PAPER B: DIMENSIONAL REDUCTION THEOREM

Rigorous Projection from 6D to 4D Observable Physics

Series: The 6D Atomic Physics Program on $M^4 \times T^2(\tau = i/\varphi)$

Paper: B of 5 (Technical Paper)

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Related Papers: Paper 0 (Master), Paper A, C, D

Abstract

We establish the rigorous mathematical framework for dimensional reduction from 6D quantum mechanics on $M^4 \times T^2$ to 4D observable physics. We define the projection operator $\Pi: H_{6D} \to H_{4D}$, prove its properties, and derive the effective 4D Hamiltonian for each T^2 mode. This provides the formal foundation for relating T^2 spectral theory to observable atomic quantities.

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1. INTRODUCTION

This paper establishes **Theorem 4** of Paper 0: the dimensional reduction framework.

The key question is: How do 6D quantum states relate to 4D measurements?

We construct the projection Π that maps 6D Hilbert space to the 4D observable subspace, preserving the mode structure that determines ionization energies.

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2. HILBERT SPACE STRUCTURE

2.1 The 6D Hilbert Space

Definition 2.1:

$$\mathcal{H}_{6D} = L^2(\mathbb{R}^3) \otimes L^2(T^2) \otimes \mathbb{C}^2$$

This is the space of square-integrable functions on $M^4 \times T^2$ with spin.

2.2 The 4D Observable Space

Definition 2.2:

$$\mathcal{H}_{4D}^{obs} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$$

This is the standard quantum mechanical Hilbert space for a spin-1/2 particle in 3D.

Physical interpretation: 4D measurement apparatus cannot directly resolve T^2 coordinates. They access only 3D spatial position and spin.

2.3 Tensor Product Decomposition

Theorem 2.3: The 6D Hilbert space decomposes as:

$$\mathcal{H}_{6D} = \bigoplus_{n_2, n_3 \in \mathbb{Z}} \mathcal{H}_{4D}^{obs} \otimes |n_2, n_3\rangle$$

where $|n_2, n_3\rangle$ are T^2 eigenstates.

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3. THE PROJECTION OPERATOR

3.1 Mode Projection

Definition 3.1 (Mode Projection):

For each $(n_2, n_3) \in \mathbb{Z}^2$, define:

$$\Pi_{n_2,n_3} : \mathcal{H}_{6D} \to \mathcal{H}_{4D}^{obs}$$

by:

$$\Pi_{n_2,n_3}[\Psi](\mathbf{r}, \sigma) = \int_{T^2} \bar{\chi}_{n_2,n_3}(\theta) \Psi(\mathbf{r}, \theta, \sigma) \frac{d^2\theta}{4\pi^2}$$

3.2 Properties

Theorem 3.2 (Projection Properties):

(a) Boundedness: $\|\Pi_{n_2,n_3}\| = 1$

(b) Mode selectivity: For $|\Psi\rangle = |\psi_{3D}\rangle \otimes |n_2', n_3'\rangle \otimes |\sigma\rangle$:

$$\Pi_{n_2,n_3}[|\Psi\rangle] = \delta_{n_2,n_2'}\delta_{n_3,n_3'} \cdot |\psi_{3D}\rangle \otimes |\sigma\rangle$$

(c) Orthogonality:

$$\Pi_{n_2,n_3} \circ \Pi^*_{m_2,m_3} = \delta_{n_2,m_2}\delta_{n_3,m_3} \cdot \Pi_{n_2,n_3}$$

Proof:

(a) By Cauchy-Schwarz and orthonormality of T^2 modes.

(b) Direct computation:

$$\int_{T^2} \bar{\chi}_{n_2,n_3}\chi_{n_2',n_3'} \frac{d^2\theta}{4\pi^2} = \delta_{n_2,n_2'}\delta_{n_3,n_3'}$$

(c) Follows from (b). ■

3.3 Completeness

Theorem 3.3 (Completeness):

$$\sum_{n_2,n_3} \Pi^*_{n_2,n_3}\Pi_{n_2,n_3} = 1_{\mathcal{H}_{6D}}$$

Proof: Completeness of Fourier modes on T^2. ■

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4. THE ADJOINT OPERATOR

4.1 Definition

Definition 4.1 (Embedding):

$$\Pi^*_{n_2,n_3} : \mathcal{H}^{obs}_{4D} \to \mathcal{H}_{6D}$$

$$\Pi^*_{n_2,n_3}[|\psi\rangle \otimes |\sigma\rangle] = |\psi\rangle \otimes |n_2, n_3\rangle \otimes |\sigma\rangle$$

4.2 Adjointness

Theorem 4.2:

$$\langle\phi|\Pi_{n_2,n_3}|\Psi\rangle_{4D} = \langle\Pi^*_{n_2,n_3}\phi|\Psi\rangle_{6D}$$

for all $|\phi\rangle \in \mathcal{H}_{4D}^{obs}$ and $|\Psi\rangle \in \mathcal{H}_{6D}$.

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5. EFFECTIVE 4D HAMILTONIAN

5.1 The 6D Hamiltonian

Definition 5.1:

$$\hat{H}_{6D} = \hat{H}_{3D} \otimes 1_{T^2} + 1_{3D} \otimes \hat{H}_{T^2}$$

where:

- $\hat{H}_{3D} = -\frac{\hbar^2}{2m_e}\nabla^2 + V(r)$
- $\hat{H}_{T^2} = -\frac{\hbar^2}{2m_e}\Delta_{T^2}$

5.2 Dimensional Reduction

Theorem 5.2 (Effective Hamiltonian):

The effective 4D Hamiltonian for mode (n_2, n_3) is:

$$\hat{H}_{eff}^{(n_2,n_3)} = \Pi_{n_2,n_3}\hat{H}_{6D}\Pi_{n_2,n_3}^* = \hat{H}_{3D} + E_{T^2}(n_2, n_3)$$

where:

$$E_{T^2}(n_2, n_3) = \frac{\hbar^2}{2m_e}\left(\frac{n_2^2}{R_2^2} + \frac{n_3^2}{R_3^2}\right)$$

Proof:

For $|\psi\rangle \in \mathcal{H}_{4D}^{obs}$:

$$\hat{H}_{6D}\Pi_{n_2,n_3}^*|\psi\rangle = \hat{H}_{6D}(|\psi\rangle \otimes |n_2, n_3\rangle)$$

$$= (\hat{H}_{3D}|\psi\rangle) \otimes |n_2, n_3\rangle + |\psi\rangle \otimes (\hat{H}_{T^2}|n_2, n_3\rangle)$$

$$= (\hat{H}_{3D}|\psi\rangle) \otimes |n_2, n_3\rangle + E_{T^2}(n_2, n_3)|\psi\rangle \otimes |n_2, n_3\rangle$$

Projecting with Π_{n_2,n_3}:

$$\Pi_{n_2,n_3}\hat{H}_{6D}\Pi_{n_2,n_3}^*|\psi\rangle = (\hat{H}_{3D} + E_{T^2})|\psi\rangle$$

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5.3 Energy Scale

Numerical estimate:

$$E_{T^2}^{(0)} = \frac{\hbar^2}{2m_e R_3^2} \approx 10^{-53} \text{ eV}$$

This is **completely negligible** compared to atomic energies (~eV).

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6. OBSERVABLE QUANTITIES

6.1 Expectation Values

Theorem 6.1: For an observable $\hat{O}$ acting on $\mathcal{H}_{3D}$ and a 6D state in mode (n_2, n_3):

$$\langle \hat{O} \rangle_{6D} = \langle \hat{O} \rangle_{4D}^{(n_2, n_3)}$$

The T² mode enters only through mode-dependent parameters.

6.2 The Coupling Observation

Key result (from Paper D): While E_{T^2} is negligible, the T² mode affects the **electromagnetic coupling** through the circulation mechanism:

$$\alpha_{eff}(n_2, n_3) = \alpha \cdot \tilde{\lambda}(n_2, n_3)$$

This is where the mode-dependence of ionization energies originates.

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7. TRANSITION AMPLITUDES

7.1 Mode Conservation

Theorem 7.1 (Mode Conservation):

For electromagnetic transitions mediated by 4D photons:

$$(n_2, n_3)_{final} = (n_2, n_3)_{initial}$$

Proof: The electromagnetic vertex $e\bar{\psi}\gamma^\mu\psi A_\mu$ is diagonal in T² modes (see Paper D). ■

7.2 Selection Rules

Corollary 7.2: Atomic transitions conserve T² mode numbers. Different modes represent different "sectors" that do not mix under electromagnetic interactions.

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8. SYMMETRIES

8.1 T² Symmetries

Theorem 8.1: The T² sector has symmetries:

(a) Translation: $\theta_a \to \theta_a + c$ (periodic boundary conditions)

(b) Reflection: $\theta_a \to -\theta_a$ (orientation reversal)

(c) Exchange: $\theta_4 \leftrightarrow \theta_5$ (broken by $R_2 \neq R_3$)

8.2 Degeneracies

Theorem 8.2: Modes (n_2, n_3) and $(n_2, -n_3)$ have identical eigenvalues:

$$\tilde{\lambda}(n_2, n_3) = \tilde{\lambda}(n_2, -n_3)$$

This degeneracy corresponds to the $\mathbb{Z}_2$ reflection symmetry.

Physical consequence: Observable quantities (IE, transition rates) are independent of the sign of n_3.

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9. RECOVERY OF STANDARD QM

9.1 Decompactification Limit

Theorem 9.1: In the limit $R_2, R_3 \to \infty$:

(a) T² modes become degenerate: $E_{T^2}(n_2, n_3) \to 0$ for all modes

(b) Mode selection becomes irrelevant

(c) Standard 4D quantum mechanics is recovered exactly

9.2 Physical Interpretation

The 6D framework **contains** standard QM as a limiting case. It provides additional structure (mode selection) that explains properties (like IE values) that are unexplained in 4D.

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10. SUMMARY

Main Result (Theorem 4 of Paper 0):

The projection $\Pi : \mathcal{H}_{6D} \to \mathcal{H}_{4D}^{obs}$ satisfies:

1. **Well-defined:** Bounded linear operator
2. **Mode-selective:** Projects onto specific T^2 sectors
3. **Complete:** Preserves all information through mode decomposition
4. **Physical:** Effective Hamiltonian $\hat{H}_{eff} = \hat{H}_{3D} + E_{T^2}$

Combined with the circulation coupling (Paper D), this establishes:

$$IE = \mathrm{Ry} \times \tilde{\lambda}(n_2, n_3)$$

END OF PAPER B

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