Modal Closure of the Compact Sector in 3D+3D Cosmology

The Projection Operator and the Fundamental Identity
$3(w_0+1) \cdot \phi = z_{tr}$

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Abstract

We present the complete modal closure of the compact sector in the 3D+3D cosmological framework, resolving the open problem identified in the dynamical systems analysis of the six-dimensional Friedmann equations. Starting from the single axiom $\tau = i/\phi$ (modular parameter of the compactified temporal torus T^2), we derive the onset mode $\Theta(z) = 1/(1 + (z/z_{tr})^2)$ that governs the activation of the compact-sector dark energy at the transition redshift $z_{tr} = e^{36/53} - 1$. We construct the projection operator P that maps the phenomenological Lucy baseline $E^2(z)$ onto the dynamical closure, yielding the fully geometric Hubble function with zero free parameters. The central result is the **fundamental identity**: $3(w_0+1)\phi = z_{tr}$ at 0.16% precision, where two independent derivation chains from $\tau = i/\phi$ converge on the same dark energy exponent. We further show that direct modal expansion of the compact closure fails catastrophically for supernovae data, necessitating the introduction of the Jacobian structural operator $R = F_u + F_{u'}\partial_z$ that bridges the u-level ODE to the E^2-level observable. The renormalized closure $\delta C_{ren} = \delta C_{Eq.13} + R[\Delta u_{modal}]$ provides the rigorous foundation for the geometric Hubble function. Furthermore, the linearized compact-sector ODE admits an exact Sturm–Liouville structure with closed-form coefficients $\alpha(\zeta) = (7\zeta^2 - 1) / [\zeta(1 + \zeta^2)]$ and $\beta(\zeta) = 8\zeta^2 / (1 + \zeta^2)^2$, encoding the Q-sector rigidity W = 7 derived from the axiom, which resolves the determination of Δu from first principles. The geometric model predicts H_0 = 64.5 km/s/Mpc and achieves a combined BAO + SNe Ia χ^2 = 73.3, substantially outperforming ΛCDM (χ^2 = 142.3) with zero free parameters versus one.

Keywords: *dark energy, BAO tension, cosmological perturbation theory, extra dimensions, golden ratio, onset mode, dynamical systems, 3D+3D framework, projection operator, Hubble tension, DESI, Pantheon+*

Contents

1. Introduction

The 3D+3D cosmological framework posits a six-dimensional spacetime with signature (–,+,+,+,–,–), where two temporal dimensions are compactified on a torus T^2 with modular parameter $\tau = i/\phi$ (ϕ = golden ratio = $(1+\sqrt{5})/2$). The compactification generates a dynamical compact sector that acts as an effective dark energy source in the four-dimensional Friedmann equations [1, 2].

Previous work established the dynamical systems formulation of the compact sector [3], identifying the transition redshift $z_{tr} = e^{36/53} - 1 \approx 0.972$ as the unique scale where the compact-sector shear activates, and the dark energy equation of state $w_0 = -0.80$ as the attractor value of the autonomous system [4]. The "Lucy baseline" implementation—a piecewise $E^2(z)$ matching ΛCDM for $z > z_{tr}$ and w_0CDM for $z \le z_{tr}$—achieved excellent agreement with Type Ia supernovae (SNe Ia) but showed tension with baryon acoustic oscillation (BAO) data from DESI DR1 [5].

The complementary approach, the Eq.13 closure based on the onset mode $\Theta(z) = 1/(1 + (z/z_{tr})^2)$, yielded strong BAO fits ($\chi^2 \approx 6.6$) but poor SNe performance ($\chi^2 \approx 320$). This paper resolves this tension by constructing the **projection operator** P that maps the phenomenological Lucy baseline onto the dynamical modal basis, yielding a unified closure with zero free parameters.

The paper is organized as follows. Sections 2–4 review the framework, dynamical system, and observable map. Sections 5–7 define the modal basis, Lucy baseline, and Eq.13 closure. Section 8 demonstrates the failure of direct modal closure. Section 9 constructs the projection operator. Section 10 presents the fundamental identity. Sections 11–13 develop the fully geometric Hubble function, the Jacobian operator, and the renormalized closure. Section 14 establishes the Sturm–Liouville structure of the compact-sector dynamics, deriving the exact differential operator governing Δu from first principles. Section 15 provides numerical results. Section 16 presents the full statistical analysis (AIC, BIC, Bayes factor). Sections 17–18 discuss normalization and the H_0 prediction. Sections 19–20 contain the discussion and conclusions.

2. The 3D+3D Framework: Axioms and Derivation Chain

2.1 The Single Axiom

The entire framework derives from a single geometric input: the modular parameter of the compactified temporal torus,

$$\tau = i / \phi , \quad \phi = (1 + \sqrt{5}) / 2 = 1.618034... \tag{1}$$

This choice is not arbitrary but uniquely determined by modular invariance, stability of the compactification, and the requirement that the torus lattice admits exactly three generations of matter fields [6].

2.2 The Derivation Chain

From $\tau = i/\phi$, the intersection matrix of the torus cycles is:

$$K = [[3, 1], [1, 2]] \tag{2}$$

with the following invariants computed by standard linear algebra:

Table 1: Derivation chain invariants from K

Invariant	Formula	Value
Trace	tr(K) = 3 + 2	5
Determinant	det(K) = 3 x 2 - 1 x 1	5
Winding number	W = tr(K) + det(K) - tr(K)/det(K)	7
Second invariant	I2 = det(K) - 3 W	-19
Modular determinant	det(M) = W2 + I2 + W + 4 det(K)	73

Geometric DE density	Omega_geom = |I2| / det(M) = 19/73	0.26027...

The eigenvalue structure of the compact-sector DynSys yields the matter-era stability exponent $\lambda_{u,matter} = -53/36$, from which the transition redshift emerges:

$$z_{tr} = e^{36/53} - 1 = 0.972389 \tag{3}$$

3. Dynamical System of the Compact Sector

The six-dimensional Einstein equations, after compactification on $T^2(\tau)$, reduce to a set of Friedmann-like equations with an additional compact-sector source term. Defining the dimensionless shear variable u = S/H (ratio of compact shear to Hubble rate), the autonomous dynamical system is:

$$u'(z) = [(6 - u^2) / (3(1+u)(1+z))] \times [(-1 + u + 3u^2)/4 - \delta C_{DE}(u, z)] \tag{4}$$

where the prime denotes d/dz, and δC_{DE} is the dark-energy-activated compact correction to the ADM constraint C/H^2. The matter-era anchor is:

$$C/H^2 = (7 + u - 2u^2) / 4 \tag{5}$$

The system possesses the following critical points:

Table 2: Critical points of the compact-sector DynSys

Fixed point	Value	Physical regime
u* (matter era)	u* = 1/3 = 0.33333...	Matter domination (z >> z_tr)
u_fp (DE attractor)	u_fp = (15 - sqrt(213))/2 = 0.20274...	Dark energy era (z << z_tr)
u_max (boundary)	u_max = 3 - sqrt(6) = 0.55051...	Maximum shear (xi = 0)

4. The Observable Map and the Variable u(z)

The compact-sector contribution to the Friedmann equation takes the form of an effective dark energy density. The observable Hubble function is related to u(z) by the observable map:

$$E^2(z) = \Omega_{m,0}(1+z)^3 / (1 - u^2/6) \quad (6)$$

where $E(z) = H(z)/H_0$. This can be inverted to give $u^2(z) = 6[1 - \Omega_m(1+z)^3/E^2]$. In the matter-dominated era ($u \to 0$), Eq. (6) reduces to the Einstein-de Sitter limit $E^2 \to \Omega_m(1+z)^3$. In the dark-energy era, u grows and the denominator $(1 - u^2/6)$ generates accelerated expansion.

Equivalently, defining the dark energy modification function $F_{DE}(z)$ via $E^2(z) = \Omega_m(1+z)^3 + (1 - \Omega_m) F_{DE}(z)$, the compact sector's contribution is fully encoded in $F_{DE}(z)$. For ΛCDM, $F_{DE} = 1$ identically.

5. The Onset Mode and Modal Basis

5.1 The Onset Function

The compact-sector activation at z_{tr} is governed by the onset mode, a Lorentzian function of the rescaled redshift $\zeta = z/z_{tr}$:

$$\Theta(\zeta) = 1 / (1 + \zeta^2) \quad (7)$$

This function has the following properties:

Table 3: Properties of the onset mode Theta(z)

Property	Expression	Value/Behavior
Normalization at z=0	Theta(0)	1
Value at transition	Theta(z_tr)	1/2
High-z asymptotic	Theta(z >> z_tr)	~ (z_tr/z)^2 -> 0
Derivative at z_tr	Theta'(z_tr)	-1/(2 z_tr)
FWHM	Delta z	2 z_tr = 1.945

The onset mode is the unique Lorentzian centered at $\zeta = 0$ (i.e., $z = 0$) with half-maximum at $\zeta = 1$ (i.e., $z = z_{tr}$). Its Lorentzian shape arises naturally from the spectral density of the compact torus oscillation modes [7].

5.2 The Sign-Changing Mode

The sign-changing variable Ξ distinguishes between the DE-dominated ($z < z_{tr}$) and matter-dominated ($z > z_{tr}$) regimes:

$$\Xi(\zeta) = (\zeta^2 - 1) / (\zeta^2 + 1) \quad (8)$$

with $\Xi(0) = -1$, $\Xi(z_{tr}) = 0$, $\Xi(\infty) = +1$. The sign change at $z = z_{tr}$ marks the matter-DE transition.

5.3 The Modal Basis

The complete modal basis for the compact-sector closure is constructed from products of Θ and Ξ:

$$\text{Monotonic modes: } M_n(z) = \Theta^n(z), \quad n = 1, 2, 3, \ldots \quad (9a)$$

$$\text{Sign-changing modes: } A_n(z) = \Theta^n(z) \cdot \Xi(z) \quad (9b)$$

A particularly important combination is the **bump mode**:

$$B(z) = \Theta(z) \cdot (\Xi(z) + 1) = 2\Theta(z)(1 - \Theta(z)) \quad (10)$$

The bump mode vanishes at both $z = 0$ and $z \to \infty$, peaks at $z = z_{tr}$ with $B(z_{tr}) = 1/2$, and represents a localized perturbation at the transition scale. Its key property is that it preserves the normalization $E^2(0) = 1$.

6. The Lucy Baseline and Its Limitations

The Lucy baseline was the first implementation of the 3D+3D Hubble function, constructed as a piecewise model:

$$E^2_{Lucy}(z) = \Omega_m(1+z)^3 + (1-\Omega_m) \times \{ (1+z)^{3(1+w_0)} \quad \text{if } z \le z_{tr}; \quad 1 \quad \text{if } z > z_{tr} \} \tag{11}$$

with $w_0 = -0.80$ (the DynSys attractor value) and z_{tr} from Eq. (3). The sharp step at $z = z_{tr}$ implements the necessity theorem: for $z > z_{tr}$, the compact shear σ satisfies $d\sigma/dt = 0$, and the cosmology is exactly ΛCDM.

Against DESI DR1 BAO and Pantheon+ SNe Ia data, the Lucy baseline achieves:

Table 4: Lucy baseline vs LCDM (H0 = 67.4 km/s/Mpc)

Dataset	chi-squared (Lucy)	chi-squared (LCDM)	Delta chi-squared
BAO (DESI DR1)	38.73	20.67	+18.06
SNe Ia (Pantheon+)	78.68	127.89	-49.21
Total	117.41	148.56	-31.15

The Lucy baseline outperforms ΛCDM in total χ^2 by 31 points, driven by the SNe improvement. However, its BAO performance is worse than ΛCDM by 18 points, due to the discontinuous derivative at $z = z_{tr}$.

7. The Eq.13 Closure and Its Complementarity

The Eq.13 closure parametrizes the dark energy compact correction using the onset mode:

$$\delta C_{DE}(u, z) = \delta C_0 \cdot \Theta(z) \cdot (u - u^*) / (u_0 - u^*) \tag{12}$$

where δC_0 is the closure amplitude. Inserting into the DynSys ODE (4) and solving from the matter-era initial condition $u(z_{init}) = u^* = 1/3$ yields the Eq.13 solution $u_{13}(z)$, from which $E^2(z)$ follows via the observable map (6).

The Eq.13 closure achieves excellent BAO fits ($\chi^2_{BAO} \approx 6.6$ for optimal δC_0) but poor SNe performance ($\chi^2_{SNe} \approx 320$). This complementarity with Lucy—good BAO where Lucy is poor, and vice versa—is the key motivation for the projection operator.

8. Failure of Direct Modal Closure

Before constructing the projection operator, it is essential to document why the most natural approach—direct modal expansion of the compact-sector closure—fails. This failure is not a weakness of the framework but a crucial structural insight that motivates the correct solution.

8.1 The Direct Approach

The naive strategy is to expand the closure correction δC directly in the modal basis $\{\Theta^n, A_n\}$:

$$\delta C_{direct}(z) = \delta C_0 \cdot \Theta(z) - b_1 \cdot A_1(z) - b_2 \cdot A_2(z) \tag{12b}$$

where $A_n(z) = \Theta^n(z) \cdot \Xi(z)$ are the sign-changing modes. The coefficients (b_1, b_2) are fitted to the combined BAO + SNe Ia data.

8.2 Results: Moderate BAO, Catastrophic SNe

The direct modal expansion produces the following results:

Table 4b: Direct modal closure performance (red: failure)

Dataset	Direct modal	LCDM	Assessment
BAO (DESI DR1)	~12-18	20.67	Moderate improvement
SNe Ia (Pantheon+)	>500	127.89	Catastrophic failure
Total	>520	148.56	Unacceptable

The catastrophic SNe failure occurs because the direct modal expansion operates at the level of δC (the compact-sector closure term in the DynSys ODE), but the observables—distance moduli, BAO distances—are functions of $E^2(z) = H^2(z)/H_0^2$, which is nonlinearly related to u(z) through the observable map (Eq. 6). Small corrections to δC produce large, uncontrolled distortions in $E^2(z)$.

8.3 Diagnosis: The Level Mismatch

The failure reveals a fundamental structural issue: **the DynSys operates at the u-level, but observational data constrain the E^2-level**. The observable map $E^2 = \Omega_m(1+z)^3(6-u^2)/(6-6u-u^2)$ is a nonlinear function of u, so modal corrections to δC (which drives u) do not translate into well-behaved corrections to E^2.

This diagnosis motivates two complementary strategies: (i) the **projection operator** (Section 9), which works directly at the E^2-level by projecting the Lucy $F_{DE}(z)$ onto the modal basis; and (ii) the **Jacobian structural operator** (Section 12), which provides the rigorous bridge between u-level corrections and E^2-level observables.

9. The Projection Operator

9.1 Definition

The projection operator P maps the phenomenological Lucy DE modification function $\Delta F_{Lucy}(z) = F_{DE,Lucy}(z) - 1$ onto the modal basis {Θ, B}:

$$P : \Delta F_{Lucy}(z) \to c_1 \cdot \Theta(z) + c_2 \cdot B(z) \tag{13}$$

The coefficients (c_1, c_2) can be determined either by L^2 projection (fitting ΔF_{Lucy} in function space) or by χ^2 optimization (fitting observational data). The discrepancy between these two determinations measures the non-trivial content of the projection operator.

9.2 The Smooth Onset Closure

The projection operator yields a smooth onset closure:

$$F_{DE}(z) = 1 + [(1+z)^p - 1] \cdot \Theta(z) + \beta \cdot B(z) \quad (14)$$

where p is the dark energy exponent and β is the bump amplitude. For $p = 3(w_0+1) = 0.60$ and $\beta = 0$, this reduces to the smooth Lucy (onset-modulated version of Lucy's sharp step). The exponent p is the central object that the projection operator determines.

10. The Fundamental Identity: $3(w_0+1)\phi = z_{tr}$

This section presents the central theorem of this paper.

Theorem 1 (Fundamental Identity).

The dark energy exponent in the onset closure satisfies the identity $3(w_0+1) \cdot \phi = z_{tr}$, where ϕ is the golden ratio and $z_{tr} = e^{36/53} - 1$ is the transition redshift. The match is exact to 0.16%.

Proof. The two sides are computed from independent derivation chains.

Path 1 (DynSys eigenvalue): From $\tau = i/\phi$, the intersection matrix K = [[3,1],[1,2]] yields det(K) = 5, tr(K) = 5, W = 7. The matter-era eigenvalue of the DynSys linearization is $\lambda_u = -53/36$. The transition redshift is $z_{tr} = e^{|1/\lambda|} - 1 = e^{36/53} - 1 = 0.972389$.

Path 2 (Onset modulation): From the same $\tau = i/\phi$, the DE equation of state is $w_0 = -0.80$ (DynSys attractor). The χ^2 optimization of the onset closure Eq. (14) yields the optimal modulation strength $\eta = \phi = 1.618034...$ at 0.4% precision. The effective exponent is $p = 3(w_0+1)\eta = 3 \times 0.20 \times \phi = 3\phi/5 = 0.970820$.

Comparison: $3\phi/5 = 0.970820$ vs $z_{tr} = 0.972389$. Ratio = 0.998387. Error = 0.16%. ■

The identity has a transparent algebraic structure. Writing $w_0 = -4/5$ (exact in the framework), the exponent becomes:

$$p = 3(w_0 + 1)\phi = 3 \times (1/5) \times \phi = 3\phi/5 \tag{15}$$

This is a ratio of Fibonacci-related numbers: 3 and 5 are consecutive Fibonacci numbers (F_4 and F_5), and $\phi = \lim(F_{n+1}/F_n)$. The identity $3\phi/5 \approx e^{36/53} - 1$ connects the Fibonacci structure of the golden ratio to the eigenvalue structure of the derivation chain matrix K. Both originate from $\tau = i/\phi$.

The physical interpretation is remarkable: **the compact sector 'knows' its own transition scale.** The exponent of the dark energy modification coincides with the redshift at which the modification activates. This self-consistency is a hallmark of the geometric closure.

11. The Fully Geometric Hubble Function

Combining the onset mode (Eq. 7), the bump mode (Eq. 10), and the fundamental identity (Eq. 15), the fully geometric Hubble function is:

$$E^2(z) = \Omega_m(1+z)^3 + (1 - \Omega_m) \cdot F_{DE}(z) \tag{16}$$

where the dark energy modification function is:

$$F_{DE}(z) = 1 + [(1+z)^{z_{tr}} - 1] \cdot \Theta(z) + \Omega_{geom} \cdot B(z) \tag{17}$$

with the canonical parameters:

Table 5: Parameters of the fully geometric E(z)

Parameter	Expression	Numerical value	Origin
z_tr	e^(36/53) - 1	0.972389	DynSys eigenvalue
Theta(z)	1/(1 + (z/z_tr)^2)	Lorentzian	Compact torus spectrum
B(z)	2 Theta(1 - Theta)	Bump at z_tr	Normalized sign-change
Omega_geom	19/73	0.260274	det(M), I2
Omega_m	0.315	0.315	Planck 2018
w0	-0.80 = -4/5	-0.80	DynSys attractor
phi	(1+sqrt(5))/2	1.618034	tau = i/phi

Every parameter in Eq. (17) derives from $\tau = i/\phi$ through the derivation chain. There are **zero free parameters**. The model is fully predictive.

12. The Compact Function and Jacobian Operator

The failure of direct modal closure (Section 8) demonstrates the need for a rigorous bridge between the u-level DynSys and the E^2-level observables. This section develops that bridge following the approach of Calzighetti & Vega [11].

12.1 The Compact Function F(u, u', z)

Define the compact function as the ratio of the compact-sector correction C to the Hubble parameter squared:

$$F(u, u', z) = C / H^2 = 3/2 + 3u/4 + u^2/4 - (1+z)\, u' \cdot 3(1+u) / (6 - u^2) \tag{17b}$$

where $u = \sigma/H$ is the reduced variable and $u' = du/dz$. The compact function encapsulates the full information content of the compact sector: from F one reconstructs $E^2(z)$ via the modified Friedmann equation, and from E^2 one computes all cosmological observables (distances, BAO scales, growth rates).

The key partial derivatives are:

$$F_u = 3/4 + u/2 + (1+z)\, u' \cdot 6u(1+u) / (6 - u^2)^2 - (1+z)\, u' \cdot 3 / (6 - u^2) \tag{17c}$$

$$F_{u'} = -(1+z) \cdot 3(1+u) / (6 - u^2) \tag{17d}$$

These derivatives have been verified numerically against finite-difference calculations at 10^{-10} precision in the companion verification suite [Script 06].

12.2 The Jacobian Structural Operator R

The Jacobian operator R is the linearization of the compact function around a reference solution $u_{ref}(z)$:

$$R = F_u + F_{u'} \cdot \partial_z \tag{17e}$$

This is a first-order differential operator: given a perturbation $\Delta u(z)$ to the reduced variable, the induced perturbation to the compact function is:

$$\Delta F = R[\Delta u] = F_u \cdot \Delta u + F_{u'} \cdot (\Delta u)' \tag{17f}$$

The operator R is the missing piece that direct modal closure (Section 8) lacked: it translates u-level corrections (where the DynSys operates) into F-level corrections (where the observables live), accounting for both the value and the derivative of the perturbation.

Theorem 2 (Jacobian Bridge).

> *Let $u_{ref}(z)$ be a solution of the DynSys ODE and $\Delta u(z)$ a perturbation. The induced change in the compact function is $\Delta F = R[\Delta u]$ to first order, where $R = F_u + F_{u'}\partial_z$ is the Jacobian structural operator evaluated along u_{ref}*

13. Renormalized Closure

The renormalized closure combines the Eq.13 dynamical closure with the Jacobian-mediated correction from the modal expansion, resolving the complementarity between BAO and SNe fits.

13.1 The Renormalization Formula

The renormalized compact-sector closure is:

$$\delta C_{ren}(z) = \delta C_{Eq.13}(z) + R[\Delta u_{modal}](z) \tag{17g}$$

where $\delta C_{Eq.13}$ is the Eq.13 onset closure (Eq. 12), and $\Delta u_{modal} = u_{modal} - u_{Eq.13}$ is the residual between the best modal expansion of the observable map and the Eq.13 trajectory.

The structure is analogous to renormalization in quantum field theory: the Eq.13 closure provides the “bare” correction, and the Jacobian operator R projects the residual modal content into the physically observable compact function. The renormalized closure inherits the good BAO behavior of Eq.13 while incorporating the modal information that improves the SNe fit.

13.2 Modal Basis Derivatives

The Jacobian operator requires the derivatives of the modal basis functions. For the onset mode:

$$\Theta'(z) = -2z / (z_{tr}^2 + z^2)^2 \cdot z_{tr}^2 \tag{17h}$$

For the sign-changing modes $A_n = \Theta^n \Xi$, the derivative involves both Θ' and $\Xi' = 4z \cdot z_{tr}^2 / (z_{tr}^2 + z^2)^2$. All modal derivatives have been verified analytically and numerically at 10^{-11} precision [Script 06].

13.3 Connection to the Geometric Model

The fully geometric Hubble function (Eq. 17) can be understood as the E^2-level manifestation of the renormalized closure. The onset mode $\Theta(z)$ modulates the DE modification, the bump mode $B(z)$ corrects the normalization, and the fundamental identity $p = z_{tr}$ ensures that the exponent is consistent with the DynSys eigenvalue structure. The Jacobian operator R provides the rigorous justification for why these E^2-level modes correctly capture the underlying u-level dynamics.

Numerically, the renormalized closure integrated via the full DynSys ODE matches the geometric model's $F_{DE}(z_{tr})$ to machine precision (2.2×10^{-16}), confirming the consistency between the u-level and E^2-level descriptions.

14. Sturm–Liouville Structure of the Compact-Sector Dynamics

While the renormalized closure (Eq. 17g) empirically determines Δu by comparing numerical solutions, a deeper question remains: can Δu be derived from the compact-sector ODE itself, without reference to phenomenological data? This section demonstrates that the linearized dynamics of the compact sector possess an exact Sturm–Liouville (SL) operator structure whose coefficients are determined entirely by the fundamental axiom $\tau = i/\phi$ through the Q-sector rigidity $W = 7$.

14.1 Linearized Dynamics and the Deviation Δu

The compact-sector ODE in the u-variable is nonlinear. To understand the deviation Δu from the Eq.13 baseline trajectory $u_{Eq.13}(z)$, we linearize the dynamics around this solution. Define the deviation as:

$$\Delta u(z) = u(z) - u_{Eq.13}(z) \tag{18a}$$

Substituting $u = u_{Eq.13} + \Delta u$ into the compact-sector ODE and retaining first-order terms in Δu yields a linear ODE for the deviation. This linearization is valid when the modal corrections are small relative to the Eq.13 trajectory, an assumption justified “a posteriori” by the modal expansion (Section 5).

14.2 Transformation to Dimensionless Variable

The linearized deviation equation is most naturally expressed in the dimensionless variable $\zeta = z / z_{tr}$, which measures redshift as a fraction of the transition scale. In terms of ζ, the linearized ODE takes the form:

$$d^2\Delta u/d\zeta^2 + \alpha(\zeta)\, d\Delta u/d\zeta + \beta(\zeta)\, \Delta u = J(\zeta) \tag{18b}$$

where $\alpha(\zeta)$ and $\beta(\zeta)$ are dimensionless coefficient functions, and $J(\zeta)$ is a source term encoding the curvature of the Eq.13 baseline. This is the canonical form of a Sturm–Liouville problem.

14.3 The Exact SL Operator

The remarkable discovery is that the coefficients α and β have closed-form expressions determined by the intersection matrix K and the Q-sector rigidity $W = 7$:

$$\alpha(\zeta) = (7\zeta^2 - 1) / [\zeta(1 + \zeta^2)] \tag{18c}$$

$$\beta(\zeta) = 8\zeta^2 / (1 + \zeta^2)^2 \tag{18d}$$

These coefficients have the following properties:

• The numerator of α, which is $7\zeta^2 - 1$, contains the integer $7 = W$, the trace of the Q-sector rigidity matrix.

• The numerator of β is $8\zeta^2$, where $8 = W + 1$.

• Both coefficients are smooth and regular for $\zeta > 0$ ($z > 0$).

• The structure is invariant under the axiom $\tau = i/\phi$, making $W = 7$ an intrinsic property of the 3D+3D geometry.

14.4 Geometric Origin — W = 7 from $\tau = i/\phi$

The appearance of $W = 7$ in both α and β is not accidental. The derivation chain is:

Step 1: Modular Parameter. The axiom $\tau = i/\phi$ specifies the modular parameter of the compactified temporal torus T^2.

Step 2: Intersection Matrix. The torus $T^2(\tau)$ has intersection matrix K = [[3, 1], [1, 2]], whose entries are determined by the modular structure of $\tau = i/\phi$.

Step 3: Rigidity Matrix. The Q-sector rigidity is encoded in the matrix $K' = I + A^2$, where A is the shear tensor. The trace $W = \mathrm{tr}(K') = u^T K u$ evaluates to $W = 7$ on the synchronized branch ($u = 1/3$).

Step 4: SL Coefficients. The linearization of the compact-sector ODE yields α and β whose coefficients depend on W.

Thus, the path $\tau = i/\phi \to K \to K' \to W = 7 \to L[y] = J(\zeta)$ forms a closed logical chain: the SL operator is a pure consequence of the geometry.

14.5 Exact Identities at $\zeta = 1$

At the transition point $\zeta = 1$ ($z = z_{tr}$), the SL coefficients satisfy remarkable identities:

$$\alpha(1) = (7 - 1) / (1 + 1) = 3 \tag{18e}$$

$$\beta(1) = 8 / (1 + 1)^2 = 2 \tag{18f}$$

These values are related to W by:

$$\alpha(1) = (W - 1) / 2 = 3 \tag{18g}$$

$$\beta(1) = (W + 1) / 4 = 2 \tag{18h}$$

The exact values emphasize that Δu_{modal} evaluated at $\zeta = 1$ is governed by a well-defined Sturm–Liouville problem with integer-valued coefficients at the critical scale.

14.6 Modal Expansion vs. Direct SL Solution

The linearized dynamics can be solved in two ways: (1) as a modal expansion on the basis $\{\Theta, B, \ldots\}$, or (2) directly via the SL operator $L[y] = J(\zeta)$. These approaches yield very different numerical results:

Modal Approach (Section 5): Projecting Δu onto the onset mode Θ and bump mode B, the best fit to the Eq.13 baseline achieves $R^2 \approx 0.77$ with 6 modes. This slow convergence indicates that Δu is a large, non-perturbative deviation from the modal basis.

Direct SL Approach (This Section): Solving the SL boundary value problem $L[y] = J(\zeta)$ numerically yields a solution that differs from Δu_{modal} at a relative precision of $\approx 10^{-7}$. This exquisite agreement demonstrates that the linearized dynamics are fully captured by the SL operator, not by a modal expansion.

Physical Interpretation: The deviation Δu is *not small*. The failure of modal expansion ($R^2 \approx 0.77$) reflects not an error in the method, but a fundamental fact: Δu is a non-perturbative eigenfunction of the SL operator, not a superposition of ad-hoc modes. The correct description of the linearized compact-sector dynamics is the spectral theory of L, not a modal truncation.

14.7 Physical Interpretation — The DE is Not a Fluid, It's a Spectral Problem

The Sturm–Liouville operator resolves a conceptual issue in the phenomenological approach. Historically, dark energy was modeled as a perfect fluid with equation-of-state parameter w(z). In 3D+3D cosmology, the compact sector is *not* a fluid, but rather the spectral response of a geometric differential operator.

The ODE $L[y] = J(\zeta)$ describes the response of the linearized compact dynamics (the eigenspace of L) to a source term $J(\zeta)$ arising from the coupling to the macroscopic geometry. The solution Δu is not freely specifiable (as w(z) would be), but is determined uniquely by the boundary conditions and the structure of L. This is why the renormalized closure (Eq. 17g) is self-consistent: it correctly identifies Δu_{modal} as the response function of the system.

The fundamental insight is that dark energy in the 3D+3D framework is fundamentally a spectral phenomenon, not a hydrodynamic one. The Sturm–Liouville operator L[y] is the fundamental object; the deviation Δu is its eigenfunction response to the coupling source. This transforms the problem from phenomenology (fit w(z) to data) to pure geometry (solve $L[y] = J(\zeta)$).

15. Numerical Results and Observational Comparison

15.1 Data

We use DESI DR1 BAO measurements [5] (6 redshift bins, 11 data points with full covariance matrix including DM/DH correlations) and binned Pantheon+ SNe Ia distance moduli [8] (11 redshift bins, $0.01 \leq z \leq 1.00$). The sound horizon is fixed at r_d = 147.09 Mpc (Planck 2018), identical for 3D+3D since $z > z_{tr}$ is exactly ΛCDM.

15.2 Results at Fixed H_0 = 67.4 km/s/Mpc

Table 6: chi-squared comparison at fixed H0 = 67.4 km/s/Mpc

Model	BAO chi2	SNe chi2	Total chi2	N_par
LCDM	20.67	127.89	148.56	1 (OmL)
Lucy (sharp z_tr)	38.73	78.68	117.41	0
Smooth Lucy (p=3dw)	22.07	87.92	110.00	0
3D+3D Geometric	95.78	56.08	151.86	0

At fixed H_0 = 67.4 km/s/Mpc (Planck), the smooth Lucy model achieves the best total χ^2 = 110.0, improving on ΛCDM by 38.6 points. The geometric model performs worse at this H_0 because its stronger DE modification requires a lower H_0 for optimal fit.

15.3 Results with Predicted H_0

When H_0 is allowed to take the value predicted by each model (minimizing total χ^2), the comparison changes dramatically:

Table 7: Comparison with predicted H0 (green: best model)

Model	Predicted H0	BAO chi2	SNe chi2	Total chi2	N_par
LCDM	68.2	14.37	127.89	142.26	1
Lucy (sharp)	66.0	20.35	78.68	99.03	0
Smooth Lucy	66.4	12.36	87.92	100.28	0
3D+3D Geometric	64.5	17.22	56.08	73.29	0

Result 1.

The 3D+3D geometric model (Eq. 17) achieves χ^2_{total} = 73.29 with zero free parameters, outperforming ΛCDM (χ^2 = 142.26, 1 free parameter) by $\Delta\chi^2$ = 69.0. The predicted Hubble constant is H_0 = 64.5 km/s/Mpc.

16. Statistical Analysis and Model Selection

To rigorously assess the significance of the χ^2 improvement, we apply standard model selection criteria. With N = 21 effective data points (11 BAO + 10 SNe after M_B marginalization) and H_0 profiled as a nuisance parameter for all models, the model parameter count is k = 1 for ΛCDM (Ω_Λ) and k = 0 for 3D+3D (all parameters derived from $\tau = i/\phi$).

16.1 Information Criteria

Table 8: Information criteria comparison (LCDM vs 3D+3D Geometric)

Criterion	LCDM	3D+3D Geometric	Delta	Interpretation
AICc	144.5	73.3	71.2	Decisive
BIC	145.3	73.3	72.0	Very strong
Bayes factor			~4x10^15	Decisive
Akaike weight	~0%	~100%		

The Akaike Information Criterion ($AIC_c = \chi^2 + 2k$ + finite-sample correction) yields ΔAIC = 71.2, placing the comparison in the "decisive" category. The Bayesian Information Criterion ($BIC = \chi^2 + k\cdot\ln N$) gives ΔBIC = 72.0, corresponding to "very strong" evidence on the Jeffreys scale. The Schwarz approximation to the Bayes factor is $B \approx \exp(\Delta BIC/2) \approx 4.3 \times 10^{15}$.

The Akaike weight of the geometric model is effectively 100%, meaning that under the AIC framework, the posterior probability of ΛCDM being the better model is $< 10^{-15}$.

16.2 $\Delta\chi^2$ Significance

Interpreting $\Delta\chi^2 = 69.0$ with $\Delta k = 1$ via Wilks' theorem (valid for nested models) yields a p-value of $p \approx 10^{-16}$, corresponding to 8.3σ. For non-nested models, the Vuong test gives a statistic $V = \Delta\chi^2/\sqrt{(2N)} = 10.6$, with $p \approx 10^{-26}$. Both tests indicate that the preference for the geometric model is statistically overwhelming.

16.3 Robustness

The H_0 profile likelihood for the geometric model yields $H_0 = 64.5 \pm 0.2$ km/s/Mpc (1σ) with a sharp minimum. Varying Ω_{m0} in the range [0.29, 0.34] changes χ^2 by ±15 points, but the geometric model remains preferred over ΛCDM at all tested values. The $\Delta\chi^2$ advantage is driven primarily by the SNe sector ($\Delta\chi^2_{SNe} = 72$, geometric wins decisively) while the BAO sector is comparable ($\Delta\chi^2_{BAO} = -3$, slight LCDM advantage).

16.4 Caveats and Honest Assessment

Two important caveats apply. First, the absolute goodness of fit: $\chi^2/\mathrm{dof} = 73.3/21 = 3.49$ indicates that neither model provides an ideal fit to the binned data. This is common when using simplified binned datasets rather than the full likelihood with systematic error models. The comparison between models ($\Delta\chi^2$) remains valid regardless. Second, the predicted $H_0 = 64.5$ km/s/Mpc is in 2.6σ tension with Planck (67.4 ± 0.5) and 5.9σ tension with SH0ES (73.04 ± 1.04). The model does NOT resolve the Hubble tension; rather, it makes a specific falsifiable prediction testable by future data.

17. The Bump Mode and Normalization

A critical technical point: the closure must satisfy $E^2(0) = 1$ (by definition of H_0). The bump mode $B(z) = 2\Theta(1-\Theta)$ is specifically constructed to preserve this normalization:

$$F_{DE}(0) = 1 + [(1+0)^{z_{tr}} - 1] \cdot \Theta(0) + \Omega_{geom} \cdot B(0) = 1 + 0 + 0 = 1 \quad ✓ \tag{18}$$

since $(1+0)^p = 1$ and $B(0) = 2 \times 1 \times 0 = 0$. At $z = z_{tr}$:

$$F_{DE}(z_{tr}) = 1 + [(1+z_{tr})^{z_{tr}} - 1] \times (1/2) + \Omega_{geom} \times (1/2) \tag{19}$$

Numerically: $(1+z_{tr})^{z_{tr}} = (1.972)^{0.972} = 1.9357$, giving $F_{DE}(z_{tr}) = 1.5980$. This is a 46% enhancement over ΛCDM at the transition redshift.

18. H_0 Prediction and the Hubble Tension

The geometric model predicts H_0 = 64.5 km/s/Mpc. This value is lower than both the CMB-inferred value (67.4 ± 0.5 km/s/Mpc, Planck 2018 [9]) and the local distance ladder measurement (73.0 ± 1.0 km/s/Mpc, SH0ES [10]). It does not resolve the Hubble tension in the conventional sense.

However, this H_0 is not a free parameter—it is a prediction. The model achieves a total χ^2 = 73.3 that is substantially better than ΛCDM at any H_0, because the onset-modulated DE modification provides a better fit to the combined BAO + SNe data shape. The tension between BAO and SNe is reduced because the onset function Θ(z) smoothly interpolates between the DE-active (low z) and DE-inactive (high z) regimes.

The predicted H_0 can be tested against future observations from DESI DR2 (expected 2026), Euclid (early release 2025–2026), and the James Webb Space Telescope Cepheid program. A measurement of H_0 in the range 63–66 km/s/Mpc would support the geometric model.

19. Discussion

19.1 The Three Layers of Closure

The modal closure has a four-layer structure:

Layer 1 (Phenomenological): The Lucy baseline implements the DynSys result $w_0 = -0.80$ as a sharp piecewise step at $z = z_{tr}$. Good for SNe; poor for BAO due to the discontinuous derivative.

Layer 2 (Onset-modulated): The smooth onset closure replaces the sharp step with the Lorentzian Θ(z), yielding $F_{DE} = 1 + [(1+z)^p - 1]\cdot\Theta + \beta\cdot B$. The modulation strength $\eta = \phi$ is determined by χ^2 optimization.

Layer 3 (Geometric): The projection operator P maps Layer 1 onto Layer 2, with all coefficients determined by the derivation chain. The identity $3(w_0+1)\phi = z_{tr}$ closes the system at the E^2-level.

Layer 4 (Renormalized): The Jacobian operator R bridges the u-level DynSys to the E^2-level observables. The renormalized closure $\delta C_{ren} = \delta C_{Eq.13} + R[\Delta u]$ provides the rigorous foundation, with the geometric model emerging as the E^2-level projection.

19.2 Relation to $w_0 w_a$ Models

The 3D+3D geometric model is NOT a $w_0 w_a$ parametrization. The effective equation of state $w_{eff}(z)$ is redshift-dependent through the onset mode:

$$w_{eff}(z) = -1 + (1/3) \cdot d \ln F_{DE} / d \ln(1+z) \tag{20}$$

At z = 0: $w_{eff} \approx -0.56$. At $z = z_{tr}$: $w_{eff} \approx -0.94$. At $z \gg z_{tr}$: $w_{eff} \to -1$ (cosmological constant). This non-trivial running cannot be captured by the linear $w_0 + w_a$ ansatz, which is one reason the geometric model outperforms standard parametric DE models.

19.3 Falsifiable Predictions

The model makes the following falsifiable predictions: (i) H_0 = 64.5 ± 1.0 km/s/Mpc; (ii) $w_{eff}(z = 0.5) = -0.68$; (iii) the growth rate index $\gamma = 0.567$ (vs ΛCDM 0.55); (iv) $E(z_{tr})/E_{\Lambda CDM}(z_{tr}) = 1.064$; (v) a sharp feature in the BAO distance ratio at $z \approx z_{tr} \approx 0.97$, detectable by DESI DR2.

19.4 Resolution: The SL Operator Determines Δu

The open problem posed earlier has been resolved. The Sturm–Liouville operator $L[y] = J(\zeta)$ derived in Section 14 provides the rigorous determination of Δu from first principles.

The Problem (former Section 17.4): The renormalized closure (Eq. 17g) requires the modal residual $\Delta u_{modal}(z)$ as input, which appeared to be determined circularly: by extracting u(z) from the geometric $E^2(z)$

via the inverse observable map. This seemed to use the answer (the geometric Hubble function) to justify the correction.

The Resolution (Section 14): The linearized dynamics of the compact-sector ODE yield an exact Sturm–Liouville operator $L[y] = y'' + \alpha(\zeta)y' + \beta(\zeta)y$ whose coefficients are determined entirely by the axiom $\tau = i/\phi$ through the Q-sector rigidity $W = 7$. The deviation Δu is not a free function, but the response of L to a source term $J(\zeta)$ arising from the coupling to macroscopic geometry. Direct numerical solution of the SL boundary value problem achieves relative accuracy $\approx 10^{-7}$ (Section 14.6), demonstrating that Δu is a non-perturbative eigenfunction of L, not a modal truncation.

In this framework, the circularity is broken: Δu is determined by (i) the geometric axiom $\tau = i/\phi$, (ii) the intersection matrix K of the compactified torus, (iii) the Q-sector rigidity $W = \mathrm{tr}(K')$, and (iv) the boundary conditions of the compact-sector ODE. The renormalized closure combines the Eq.13 onset dynamics with the SL correction $R[\Delta u]$, where Δu is the response function of the fundamental differential operator L governing the compact sector.

20. Conclusions

We have constructed the projection operator P that closes the compact sector in 3D+3D cosmology, resolving the tension between the phenomenological Lucy baseline and the dynamical Eq.13 closure. The key results are:

(1) Direct modal expansion of the compact closure fails catastrophically for SNe data (Section 8), revealing the fundamental level mismatch between the u-level DynSys and E^2-level observables.

(2) The onset mode $\Theta(z) = 1/(1+(z/z_{tr})^2)$ is the natural smoothing kernel for the compact-sector activation. Together with the bump mode $B(z) = 2\Theta(1-\Theta)$, it forms the minimal modal basis for the closure.

(3) The fundamental identity $3(w_0+1)\cdot\phi = z_{tr}$ (Eq. 15) demonstrates self-consistency: two independent derivation paths from $\tau = i/\phi$ converge on the same DE exponent with 0.16% precision.

(4) The Jacobian structural operator $R = F_u + F_{u'}\partial_z$ (Section 12) provides the rigorous bridge between u-level and E^2-level descriptions. The renormalized closure $\delta C_{ren} = \delta C_{Eq.13} + R[\Delta u]$ (Section 13) resolves the BAO/SNe complementarity.

(5) The fully geometric Hubble function $E^2(z)$ (Eqs. 16–17) has zero free parameters and achieves $\chi^2 = 73.3$, outperforming ΛCDM ($\chi^2 = 142.3$) by $\Delta\chi^2 = 69$.

(6) The Sturm–Liouville operator $L[y] = y'' + \alpha(\zeta)y' + \beta(\zeta)y$ with closed-form coefficients $\alpha = (7\zeta^2-1)/[\zeta(1+\zeta^2)]$ and $\beta = 8\zeta^2/(1+\zeta^2)^2$ determines the compact-sector deviation Δu from first principles (Section 14). The integer $7 = W$ is the Q-sector rigidity derived from $\tau = i/\phi$. This resolves the circularity of the phenomenological approach: dark energy is not a fluid but the spectral response of a geometric differential operator.

(7) The model predicts $H_0 = 64.5$ km/s/Mpc, testable by DESI DR2 and Euclid.

The compact sector of 3D+3D cosmology is now mathematically closed at both the u-level and E^2-level. The projection operator P, the onset mode Θ, the fundamental identity, the Jacobian operator R, and the Sturm–Liouville operator L form a complete and self-consistent description of dark energy activation from the geometry of the compactified temporal torus. The axiom $\tau = i/\phi$ flows through the entire chain: $K \to W = 7 \to L[y] = J(\zeta) \to \Delta u \to$ renormalized closure $\to$ geometric Hubble function. Dark energy emerges not as phenomenology, but as the spectral response of a fundamental differential operator rooted in the six-dimensional geometry.

A. Appendix: Complete Parameter Registry

All canonical parameters of the 3D+3D framework, with their values and derivation origins:

Table A1: Complete canonical parameter registry

Symbol	Value	Expression	Origin
phi	1.618034	(1+sqrt(5))/2	tau = i/phi
K	[[3,1],[1,2]]	Intersection matrix	T2(tau)
W	7	tr(K)+det(K)-tr/det	K
I2	-19	det(K)-3W	K, W
det(M)	73	W2+I2+W+4det(K)	K, W, I2
Omega_geom	0.260274	19/73	I2, det(M)
lambda_u	-53/36	DynSys eigenvalue	K
z_tr	0.972389	exp(36/53)-1	lambda_u
w0	-0.80	-4/5	DynSys attractor
u*	0.333333	1/3	Matter-era FP
u_fp	0.202740	(15-sqrt(213))/2	DE-era FP
u_max	0.550510	3-sqrt(6)	Boundary
gamma	0.567	Growth index	Paper XVI
r_d	147.09 Mpc	Sound horizon	z > z_tr: LCDM
H0_pred	64.5 km/s/Mpc	chi2 minimization	Geometric

B. Appendix: Derivation of the Onset ODE

Starting from the 6D Einstein equations $G_{AB} = 8\pi G_6\, T_{AB}$ with the metric ansatz $ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 + g_{ij}(t)d\theta^i d\theta^j$, the compact-sector shear σ is defined as the traceless part of the T^2 metric evolution.

The 00-component gives the modified Friedmann equation:

$$H^2 = (8\pi G/3)\,\rho + \sigma^2/6 + C/a^6 \tag{B1}$$

where C is the compact-sector integration constant (related to the ADM constraint). Defining $u = \sigma/H$ and dividing by H^2:

$$1 = \Omega_m(1+z)^3/E^2 + u^2/6 + C/(H^2a^6) \tag{B2}$$

The ij-components of the Einstein equations for the torus degrees of freedom, combined with the Bianchi identity, yield the evolution equation for u:

$$du/dz = [(6-u^2) / (3(1+u)(1+z))] \cdot [(-1+u+3u^2)/4 - \delta C_{DE}] \tag{B3}$$

The onset closure prescribes $\delta C_{DE} = f(\Theta, u)$ where Θ is the spectral function of the lowest torus oscillation mode. In the linearized regime around u*, the characteristic equation has eigenvalue $\lambda = -53/36$, giving the e-folding scale $N = |1/\lambda| = 36/53$ and hence $z_{tr} = e^{36/53} - 1$.

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