PAPER C: EFFECTIVE COULOMB COUPLING

6D Electromagnetic Theory and 4D Reduction

Series: The 6D Atomic Physics Program on $M^4 \times T^2(\tau = i/\varphi)$

Paper: C of 5 (Technical Paper)

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Related Papers: Paper 0 (Master), Paper A, B, D

Abstract

We analyze the 6D electromagnetic theory on $M^4 \times T^2$ and perform dimensional reduction to derive the effective 4D Coulomb potential. We examine the 6D Green's function, Kaluza-Klein mode expansion, and the structure of electron-photon coupling. This paper provides the field-theoretic foundation for the mode-dependent coupling derived in Paper D.

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1. INTRODUCTION

This paper establishes the electromagnetic field theory framework underlying Papers A-D. We analyze:

1. The 6D Maxwell equations on $M^4 \times T^2$
2. Kaluza-Klein decomposition of the gauge field
3. The 6D Green's function and Coulomb potential
4. Dimensional reduction to 4D effective theory

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2. 6D ELECTROMAGNETISM

2.1 The 6D Maxwell Lagrangian

Definition 2.1:

$$\mathcal{L}^{(6D)}_{Maxwell} = -\frac{1}{4g_6^2} F_{MN} F^{MN}$$

where:

- $F_{MN} = \partial_M A_N - \partial_N A_M$
- $M, N = 0, 1, 2, 3, 4, 5$
- g_6 is the 6D gauge coupling with $[g_6] = \text{length}$

2.2 The 6D Gauge Field

Decomposition 2.2:

$$A_M = (A_\mu, A_a)$$

where:

- A_μ (μ = 0,1,2,3): 4D gauge field components
- A_a (a = 4,5): T^2 components (scalars from 4D viewpoint)

2.3 Field Strength Components

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{(4D field strength)}$$

$$F_{\mu a} = \partial_\mu A_a - \partial_a A_\mu \quad \text{(KK mixing)}$$

$$F_{ab} = \partial_a A_b - \partial_b A_a \quad \text{(internal flux)}$$

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3. KALUZA-KLEIN DECOMPOSITION

3.1 Mode Expansion

Definition 3.1: The gauge field expands as:

$$A_M(x, \theta) = \sum_{n_2, n_3 \in \mathbb{Z}} A^{(n)}_M(x) \cdot \chi_{n_2, n_3}(\theta)$$

where $\chi_n(\theta) = e^{i(n_2\theta_4 + n_3\theta_5)}/(2\pi)$.

3.2 Zero Mode

Definition 3.2: The zero mode $(n_2, n_3) = (0, 0)$ gives the **4D photon**:

$$A_\mu^{(0,0)}(x) = \frac{1}{V_{T^2}} \int_{T^2} A_\mu(x, \theta) d^2\theta$$

3.3 Massive KK Modes

Theorem 3.3: For $(n_2, n_3) \neq (0, 0)$, the mode $A_\mu^{(n)}$ has mass:

$$m_{KK}^2(n) = \frac{n_2^2}{R_2^2} + \frac{n_3^2}{R_3^2}$$

3.4 The 4D Gauge Coupling

Theorem 3.4: The 4D gauge coupling is:

$$g_4 = \frac{g_6}{\sqrt{V_{T^2}}} = \frac{g_6}{2\pi\sqrt{R_2 R_3}}$$

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4. THE 6D GREEN'S FUNCTION

4.1 Definition

Definition 4.1: The 6D Green's function satisfies:

$$\nabla_{6D}^2 G_{6D}(x, \theta; x', \theta') = -\delta^{(4)}(x - x')\delta^{(2)}(\theta - \theta')$$

4.2 Mode Expansion

Theorem 4.2:

$$G_{6D}(x, \theta; x', \theta') = \sum_{n_2, n_3} G_n^{(4D)}(x - x') \cdot \chi_n(\theta)\bar{\chi}_n(\theta')$$

where $G_n^{(4D)}$ satisfies:

$$(\nabla_{4D}^2 - m_{KK}^2(n))G_n^{(4D)}(x) = -\delta^{(4)}(x)$$

4.3 Solutions

(a) Zero mode $(0, 0)$:

$$G_0^{(4D)}(r) = \frac{1}{4\pi^2 r^2} \quad \text{(massless, 4D Coulomb)}$$

(b) Massive modes:

$$G_n^{(4D)}(r) \propto \frac{e^{-m_{KK}|x|}}{|x|} \quad \text{(Yukawa, short-range)}$$

4.4 Long-Range Dominance

Theorem 4.4: At atomic scales $r \sim a_0 \ll R_3$:

$$G_{6D} \approx G_0^{(4D)} \cdot \frac{1}{V_{T^2}}$$

Only the zero mode contributes to long-range physics.

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5. THE 4D COULOMB POTENTIAL

5.1 Derivation

Theorem 5.1: The effective 4D Coulomb potential is:

$$V_{Coulomb}^{(4D)}(r) = -\frac{e_4^2}{4\pi\varepsilon_0 r}$$

where $e_4 = e_6/\sqrt{V_{T^2}}$ is the 4D charge.

5.2 Mode Independence (Naive)

Observation 5.2: The naive KK reduction gives a mode-independent Coulomb potential!

This appears to contradict the observed mode-dependence of IE.

5.3 Resolution

The mode-dependence enters through the **current structure**, not the potential. See Paper D for the complete derivation via T^2 circulation.

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6. ELECTRON-PHOTON COUPLING

6.1 The 6D Coupling

Definition 6.1: The 6D QED interaction is:

$$\mathcal{L}_{int}^{(6D)} = e_6 \bar{\Psi} \Gamma^M \Psi A_M$$

6.2 Decomposition

$$\mathcal{L}_{int}^{(6D)} = e_6 \bar{\Psi} \gamma^\mu \Psi A_\mu + e_6 \bar{\Psi} \Gamma^a \Psi A_a$$

The first term is the 4D electromagnetic coupling.
The second term couples to T^2 gauge components.

6.3 4D Effective Coupling

Theorem 6.3: For zero-mode photon and electron in mode (n^e):

The overlap integral:

$$\int_{T^2} |\chi_{n^e}|^2 \chi_{0,0} d^2\theta = 1$$

gives 4D coupling:

$$\mathcal{L}_{int}^{(4D)} = e_4 \bar{\psi} \gamma^\mu \psi A_\mu^{(0,0)}$$

Note: This is mode-independent! The mode-dependence arises from the T^2 current structure (Paper D).

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7. THE T^2 CURRENT

7.1 Definition

Definition 7.1: The T^2 electromagnetic current is:

$$J^a = e_6 \bar{\Psi} \Gamma^a \Psi$$

7.2 Mode Structure

Theorem 7.2: For an electron in mode (n_2, n_3):

$$\langle J^a \rangle = e_6 |\psi|^2 \cdot \frac{n_a}{R_a}$$

The T² current is proportional to the mode numbers!

7.3 Current Magnitude

$$|J_{T^2}|^2 = e_6^2 |\psi|^4 \left(\frac{n_2^2}{R_2^2} + \frac{n_3^2}{R_3^2} \right) = \frac{e_6^2 |\psi|^4}{R_3^2} \tilde{\lambda}(n_2, n_3)$$

This is where $\tilde{\lambda}$ enters the physics!

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8. CONNECTION TO PAPER D

8.1 The Key Insight

Paper D shows that the T² current determines the effective electromagnetic coupling:

$$e_{eff}(n_2, n_3) = e_4 \cdot \sqrt{\tilde{\lambda}(n_2, n_3)}$$

8.2 The Chain of Logic

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6D QED Lagrangian
     ↓
T² current: J^a ∝ n_a/R_a
     ↓
Current magnitude: |J| ∝ √λ̃
     ↓
Effective coupling: e_eff = e₄√λ̃
     ↓
Effective α: α_eff = α × λ̃
     ↓
Ionization energy: IE = Ry × λ̃
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8.3 Physical Picture

An electron "circulating" on T² generates a current. This current couples to the electromagnetic field. Faster circulation (higher mode) means stronger coupling.

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9. SUMMARY

Key Results:

1. **6D Maxwell theory** reduces to 4D plus massive KK modes
2. **Zero-mode photon** mediates long-range Coulomb interaction
3. **Naive reduction** gives mode-independent coupling
4. **T^2 current structure** introduces mode-dependence through $|J| \propto \sqrt{\tilde{\lambda}}$
5. **Final result:** $\alpha_{eff} = \alpha \cdot \tilde{\lambda}$, hence IE = Ry × $\tilde{\lambda}$

This paper provides the field-theoretic foundation. Paper D completes the derivation by analyzing the circulation mechanism.

END OF PAPER C

Theory origin: September 14, 2025
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