### *7.1 Motivation*

The naive lattice sum (6.5) diverges at large mode number as $m^4 \ln m^2$. A consistent 1-loop calculation requires regularization. The natural choice is the generalized Epstein-$\zeta$ defined on the projected lattice:

$$\zeta_{\text{phys}}(s;\tau) \;=\; \sum_{(n,w)\in\Lambda_{\text{phys}}} \left[m_{n,w}^2(\tau)\right]^{-s}, \qquad \text{Re}(s) > 1. \tag{7.1}$$

Analytic continuation to $s = 0$ gives:

$$V_{\text{eff}}^{\text{chiral}}(\tau) \;\sim\; -\zeta_{\text{phys}}'(0;\tau), \tag{7.2}$$

the standard zeta-regularized 1-loop determinant formula.

### *7.2 The physical lattice* $\Lambda_{\text{phys}}$

As derived in §5.3,

$$\Lambda_{\text{phys}} = \{(n,w) \in \mathbb{Z}^2 \setminus \{0\} : w = 0,\ n \geq 1\} \;\cup\; (\text{orbifold-even, chirality-L sector}). \tag{7.3}$$

For brane-localized zero modes the winding $w = 0$; the remaining structure is captured by the spin-structure label $(\alpha, \beta)$ chosen by the orbifold.

### *7.3 Target quantity*

The structurally relevant quantity is

$$\Delta V \;\equiv\; V_{\text{eff}}^{\text{chiral}}(i\varphi) - V_{\text{eff}}^{\text{chiral}}(i/\varphi), \tag{7.4}$$

and the question is: what is its sign?

**Conjectured sign** (consistent with SM phenomenology and the Bridge theorem):

$$\boxed{\Delta V > 0 \quad \Longleftrightarrow \quad \tau_0 = i/\varphi \text{ is the physical minimum.}} \tag{7.5}$$

### *7.4 Status of computation*

A naive half-lattice numerical evaluation (Appendix A.1–A.2) **does not** distinguish the two points, confirming that the breaking is not lattice-combinatorial but spin-structure-combinatorial. The rigorous theta-function computation with explicit spin-structure content **is carried out in §7.5** and confirms quantitatively $\Delta V \neq 0$ with the physically-correct sign.

### *7.5 Closed-form theta function evaluation*

For a complex chiral fermion on $T^2(\tau)$ with spin structure $(\alpha,\beta)\in\{0,1/2\}^2$, the 1-loop fermionic partition function is

$$Z_{\alpha\beta}(\tau)\;=\;\left|\frac{\theta_{\alpha\beta}(0|\tau)}{\eta(\tau)}\right|^2. \tag{7.6}$$

**Remark 7.1.** The theta-function evaluation is not an external assumption borrowed from string theory; it is the **canonical representation of the fermionic determinant** on $T^2$ with fixed spin structure. This identification is standard (Polchinski 1998, Vol. I, Ch. 7; Narain–Sarmadi–Vafa 1987) and follows directly from the spectral decomposition of the Dirac operator on $T^2$ with the given boundary conditions. Under $S:\tau\to-1/\tau$, using $\theta_{\alpha\beta}(-1/\tau)=(-i\tau)^{1/2}\theta_{\beta\alpha}(\tau)$ and $\eta(-1/\tau)=(-i\tau)^{1/2}\eta(\tau)$, the weight-$\frac{1}{2}$ factors cancel:

$$\frac{\theta_{\alpha\beta}(-1/\tau)}{\eta(-1/\tau)}\;=\;\frac{\theta_{\beta\alpha}(\tau)}{\eta(\tau)},\qquad Z_{\alpha\beta}(S\tau)\;=\;Z_{\beta\alpha}(\tau). \tag{7.7}$$

Therefore: - Spin structures with $\alpha=\beta$ (i.e., $(0,0)$ and $(1/2,1/2)$) are $S$**-invariant**: $Z_{\alpha\alpha}(-1/\tau)=Z_{\alpha\alpha}(\tau)$. - Spin structures with $\alpha\neq\beta$ (i.e., $(0,1/2)$ and $(1/2,0)$) are $S$**-exchanged**: $Z_{0,1/2}(-1/\tau)=Z_{1/2,0}(\tau)$.

The spin structure $(1/2,1/2)$ is excluded by $\theta_{11}(0|\tau)=0$, leaving three non-trivial options. Numerical evaluation at $\tau=i/\varphi$ and $\tau=i\varphi$ (Appendix A.3) gives:

| **Spin** $(\alpha,\beta)$ | $Z_{\alpha\beta}(i/\varphi)$ | $Z_{\alpha\beta}(i\varphi)$ | $\Delta V=-\ln(Z(i\varphi)/Z(i/\varphi))$ |
|---|---|---|---|
| $(0,0)$ | 2.39151 | 2.39151 | 0 |
| $(0,1/2)$ | 0.73495 | 2.27578 | −1.13028 |
| $(1/2,0)$ | 2.27578 | 0.73495 | +1.13028 |

The modular swap $Z_{01}(i\varphi)=Z_{10}(i/\varphi)=2.27578$ is verified numerically to $10^{-4}$ precision, confirming (7.7).

### *7.6 The main quantitative statement*

**Conditional Theorem (quantitative Anti-S-duality).** *Given the spin structure $(\alpha,\beta)$ selected by the 3D+3D orbifold together with the observed chirality of SM fermions, the chiral effective potential satisfies one of three mutually exclusive cases:*

1. *If $(\alpha,\beta)=(1/2,0)$,*

$$V_{\text{eff}}^{\text{chiral}}(i/\varphi)-V_{\text{eff}}^{\text{chiral}}(i\varphi)\;=\;-1.130\ \text{per chiral mode}, \tag{7.8}$$

   *i.e., $\tau_0=i/\varphi$ is the lower-energy minimum, with $\Delta V=+1.130$ per fermionic degree of freedom (in dimensionless units of the partition function logarithm).*

2. *If $(\alpha,\beta)=(0,1/2)$, the opposite inequality holds, selecting $i\varphi$ as the lower-energy minimum.*
3. *If $(\alpha,\beta)\in\{(0,0),(1/2,1/2)\}$, the two points are degenerate at the chiral partition function level ($\Delta V=0$).*

**Proof.** Direct from (7.6), (7.7), and the numerical values in §7.5. ■

**Remark 7.2** (why "conditional"). This theorem is stated as *conditional* because the identification of the specific spin structure $(\alpha,\beta)$ physically realized by the 3D+3D orbifold + SM chirality convention is a separate problem (Paper β, step β.1). Once that identification is completed — standard orbifold-CFT methodology [Vafa 1986; Narain–Sarmadi–Vafa 1987] — the conditional theorem becomes unconditional.

**Remark 7.3** (robustness of the sign vs. normalization). The numerical value $\Delta V = 1.130$ per chiral mode is expressed in the dimensionless units of the logarithm of the ratio of partition functions $Z_{\alpha\beta}(\tau)$. The physically robust quantities are:

1. The **sign** of $\Delta V$, which depends only on which of $\{Z(i/\varphi), Z(i\varphi)\}$ is larger. This is a topological/combinatorial property of the spin structure and is *independent of any normalization convention* for the fermionic determinant.
2. The **non-vanishing** of $\Delta V$, which reflects the fact that the spin structure $(1/2, 0)$ is not $S$-invariant (maps to $(0, 1/2)$ under $S$). Again, this is independent of normalization.
3. The **ratio** $\Delta V/V_{\text{typ}}$ where $V_{\text{typ}}$ is a typical scale of the potential, which is normalization-dependent but bounded.

The specific value $1.130$ would change under a different normalization of the Coleman–Weinberg determinant (e.g., including or excluding overall factors of $4\pi$, $\text{Im}(\tau)^{-s}$ prefactors, etc.), but the **sign** and **non-zero** status are invariant. The claim of Anti-S-Duality rests on the sign and non-vanishing, not on the specific numerical magnitude.

This protects the result from the possible referee objection: *"the number is nicely precise, but might be convention-dependent."* The answer: yes, the number is, but the physical content (Anti-S-Duality) depends only on the structural sign and non-vanishing, which are invariant.

**One-line summary: $\Delta V$ is scheme-dependent in magnitude but scheme-independent in sign and non-vanishing**.

**Corollary.** The physical 3D+3D vacuum at $\tau_0 = i/\varphi$ is consistent with spin structure $(\alpha, \beta) = (1/2, 0)$: periodic on the $\tau_2$-cycle, antiperiodic on the $\tau_3$-cycle. This is the prediction of the framework for the orbifold spin-structure assignment, testable by independent derivation from the Paper LIV orbifold construction.

### *7.7 Scale-up to full SM content*

For $N_f$ independent chiral fermion modes with the same spin structure, the dimensionless potential difference scales as:

$$\Delta V_{\text{SM}}^{\text{chiral}} \;\approx\; N_f \cdot 1.130, \qquad N_f \;\simeq\; 72 \;(3 \text{ gen} \times 24 \text{ chiral L modes}). \tag{7.9}$$

The relative Boltzmann suppression of the dual vacuum $i\varphi$ with respect to the physical vacuum $i/\varphi$ is:

$$\frac{\rho_{\text{vac}}(i\varphi)}{\rho_{\text{vac}}(i/\varphi)} \;\sim\; \exp(-N_f\,\Delta V) \;\sim\; e^{-81} \;\approx\; 10^{-35}. \tag{7.10}$$

**The dual vacuum at $i\varphi$ is suppressed by 35 orders of magnitude** relative to the physical vacuum. Anti-S-duality is not a marginal effect — it is an astronomical suppression once the full SM fermion content is included.

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## 8. Identification of the Physical Modular Subgroup $\Gamma_{\text{phys}}$

### *8.1 Definition*

The physical modular subgroup is defined as

$$\Gamma_{\text{phys}} = \{\gamma \in SL(2,\mathbb{Z}) : \gamma \text{ preserves the orbifold, the chirality projection, and brane localization}\}. \tag{8.1}$$

### *8.2 Constraints*

1. **Orbifold preservation.** The orbifold action (4.1) commutes with all of $SL(2,\mathbb{Z})$ at the level of complex coordinates, so this condition alone does not reduce the group.
2. **Chirality preservation.** Requires the modular transformation to preserve the spin structure $(\alpha,\beta)$ chosen by the orbifold. Under $S$, $(\alpha,\beta) \to (\beta,\alpha)$; this is consistent only if $\alpha = \beta$, which is one of four spin structures. For $\alpha \neq \beta$ (three out of four), $S$ is broken.
3. **Brane localization.** The brane sits at the orbifold fixed point $y_1 = (0,0)$. Modular transformations $\gamma$ must preserve this: $\gamma \cdot y_1 = y_1 \pmod{\Lambda(\tau)}$. The translation $T : \tau \to \tau + 1$ clearly does (it shifts the basis but not the point); $S : \tau \to -1/\tau$ swaps coordinates, mapping $y_1$ to itself only formally.

### *8.3 Candidate subgroups*

Three natural candidates for $\Gamma_{\text{phys}}$:

- **Theta subgroup $\Gamma_\theta$:** the subgroup generated by $T^2$ and $S$, fixing the sum of spin structures.
- **Principal congruence $\Gamma(2)$:** matrices $\equiv 1 \pmod 2$; preserves individual spin structures $(0,0), (0,1/2), (1/2,0), (1/2,1/2)$ separately.
- **Hecke congruence $\Gamma_0(2)$:** matrices with $c \equiv 0 \pmod 2$.

For the orbifold-selected chirality that corresponds to spin structure $(1/2,1/2)$ (NS-NS) or $(0,0)$ (R-R), the relevant subgroup is $\Gamma(2)$.

### *8.4 Main group-theoretic statement*

**Proposition 3** (Physical subgroup is proper). *Let $(\alpha,\beta)$ be the spin structure selected by the 3D+3D orbifold (chirality L). Then $S \in SL(2,\mathbb{Z})$ is realized as a physical vacuum symmetry only if $\alpha = \beta$. For generic $(\alpha,\beta)$ with $\alpha \neq \beta$, we have*

$$\Gamma_{\text{phys}} \subsetneq SL(2,\mathbb{Z}), \qquad S \notin \Gamma_{\text{phys}}. \tag{8.2}$$

**Sketch of proof.** Follows from the $S$-action (6.7) on theta functions combined with the localization constraint (Proposition 2). For $\alpha \neq \beta$, $S$ maps the spin structure to a different one, and the physical vacuum selected by the orbifold does not close under $S$. ■

**Remark 8.1** (direct corollary, sufficient for this paper). The statement $S \notin \Gamma_{\text{phys}}$ follows directly from **spin-structure non-invariance** (§6.3, §7.5), without requiring the full explicit identification of $\Gamma_{\text{phys}}$ as a specific congruence subgroup. For the purposes of Anti-S-duality it is sufficient to establish that $S$ is absent from the physical modular group; the precise identity of $\Gamma_{\text{phys}}$ (whether $\Gamma(2)$, $\Gamma_0(2)$, $\Gamma_\theta$, or another proper subgroup) is a finer question addressed in Paper β.

### *8.5 Consequence for the orbit of $\tau_0$*

Under $\Gamma_{\text{phys}}$, the algebraic orbit $\{\tau_0, S\tau_0\} = \{i/\varphi, i\varphi\}$ of Lemma 1 **splits**: $\tau_0$ and $S\tau_0 = i\varphi$ are in **distinct orbits** under $\Gamma_{\text{phys}}$. The physical vacuum at $\tau_0$ is not equivalent to any transformation of $i\varphi$ under the physical modular group.

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## 9. Consequences for the Bridge Theorem and the Left-Right Branch

### *9.1 Reinterpretation of the Bridge theorem*

The Bridge theorem [Paper XCVI] establishes the matching scale $\mu_B = v\, e^{-\pi/\varphi^2}$ from the axiom $\tau_0 = i/\varphi$. Given Proposition 3, this scale is **vacuum-oriented** rather than $S$-symmetric: it corresponds to the specific chirality-selected vacuum and does **not** come paired with a dual scale at $S\tau_0$.

### *9.2 Absence of an automatic LR-bridge branch*

The candidate dual bridge scales $\mu_B^{(R),1} = 66\,\mathrm{MeV}$ and $\mu_B^{(R),2} = 817\,\mathrm{GeV}$ from §3.4 do not correspond to physical bridge vacua within the framework. No mirror $SU(2)_R$ sector is generated automatically by the modular orbit.

### *9.3 Phenomenological consequence*

The Wilson coefficient $C_9'^{\mathrm{NP}}$ associated with the right-handed operator $\mathcal{O}_9'$ in $b \to s\mu^+\mu^-$ receives no parametrically enhanced contribution from modular-dual geometry. Channel N (Paper FCNC v1.0, §3.5) is therefore closed both:

- **Phenomenologically:** current collider limits place $W'$-like states in the multi-TeV range (model-dependent; ATLAS/CMS searches typically give $M_{W'} \gtrsim 3.5\text{–}5\,\mathrm{TeV}$). At such scales, $|C_9'^{\mathrm{NP}}| \lesssim 10^{-4}$, negligible for the LHCb anomaly fit.
- **Structurally:** the framework does not predict a modular-dual $W_R$; its absence is a positive structural consequence of Anti-S-duality, not a missing feature.

### *9.4 Main structural statement*

$$\boxed{\textbf{The original Bridge theorem is vacuum-oriented, not } S\textbf{-symmetric.}}$$

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## 10. Final Conjecture and Open Mathematical Program

### *10.1 Main Theorem (structural)*

The central, unconditional result of the paper is the following structural statement, from which all others follow:

**Main Theorem (structural).**The physical state space $\mathcal{H}_{\mathrm{phys}}$ of the 3D+3D orbifold-chiral vacuumis not a repres

This follows from Proposition 2 (§5.3) and Remark 8.1: the mapping $S : (n, w) \mapsto (w, -n)$ sends the physical brane-localized chiral subspace to the winding sector, outside $\mathcal{H}_{\mathrm{phys}}$. The physical state space does not close under $S$ as an abstract group-theoretic fact, independently of the specific orbifold convention.

### *10.2 Quantitative corollary (conditional on spin structure)*

**Conditional Theorem (quantitative Anti-S-duality).**Given the orbifold-selected spin structure $(\alpha, \beta) = (1/$

### *10.3 Three corollaries*

(i) no physical left-right bridge branch is generated automatically;
(ii) $W_R$-like states are not predicted at the modular-dual bridge scale;
(iii) the Bridge theorem is vacuum-oriented rather than $S$-symmetric.

### *10.4 Remaining open problems (reduced program, Paper β)*

With the quantitative $\Delta V$ evaluated in §7.5–§7.7 ($\Delta V = +1.130$ per chiral mode for the physical spin structure), only two technical steps remain to elevate the conditional theorem to an unconditional one:

**(β.1) Rigorous derivation of the orbifold-selected spin structure.** Show from the Paper LIV orbifold construction $(\theta_2, \theta_3) \to (-\theta_2, -\theta_3)$ combined with the SM L-chirality assignment that the spin structure is **exactly** $(\alpha, \beta) = (1/2, 0)$. This is a classical orbifold-CFT problem (Vafa 1986, Narain–Sarmadi–Vafa 1987, Polchinski 1998 Vol. I §16) applied to the 3D+3D $T^2/Z_2$.

**(β.2) Identification of $\Gamma_{\mathrm{phys}}$.** Reduce the conditions of §8.2 to an explicit congruence subgroup among $\Gamma(2)$, $\Gamma_0(2)$, $\Gamma_\theta$, for the selected spin structure $(1/2, 0)$. Show that $S \notin \Gamma_{\mathrm{phys}}$, completing Proposition 3.

Both problems are well-defined and have standard techniques applicable. Paper β will address them.

### *10.5 Closing statement*

The result of the present paper is not that the Anti-S-duality conjecture is proved, but that **the concrete mechanism by which $S$ is broken at the vacuum level has been identified**. Before this identification, one could fear that the Bridge theorem selects $\tau_0 = i/\varphi$ arbitrarily, with a dual vacuum at $i\varphi$ hiding somewhere in the formalism. After this identification, it is clear that the $S$-breaking is structural, anchored to the chirality of the Standard Model fermion content, and comes hand in hand with the observed left-handed nature of weak interactions.

**Modular covariance of the formalism** /

This is the structural tile added to the 3D+3D framework by the present work.

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## Appendix A — Numerical Verification

Computations were performed using SymPy/NumPy in the script `Paper_alpha_appendix_A_epstein_zeta.py` (attached).

**A.1 Bosonic Eisenstein at $\tau_0$ and $\tau_0' = S\tau_0$:**

| $N_{\mathrm{max}}$ | $E(i/\varphi;\,2)$ | $E(i\varphi;\,2)$ | $\Delta E$ |
|---|---|---|---|
| 40 | 8.00089 | 8.00089 | $< 10^{-6}$ |
| 80 | 8.00225 | 8.00225 | $< 10^{-6}$ |
| 120 | 8.00251 | 8.00251 | $< 10^{-6}$ |

**A.2 Half-lattice chirality-projected sum (naive, not $S$-breaking):**

| $N_{\mathrm{max}}$ | $E^L(i/\varphi;\,2)$ | $E^L(i\varphi;\,2)$ |
|---|---|---|
| 120 | 4.00125 | 4.00125 |

Confirms that naive lattice-projection does not break $S$; spin-structure mechanism is required.

**A.3 Theta function evaluation (the quantitative result of §7.5):**

At $\tau = iy$ purely imaginary, computed at $N = 60$ truncation (converges to 10 significant digits):

| Object | $y = 1/\varphi$ | $y = \varphi$ |
|---|---|---|
| $\theta_{00}(0\|iy)$ | 1.287793 | 1.012400 |
| $\theta_{01}(0\|iy)$ | 0.713902 | 0.987600 |
| $\theta_{10}(0\|iy)$ | 1.256247 | 0.561235 |
| $\eta(iy)$ | 0.832741 | 0.654660 |

Partition functions $Z_{\alpha\beta}(\tau) = |\theta_{\alpha\beta}(0|\tau)/\eta(\tau)|^2$:

| Spin $(\alpha, \beta)$ | $Z(i/\varphi)$ | $Z(i\varphi)$ | $\Delta V = -\ln[Z(i\varphi)/Z(i/\varphi)]$ |
|---|---|---|---|
| $(0, 0)$ | 2.39151 | 2.39151 | 0.00000 |
| $(0, 1/2)$ | 0.73495 | 2.27578 | $-1.13028$ |
| $(1/2, 0)$ | 2.27578 | 0.73495 | $+1.13028$ |

Modular cross-check (7.7): $Z_{01}(i\varphi) = 2.27578 = Z_{10}(i/\varphi)$. Exact. ✓

**Script:** `Paper_alpha_Chiral_Vacuum_Selection_appendix_A_theta.py` (attached).

**A.3 Level A verification:**

- $\tau_0 = i/\varphi$, $S\tau_0 = i\varphi \neq \tau_0$ ✓
- $T\tau_0 = 1 + i/\varphi$ preserves $\mathrm{Im}\,\tau$ ✓
- $C_c\tau_0 = -\bar{\tau}_0 = \tau_0$ (complex conjugation fixes purely imaginary axis) ✓

**A.4 Level B numerical consequences:**

- $\mu_B^{(L)} = v\, e^{-\pi/\varphi^2} = 74.161\,\mathrm{GeV}$ ✓
- $\mu_B^{(R),1} = v\, e^{-\pi\varphi^2} = 66.0\,\mathrm{MeV}$
- $\mu_B^{(R),2} = v\, e^{+\pi/\varphi^2} = 817.5\,\mathrm{GeV}$

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## Appendix B — Modular Notation and Form Identities

**$SL(2, \mathbb{Z})$ generators:**

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S^2 = -1, \quad (ST)^3 = 1.$$

**Non-holomorphic Eisenstein of weight 2:**

$$\widehat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi\,\mathrm{Im}\,\tau}, \qquad \widehat{E}_2(-1/\tau) = \tau^2\,\widehat{E}_2(\tau).$$

**Jacobi theta functions:**

$$\theta_{\alpha\beta}(\tau) \quad \text{for} \quad (\alpha, \beta) \in \{0, 1/2\}^2, \qquad \theta_{\alpha\beta}(-1/\tau) = (-i\tau)^{1/2}\,\theta_{\beta\alpha}(\tau).$$

**Dedekind eta:**

$$\eta(\tau) = e^{i\pi\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}), \qquad \eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau).$$

**Congruence subgroups:**

$$\Gamma(N) = \{\gamma \in SL(2,\mathbb{Z}) : \gamma \equiv 1 \pmod N\},$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c \equiv 0 \pmod N \right\}.$$

---

## Appendix C — Poisson Resummation and KK/Winding Duality

**Poisson summation formula on $T^2$:**

$$\sum_{\vec{n}\in\mathbb{Z}^2} f(\vec{n}) = \sum_{\vec{w}\in\mathbb{Z}^2} \hat{f}(\vec{w}),$$

where $\hat{f}$ is the Fourier transform. On $T^2(\tau)$, this implements $S$-duality by exchanging momentum lattice $\mathbb{Z}^2$ with winding lattice $\mathbb{Z}^2$, with action $(n, w) \mapsto (w, -n)$ (up to sign conventions).

For brane-localized field