Dual Derivation and Formal Independence of the Golden Ratio in Six-Dimensional Geometry

Structural Fixed Points of the 3D+3D Core

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Abstract

We demonstrate that the golden ratio $= (1+\sqrt{5})/2$ emerges independently from two distinct mathematical structures within a six-dimensional spacetime framework with signature (3,3). The first derivation (NT) proceeds through algebraic number theory: requiring Complex Multiplication on the compactification torus T^2 with discriminant $\Delta = D - 1 = 5$ uniquely selects the quadratic field $Q(\sqrt{5})$, whose fundamental unit is . The second derivation (INF) proceeds through convex optimization: minimizing any strictly convex f-divergence for SO(3,3) temporal mixing with isotropy target $q = 1/D$ yields $\sinh^2$ * $= 1/(D-2) = 1/4$, giving e^{ *} = . We prove that these derivations are **formally independent**: neither implies the other without additional axioms. We construct explicit counter-models demonstrating this separation. Under an optional bridge axiom = ie^{− }, the two derivations converge to the same temporal fixed point without logical collapse. The dual emergence of from independent mathematical structures strengthens the internal consistency of the six-dimensional framework and suggests that the golden ratio is a **structural attractor** of (3,3) geometry.

Keywords: Golden ratio, six-dimensional spacetime, Complex Multiplication, f-divergence, formal independence, SO(3,3), modular parameter, discriminant theorem

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1. Introduction

1.1 The Golden Ratio in Fundamental Physics

The golden ratio = (1+√5)/2 1.618 appears throughout mathematics, from Fibonacci sequences to Penrose tilings. Its appearance in fundamental physics, however, has historically been viewed with skepticism, often dismissed as numerology.

In this paper, we demonstrate that emerges **necessarily** from the geometric structure of six-dimensional spacetime with signature (3,3), through **two independent mathematical pathways**. This dual derivation transforms from a suspicious coincidence into a structural consequence of the geometry.

1.2 The Two Derivation Pathways

We establish two logically distinct routes to :

Pathway NT (Number-Theoretic): - Complex Multiplication on the temporal torus T^2 - Discriminant principle $\Delta = D - 1$ - Uniqueness of $Q(\sqrt{5})$ at $\Delta = 5$ - Fundamental unit

Pathway INF (Informational): - SO(3,3) hyperbolic mixing - Convex divergence minimization - Isotropy principle $q = 1/D$ - Fixed point e^{ *} =

1.3 Formal Independence

We prove that these pathways are **formally independent**: neither can be derived from the other without introducing additional axioms. This is established through:

1. **Separation of mathematical languages:** NT uses algebraic number theory; INF uses convex optimization
2. **Counter-models:** We construct scenarios where one pathway holds while the other fails

1.4 The Bridge Axiom

Despite their independence, the two derivations describe the **same physical object** — the temporal sector of the 6D spacetime. We introduce an optional bridge axiom = ie^{− } that establishes **compatibility** without collapsing independence.

1.5 Significance

The dual derivation of from independent mathematical structures suggests that the golden ratio is not accidental but rather a **structural attractor** of (3,3) geometry. This strengthens the internal consistency of the 3D+3D framework and provides a firm mathematical foundation for subsequent physical predictions.

1.6 Structure of This Paper

- Section 2: Establishes the geometric core (D = 6, signature (3,3))
- Section 3: Presents the number-theoretic derivation
- Section 4: Presents the informational derivation
- Section 5: Proves formal independence
- Section 6: Constructs counter-models
- Section 7: Introduces the bridge axiom
- Section 8: Discusses physical implications
- Sections 9-10: Discussion and conclusions

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2. Geometric Core: The Six-Dimensional Framework

2.1 Foundational Principles

We assume four physical principles that constrain the spacetime structure:

Principle 1 (Causality): Physics requires at least one temporal dimension.

Principle 2 (Stable Compactification): Dimensional reduction requires a compact manifold; minimally, a torus T^2 with two compactifiable dimensions.

Principle 3 (Signature Symmetry): The number of spatial and temporal dimensions are equal:

$$N_{\text{space}} = N_{\text{time}}$$

Principle 4 (Modularity): The compactification torus T^2 must admit Complex Multiplication (CM) for automatic modularity of the partition function.

2.2 Derivation of D = 6

Theorem 2.1 (Dimensional Uniqueness): The minimal spacetime dimension satisfying all four principles is D = 6.

Proof:

For a torus with CM, the modular parameter = iy must satisfy:

$$y + \frac{1}{y} = \sqrt{\Delta}$$

where Δ is the discriminant of the associated quadratic field. With the discriminant principle $\Delta = D - 1$:

Step 1: Real solutions require:

$$\sqrt{\Delta} \geq 2 \implies \Delta \geq 4 \implies D - 1 \geq 4 \implies D \geq 5$$

Step 2: Signature symmetry (Principle 3) requires D even.

Step 3: The minimal even integer 5 is D = 6.

2.3 The Metric Signature

With D = 6 and N_space = N_time = 3, the metric signature is:

$$(-, +, +, +, -, -)$$

corresponding to: - 1 observable time (−) - 3 observable space (+, +, +) - 2 compact temporal dimensions (−, −)

2.4 The Discriminant

With D = 6:

$$\boxed{\Delta = D - 1 = 5}$$

This discriminant will be central to both derivations.

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3. Derivation I: Number-Theoretic Fixed Point (NT)

3.1 Axiomatic Structure

The number-theoretic derivation proceeds from the following assumptions:

(NT1) D = 6 (from Section 2)

(NT2) The compactification torus T^2 admits Complex Multiplication (CM)

(NT3) The discriminant of the CM field equals Δ = D − 1 = 5

(NT4) CM uniqueness: The quadratic field with discriminant Δ is unique

3.2 The Discriminant Theorem

Theorem 3.1 (Discriminant Theorem): The quadratic field Q(√d) with discriminant Δ = 5 is unique: Q(√5).

Proof:

The discriminant of Q(√d) with d squarefree is:

$$\Delta = \begin{cases} d & \text{if } d \equiv 1 \pmod 4 \\ 4d & \text{if } d \equiv 2, 3 \pmod 4 \end{cases}$$

For Δ = 5:

Case 1: d 1 (mod 4) Then Δ = d = 5. Since 5 1 (mod 4), this is valid. Solution: d = 5.

Case 2: d 2, 3 (mod 4) Then Δ = 4d = 5, giving d = 5/4 . No solution.

Verification table:

d	d mod 4	Δ	Status
2	2	8	
3	3	12	
5	**1**	**5**	
6	2	24	
7	3	28	

Conclusion: Q(√5) is the unique quadratic field with Δ = 5.

3.3 The Fundamental Unit

Definition 3.2: The fundamental unit of a real quadratic field Q(√d) is the smallest unit > 1 in the ring of integers.

Theorem 3.3: The fundamental unit of Q(√5) is:

$$\boxed{\varepsilon = \varphi = \frac{1+\sqrt{5}}{2}}$$

Proof:

The ring of integers of Q(√5) is [(1+√5)/2] since 5 1 (mod 4).

A unit = a + b (where = (1+√5)/2) satisfies N( ) = ±1, i.e.:

$$a^2 + ab - b^2 = \pm 1$$

The smallest solution with > 1 is a = 0, b = 1:

$$\varepsilon = \omega = \frac{1+\sqrt{5}}{2} = \varphi$$

Verification: N( ) = · (1−) = · (−1/) = −1.

3.4 The Modular Parameter

Theorem 3.4: For a CM torus with discriminant Δ = 5, the modular parameter is:

$$\tau = \frac{i}{\varphi}$$

Proof:

The modular equation y + 1/y = √Δ = √5 has solutions:

$$y = \frac{\sqrt{5} \pm 1}{2}$$

giving y = 1.618 and y = 1/ 0.618.

Physical selection: In signature (3,3), both compact dimensions are temporal. A proper hierarchy (observable time > compact times) requires:

$$\mathrm{Im}(\tau) < 1$$

This selects y = 1/ , hence:

$$\tau = \frac{i}{\varphi}$$

3.5 Summary of NT Derivation

$$\boxed{D = 6 \xrightarrow{\Delta = D-1} \Delta = 5 \xrightarrow{\text{unique}} \mathbb{Q}(\sqrt{5}) \xrightarrow{\text{fund. unit}} \varphi \xrightarrow{\text{selection}} \tau = \frac{i}{\varphi}}$$

Key point: This derivation uses only algebraic number theory. No optimization principles, no divergences, no mixing observables.

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4. Derivation II: Informational Fixed Point (INF)

4.1 Axiomatic Structure

The informational derivation proceeds from the following assumptions:

(I1) D = 6 (from Section 2)

(I2) Hyperbolic temporal mixing admits an observable:

$$P(\theta) = \frac{\sinh^2 \theta}{1 + 2\sinh^2 \theta}, \quad \theta \geq 0$$

(I3) The isotropy target is q = 1/D

(I4) The physical optimum is obtained by minimizing any strictly convex f-divergence between Bernoulli(q) and Bernoulli(P( ))

4.2 The Mixing Observable

Lemma 4.1: The function P( ) is continuous, strictly increasing, and maps [0,∞) bijectively onto [0, 1/2).

Proof:

- sinh² is strictly increasing for 0
- The function u → u/(1+2u) is strictly increasing for u 0
- P(0) = 0 and lim_{ →∞} P( ) = 1/2

4.3 The f-Divergence Family

Definition 4.2: For a strictly convex function f with f(1) = 0, the f-divergence between distributions P and Q is:

$$D_f(P\|Q) = \sum_x Q(x) \cdot f\left(\frac{P(x)}{Q(x)}\right)$$

Examples: - KL divergence: f(t) = t ln t - **Reverse KL:** f(t) = −ln t - **^{2} divergence:** f(t) = $(t-1)^2$ - **Hellinger distance:** f(t) = $(\sqrt{t} - 1)^2$ - **Total variation:** f(t) = |t − 1|

4.4 The Isotropization Principle

Principle (Informational Isotropy): The physical vacuum selects the mixing parameter that minimizes the information distance from perfect isotropy.

In D dimensions, perfect isotropy corresponds to:

$$q = \frac{1}{D}$$

For D = 6:

$$q = \frac{1}{6}$$

4.5 The Optimization Theorem

Theorem 4.3 (Unique Informational Optimum): For any strictly convex f-divergence, the function:

$$F(\theta) = D_f(q\|P(\theta))$$

has a unique global minimum at * satisfying P(*) = q.

Proof:

Any strictly convex f-divergence D_f(q p) is strictly convex in p with unique minimum at p = q.

Since P( ) is strictly monotonic and continuous (Lemma 4.1), there exists a unique * such that:

$$P(\theta^*) = q = \frac{1}{D}$$

4.6 Explicit Solution

Solving P(*) = 1/D:

$$\frac{\sinh^2\theta^*}{1 + 2\sinh^2\theta^*} = \frac{1}{D}$$

Cross-multiplying:

$$D\sinh^2\theta^* = 1 + 2\sinh^2\theta^*$$
$$(D-2)\sinh^2\theta^* = 1$$

$$\boxed{\sinh^2\theta^* = \frac{1}{D-2}}$$

4.7 The Six-Dimensional Case

For D = 6:

$$\sinh^2 \theta^* = \frac{1}{4} \implies \sinh \theta^* = \frac{1}{2}$$

Using the inverse hyperbolic sine:

$$\theta^* = \operatorname{arsinh}\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2} + \sqrt{1 + \frac{1}{4}}\right) = \ln\left(\frac{1+\sqrt{5}}{2}\right)$$

Therefore:

$$\boxed{e^{\theta^*} = \varphi}$$

4.8 Universality Theorem

Theorem 4.4 (Universality): The result e^{ *} =   is independent of the choice of f-divergence.

Proof:

For any strictly convex f, the minimum of D_f(q   p) occurs at p = q. Since P( ) is bijective, the optimal * satisfies P(*) = 1/D regardless of f.

4.9 Summary of INF Derivation

$$\boxed{D = 6 \xrightarrow{q=1/D} q = \frac{1}{6} \xrightarrow{\text{min } D_f} P(\theta^*) = \frac{1}{6} \xrightarrow{\text{solve}} \sinh \theta^* = \frac{1}{2} \xrightarrow{\text{exp}} e^{\theta^*} = \varphi}$$

Key point: This derivation uses only convex optimization. No discriminants, no quadratic fields, no CM theory.

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5. Formal Independence of the Two Derivations

5.1 Definition of Independence

Definition 5.1: Two derivation schemes (A) and (B) are **formally independent** if: 1. Neither set of assumptions implies the other 2. Neither conclusion can be derived from the other's premises alone

5.2 Language Separation

Theorem 5.1 (Non-Derivability): The NT derivation does not logically entail the INF derivation, and conversely, unless additional axioms are introduced.

Proof:

Part 1: NT   INF

The NT pathway operates in the language of algebraic number theory: - Primitive objects: Quadratic fields, discriminants, rings of integers, units - Axioms: CM requirement, discriminant principle $\Delta = \mathrm{D} - 1$ - Conclusion: = i/

This language contains no reference to: - Mixing observables P( ) - Divergence functionals D_f - Optimization principles

Therefore, from NT alone, one cannot derive INF.

Part 2: INF   NT

The INF pathway operates in the language of convex optimization: - Primitive objects: Monotone observables, strictly convex functions, divergences - Axioms: Isotropization principle, divergence minimization - Conclusion: e^{ *} =

This language contains no reference to: - Discriminants - Quadratic fields - Complex Multiplication

Therefore, from INF alone, one cannot derive NT.

Conclusion: To derive one from the other requires a bridging axiom connecting number-theoretic structure to variational optimization.

5.3 Formal Statement

Let *{NT} denote derivability from NT axioms, and* {INF} denote derivability from INF axioms.

Corollary 5.2:

$$(\mathrm{NT1–NT4}) \nvdash_{\mathrm{NT}} (\mathrm{I2–I4})$$

$$(\mathrm{I1–I4}) \nvdash_{\mathrm{INF}} (\mathrm{NT2–NT4})$$

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6. Counter-Model Separation

To strengthen the independence claim, we construct explicit counter-models.

6.1 Counter-Model A: NT holds, INF fails

Construction:

Assume D = 6 and the full NT structure (NT1–NT4): - CM requirement holds - Discriminant $\Delta =$ 5 - Field Q(√5) selected -   = i/ follows

Now **modify** the informational sector:

Option A1: Replace P( ) with an alternative observable:

$$\widetilde{P}(\theta) = \frac{\tanh^2 \theta}{1 + \tanh^2 \theta}$$

Option A2: Replace convex minimization with a different variational principle (e.g., entropy maximization, non-convex functional).

Result: Under either modification, the INF derivation breaks. There is no theorem forcing e^{ *} = with the modified assumptions.

Conclusion: NT can hold while INF fails.

6.2 Counter-Model B: INF holds, NT fails

Construction:

Assume D = 6 and the full INF structure (I1–I4): - Observable P( ) = sinh² /(1+2sinh²) - Isotropy target q = 1/6 - Convex minimization - e^{ *} = follows

Now **modify** the number-theoretic sector:

Option B1: Drop the CM requirement. Allow the torus to have arbitrary modular parameter.

Option B2: Change the discriminant principle to Δ = D (instead of D − 1).

Result: Under either modification, the NT derivation breaks. = i/ is not enforced by number theory.

Conclusion: INF can hold while NT fails.

6.3 Summary Table

Counter-Model	NT Status	INF Status	Demonstrates
A	Holds	Fails	NT INF
B	Fails	Holds	INF NT

These counter-models establish that neither derivation is reducible to the other.

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7. Bridge Axiom and Compatibility

7.1 Motivation

The two derivations are formally independent, yet within the 3D+3D framework they describe the **same physical object**: the temporal sector of 6D spacetime.

To express their **compatibility** without collapsing their independence, we introduce an explicit bridge axiom.

7.2 The Bridge Axiom

Bridge Axiom (B): There exists a monotone identification between the modular parameter of the temporal torus and the hyperbolic mixing parameter :

$$\boxed{\tau = i \cdot e^{-\theta}}$$

Equivalently:

$$\theta = -\ln(\mathrm{Im}\ \tau), \quad \mathrm{Im}\ \tau > 0$$

7.3 Remark on Status

Remark 7.1: Axiom (B) is **not** used to derive either NT or INF. It is introduced **only** to relate the two descriptions after they are independently obtained.

7.4 Convergence Theorem

Theorem 7.1 (Convergence): Assume: 1. NT holds, yielding = i/ 2. INF holds, yielding e^{ *} = 3. Bridge axiom (B) holds

Then both derivations define the same fixed point of the temporal sector.

Proof:

From NT: Im( ) = 1/

Applying B:

$$e^{\theta} = \frac{1}{\mathrm{Im}(\tau)} = \frac{1}{1/\varphi} = \varphi$$

Thus = ln .

This coincides with INF, which yields e^{ *} = , hence* = ln .

Therefore, both schemes converge to the same temporal fixed point:

$$\theta^{*} = \ln\varphi, \quad \tau = \frac{i}{\varphi}$$

7.5 Non-Collapse Theorem

Theorem 7.2 (Independence Preserved): The bridge axiom (B) does not render NT and INF derivable from each other.

Proof:

Given (B) alone:

1. (B) does not impose CM constraints or discriminant principles → (B) does not imply = i/ → (B) NT conclusion
2. (B) does not impose convex divergence minimization or isotropy principles → (B) does not imply e^{ *} = → (B) INF conclusion

Therefore (B) establishes **compatibility** (when both hold) but does not enable **cross-derivation**.

7.6 Diagram of Logical Structure

```
NT PATHWAY                            INF PATHWAY
(Number Theory)                       (Information Theory)
```

```
NT1: D = 6                          I1: D = 6
NT2: CM requirement                 I2: P( ) observable
NT3: Δ = D − 1                      I3: q = 1/D
NT4: Field uniqueness               I4: Convex minimization
      ↓                                   ↓
  = i/                             e^{ *} =



         BRIDGE AXIOM (B)
          = i·e^{- }

                 ↓
      CONVERGENCE: Same fixed point
       * = ln  ,   = i/

Key: NT   INF (independent)
     NT   INF   B → Convergence
```

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8. Physical Implications

8.1 The Golden Ratio as Structural Attractor

The dual emergence of   from independent mathematical structures suggests that the golden ratio is not accidental but a **structural attractor** of (3,3) geometry.

Definition 8.1: A quantity x is a **structural attractor** of a geometric framework if it emerges necessarily from multiple independent derivation pathways within that framework.

Proposition 8.1: The golden ratio   is a structural attractor of 6D spacetime with signature (3,3).

8.2 Robustness

The independence of the two derivations provides **robustness**:

- If one pathway were found to be flawed, the other would still stand
- The convergence under the bridge axiom provides mutual consistency check
- No "double counting" — the derivations use different primitives

8.3 Connection to Physical Observables

The golden ratio   and the modular parameter   = i/ determine:

Observable	Dependence	Paper Reference
Proton charge radius	D (hence indirectly)	Paper L1
Neutron charge radius	directly	Paper L1

Observable	Dependence	Paper Reference
Lepton mass ratios	^9, ^17	Paper on Lepton Masses
Fine structure constant	^4 e^3 − 1/	Paper LXXII
CKM CP phase	/ ^{2}	Paper LXXII

8.4 Falsifiability

The framework makes falsifiable predictions:

1. If precision measurements contradict -dependent predictions, the framework is falsified
2. If D 6 is established (e.g., through extra-dimensional signatures), the framework is falsified
3. The counter-models in Section 6 show what **would** falsify each pathway independently

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9. Discussion

9.1 Relation to Previous Work

The number-theoretic derivation connects to classical results on CM tori and modular forms [1,2]. The informational derivation relates to the principle of maximum entropy and f-divergence theory [3,4].

The novel contribution is demonstrating their **formal independence** and **convergence** within a unified geometric framework.

9.2 Potential Objections

Objection 1: "Both derivations use D = 6, so they're not truly independent."

Response: D = 6 is a shared **base assumption**, derived from the geometric core (Section 2). The independence concerns the **mechanism** by which emerges from D = 6, not the starting point.

Objection 2: "The bridge axiom is ad hoc."

Response: The bridge axiom is **optional** and **not used for derivation**. It merely expresses the physical identification that both derivations describe the same temporal sector. One could omit it entirely and still have two independent derivations of .

Objection 3: "This is just mathematical coincidence, not physics."

Response: Whether Nature realizes D = 6 is an empirical question. The mathematical result is rigorous: **if** D = 6, **then** emerges from two independent pathways. This strengthens the theoretical framework by providing multiple self-consistency checks.

9.3 Open Questions

1. Is there a deeper principle unifying NT and INF?
2. Are there other structural attractors in (3,3) geometry?
3. Can the bridge axiom be derived from more fundamental principles?

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10. Conclusion

We have established:

10.1 Main Results

1. **Derivation NT:** emerges from algebraic number theory (CM, discriminant $\Delta = 5$, field Q(√5))
2. **Derivation INF:** emerges from convex optimization (f-divergence minimization, isotropy target 1/D)
3. **Formal Independence:** NT and INF cannot be derived from each other without additional axioms
4. **Counter-Models:** Explicit constructions demonstrate the separation
5. **Bridge Axiom:** Under = ie^{− }, both derivations converge to the same fixed point
6. **Non-Collapse:** The bridge axiom preserves independence while establishing compatibility

10.2 The Golden Ratio as Structural Attractor

The dual emergence of from independent mathematical structures establishes it as a **structural attractor** of 6D spacetime with signature (3,3).

10.3 Final Statement

$$\boxed{\varphi = \frac{1+\sqrt{5}}{2} \text{ emerges from two independent pathways in } D = 6 \text{ geometry}}$$

Whether Nature realizes this six-dimensional structure remains an empirical question. The mathematics itself is rigorous and closed.

Appendix A: Mathematical Background on Quadratic Fields

A.1 Basic Definitions

A **quadratic field** is Q(√d) where d is a squarefree integer.

The **discriminant** Δ of Q(√d) is:

$$\Delta = \begin{cases} d & d \equiv 1 \pmod 4 \\ 4d & d \equiv 2,3 \pmod 4 \end{cases}$$

The **ring of integers** is:

$$\mathcal{O} = \begin{cases} \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & d \equiv 1 \pmod 4 \\ \mathbb{Z}[\sqrt{d}] & d \equiv 2,3 \pmod 4 \end{cases}$$

A.2 Units and the Fundamental Unit

A **unit** in O is an element   with N( ) = ±1.

For real quadratic fields (d > 0), the unit group is:

$$\mathcal{O}^\times = \{\pm\varepsilon^n : n \in \mathbb{Z}\}$$

where   > 1 is the **fundamental unit**.

A.3 The Field Q(√5)

For d = 5: - Discriminant: Δ = 5 (since 5   1 mod 4) - Ring of integers: [(1+√5)/2] - Fundamental unit: = (1+√5)/2

A.4 Complex Multiplication

A torus T² = /Λ has **Complex Multiplication** if End(T²) is larger than . This occurs when the modular parameter   generates a quadratic imaginary field.

For CM at discriminant Δ, the modular parameter satisfies special arithmetic properties related to the class field theory of Q(√−Δ).

Appendix B: Proof of f-Divergence Universality

B.1 Definition

For strictly convex f with f(1) = 0, the f-divergence between Bernoulli(q) and Bernoulli(p) is:

$$D_f(q\|p) = p \cdot f\left(\frac{q}{p}\right) + (1-p) \cdot f\left(\frac{1-q}{1-p}\right)$$

B.2 Minimum Condition

Lemma B.1: D_f(q   p)   0 with equality iff p = q.

Proof: By strict convexity of f and Jensen's inequality.

B.3 Universality

Theorem B.1: The optimal p* minimizing D_f(q   p) is p* = q, independent of f.

Proof: The minimum of any strictly convex divergence occurs at distribution equality.

Corollary B.2: For any f, minimizing D_f(1/D   P( )) over   yields P(*) = 1/D.

Appendix C: Convergence Theorem Under Bridge Axiom

C.1 Setup

- NT yields: = i/ , hence Im( ) = 1/
- INF yields: e^{ } = , *hence*   = ln
- Bridge: = ie^{− }

C.2 Verification

From NT: Im( ) = 1/

From Bridge: Im( ) = e^{− }

Combining: e^{− } = 1/ , hence e^ = , hence = ln

This matches INF: * = ln

C.3 Independence Check

Bridge alone does not determine either or . It only relates them once determined by their respective derivations.

Appendix D: Numerical Verification

```
import numpy as np

# Golden ratio
phi = (1 + np.sqrt(5)) / 2

print("=== NUMERICAL VERIFICATION ===\n")

# NT Derivation
print("--- NT (Number-Theoretic) ---")
Delta = 5
y_plus = (np.sqrt(Delta) + 1) / 2
y_minus = (np.sqrt(Delta) - 1) / 2
print(f"Δ = {Delta}")
print(f"y  = {y_plus:.10f} =  ")
print(f"y  = {y_minus:.10f} = 1/ ")
print(f"  = i/  = i × {1/phi:.10f}")
print()

# INF Derivation
print("--- INF (Informational) ---")
D = 6
sinh2_theta = 1 / (D - 2)
sinh_theta = np.sqrt(sinh2_theta)
theta_star = np.arcsinh(sinh_theta)
exp_theta = np.exp(theta_star)
print(f"D = {D}")
print(f"sinh² * = 1/(D-2) = {sinh2_theta}")
print(f"sinh  * = {sinh_theta}")
print(f" * = {theta_star:.10f}")
print(f"e^ * = {exp_theta:.10f}")
print(f"  = {phi:.10f}")
print(f"Match: {np.isclose(exp_theta, phi)}")
```

```
print()

# Bridge Axiom Verification
print("--- Bridge Axiom Verification ---")
tau_imag = 1 / phi
theta_from_bridge = -np.log(tau_imag)
print(f"Im( ) from NT = {tau_imag:.10f}")
print(f"  from Bridge = -ln(Im( )) = {theta_from_bridge:.10f}")
print(f" * from INF = {theta_star:.10f}")
print(f"Convergence: {np.isclose(theta_from_bridge, theta_star)}")

# Output:
# === NUMERICAL VERIFICATION ===
#
# --- NT (Number-Theoretic) ---
# Δ = 5
# y  = 1.6180339887 =
# y  = 0.6180339887 = 1/
#   = i/  = i × 0.6180339887
#
# --- INF (Informational) ---
# D = 6
# sinh² * = 1/(D-2) = 0.25
# sinh  * = 0.5
#  * = 0.4812118250
# e^* = 1.6180339887
#   = 1.6180339887
# Match: True
#
# --- Bridge Axiom Verification ---
# Im( ) from NT = 0.6180339887
#   from Bridge = -ln(Im( )) = 0.4812118250
#  * from INF = 0.4812118250
# Convergence: True
```

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[6] Calzighetti, S. & Lucy. "Golden-Ratio Attractor from SO(3,3) Geometry." 3D+3D Laboratory (2026).

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DUAL DERIVATION OF φ — FORMALLY INDEPENDENT — CONVERGENT UNDER BRIDGE

"The golden ratio is not a coincidence. It is a structural attractor of six-dimensional geometry."

— S. Calzighetti & Lucy, 2026