A Note on the Approximate Relation $1/\alpha \approx \pi + \pi^2 + 4\pi^3$ and the Volumes of Complex Projective Spaces

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Abstract. The well-known approximation $1/\alpha \approx \pi + \pi^2 + 4\pi^3$ (accurate to 2.2 ppm, ruled out as exact at $> 27{,}000\sigma$) can be rewritten as

$$\frac{1}{\alpha} \approx \sum_{n=1}^{3} (2n-2)!\, \frac{\pi^n}{n!} = \sum_{n=1}^{3} (\dim_{\mathbb{R}} \mathbb{CP}^{n-1})! \;\times\; \mathrm{Vol}(\mathbb{CP}^n)$$

where $\mathrm{Vol}(\mathbb{CP}^n) = \pi^n/n!$ is the Fubini-Study volume of complex projective n-space. The three terms correspond to the subspaces in the natural inclusion $\mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \mathbb{CP}^3$, each weighted by the factorial of the real dimension of the previous subspace. The sum has exactly three terms because Penrose's twistor space $\mathbb{CP}^3$ has three complex dimensions. We present this as an observation about an approximate numerical identity, not a derivation. The formula is experimentally excluded as exact, and the weights are reverse-engineered from the known coefficients. The observation is motivated by recently reported empirical formulas expressing both Newton's constant G and the cosmological constant Λ in terms of α and the electron Yukawa coupling: if α is determined by the geometry of $\mathbb{CP}^3$, then G and Λ would also follow from twistor geometry. We note the connection to Penrose's twistor program and to Atiyah's unsuccessful 2018 attempt to derive α from related geometric structures.

Keywords. fine-structure constant, complex projective space, twistor space, numerical coincidence

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1. The Formula

The approximation

$$\frac{1}{\alpha} \approx \pi + \pi^2 + 4\pi^3 = 137.0363\ldots \tag{1}$$

has been known for decades. Sherbon [2012] connected it to Pauli's "World Clock" dream and formalized it in terms of three geometric layers. It appears in various catalogs of π-based approximations to α and is mentioned in the Wikipedia article on the number 137.

The formula matches the observed $1/\alpha = 137.035\,999\,206(11)$ to 2.2 parts per million. Despite this accuracy, it is **ruled out as exact** at more than 27,000 standard deviations, given the 81 parts-per-trillion precision of the most recent measurement [Morel et al. 2020; see also Parker et al. 2018].

We do not dispute this. Our observation is about the *form* of the approximation, not its exactness.

2. Rewriting in Terms of $\mathbb{CP}^n$ Volumes

The volume of complex projective space $\mathbb{CP}^n$ in the Fubini-Study metric is

$$\mathrm{Vol}(\mathbb{CP}^n) = \frac{\pi^n}{n!} \tag{2}$$

The three terms in Eq. (1) can be written:

n	Weight w_n	$\mathrm{Vol}(\mathbb{CP}^n)$	$w_n \cdot \mathrm{Vol}$	
1	$0! = 1$	$\pi/1! = \pi$	π	
2	$2! = 2$	$\pi^2/2!$	π^2	
3	$4! = 24$	$\pi^3/3!$	$4\pi^3$	
Sum			137.0363	$\approx 1/\alpha$

The weights follow the pattern $w_n = (2n-2)!$, which equals the factorial of the real dimension of the previous projective space in the inclusion chain:

$$w_n = (2n-2)! = \Big(\dim_{\mathbb{R}} \mathbb{CP}^{n-1}\Big)! \tag{3}$$

since $\dim_{\mathbb{R}} \mathbb{CP}^k = 2k$. The formula becomes:

$$\boxed{\frac{1}{\alpha} \approx \sum_{n=1}^{3} \Big(\dim_{\mathbb{R}} \mathbb{CP}^{n-1}\Big)! \times \mathrm{Vol}(\mathbb{CP}^n)} \tag{4}$$

We stress that Eq. (4) is a *rewriting* of the known coefficients (1, 1, 4) in terms of volumes and factorials. Given three small integers, many factorial or combinatorial patterns could be fitted. We single out this one because of the connection to twistor space discussed in Section 3.

3. The Truncation and Twistor Space

Penrose's twistor space [1967] is $\mathbb{CP}^3$, a complex manifold of dimension 3. The natural inclusion chain

$$\mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \mathbb{CP}^3$$

provides three nontrivial subspaces ($n = 1, 2, 3$). The sum in Eq. (4) runs over these subspaces. (We use "subspaces in the inclusion chain" rather than "strata" to avoid confusion with the standard stratification of $\mathbb{CP}^3$, whose strata are the open complements $\mathbb{C}^n \setminus \mathbb{C}^{n-1}$ rather than the $\mathbb{CP}^n$ themselves.)

If one attempts to extend the sum to $n = 4$:

$$w_4 \cdot \mathrm{Vol}(\mathbb{CP}^4) = 6!\,\frac{\pi^4}{4!} = 720 \times 4.06 = 2922$$

The fourth term alone exceeds the target by a factor of 20. The sum **must** truncate at $n = 3$. If the formula has a geometric origin, this truncation is natural: twistor space is $\mathbb{CP}^3$, not $\mathbb{CP}^4$. The number of terms in the approximation would equal the complex dimension of twistor space.

4. Interpretation of the Weights

Each weight $w_n = (\dim_{\mathbb{R}} \mathbb{CP}^{n-1})!$ counts the number of permutations of the real coordinates of the previous subspace:

- $n = 1$: $\mathbb{CP}^0$ is a point (0 real dimensions) $\to 0! = 1$ permutation
- $n = 2$: $\mathbb{CP}^1 \cong S^2$ has 2 real dimensions $\to 2! = 2$ permutations
- $n = 3$: $\mathbb{CP}^2$ has 4 real dimensions $\to 4! = 24$ permutations

A possible reading: each contribution to $1/\alpha$ involves the volume of a $\mathbb{CP}^n$ subspace, weighted by the combinatorial complexity of the previous subspace in the inclusion chain.

We do not have a first-principles derivation of why this particular weighting arises. In a path-integral formulation on $\mathbb{CP}^3$, the factorials might emerge from the integration measure — but this remains speculation.

5. Relation to Atiyah's Program

Atiyah [2018] attempted to derive α from the geometry of related mathematical structures, defining a "Todd function" (named after the algebraic geometer J.A. Todd) and claiming $1/\alpha$ equals a specific limit of this function. The attempt was widely criticized [Carroll 2018] and has not been accepted: the derivation contained gaps, and the numerical result could not be independently reproduced.

Our observation differs from Atiyah's in three ways: (1) we do not claim a derivation — only a rewriting; (2) our formula is manifestly approximate (2.2 ppm off), while Atiyah claimed exactness; (3) our geometric objects (volumes of $\mathbb{CP}^n$ in the Fubini-Study metric) are standard and unambiguous, while Atiyah's Todd function was never precisely defined in his manuscript.

Nevertheless, there is an intriguing overlap: the Todd *class* (a standard invariant in algebraic geometry, distinct from Atiyah's "Todd function") appears naturally in the Hirzebruch-Riemann-Roch theorem on $\mathbb{CP}^n$. Whether the Todd class plays a role in connecting the $\mathbb{CP}^n$ volumes to the fine-structure constant is an open question.

6. Prior Geometric Interpretations

Sherbon [2012] connected the three terms to Pauli's "World Clock" dream: three concentric rhythms mapped to π, π^2, $4\pi^3$. Zhang Xiangqian [2026] proposed a three-layer topological interpretation from spiral space theory: circle cross-section (π), helix-swept triangle (π^2), Gaussian sphere ($4\pi^3$).

The $\mathbb{CP}^n$ rewriting provides a precise mathematical framework for these earlier intuitions:

Sherbon (Pauli)	Zhang	This paper
Small rhythm	Circle (π)	$\mathrm{Vol}(\mathbb{CP}^1) = \pi$
Medium rhythm	Helix (π^2)	$2! \cdot \mathrm{Vol}(\mathbb{CP}^2) = \pi^2$
Large rhythm ($\times 4$)	Sphere ($4\pi^3$)	$4! \cdot \mathrm{Vol}(\mathbb{CP}^3) = 4\pi^3$

7. The Correction Term

The formula is 2.2 ppm too large: $\pi + \pi^2 + 4\pi^3 - 1/\alpha = 3.05 \times 10^{-4}$. If the $\mathbb{CP}^n$ rewriting has a geometric origin, the correction must also have one.

Possible sources of a correction at the 10^{-4} level:

- Higher-order curvature invariants of $\mathbb{CP}^3$ (e.g., integrals of c_2 or c_3)

- Corrections from the non-trivial topology of the twistor fibration $\mathbb{CP}^3 \to S^4$
- Quantum corrections (the formula gives α at a specific energy scale; RG running shifts it by $\sim \alpha^2 \ln(\mu/m_e)$)
- The formula is simply approximate, and the correction has no geometric meaning

We note that $\alpha/(2\pi) = 1.16 \times 10^{-3}$ is the right order of magnitude for a one-loop correction to the formula. Whether the 2.2 ppm discrepancy can be absorbed into a term proportional to α times a geometric invariant of $\mathbb{CP}^3$ is worth investigating but is beyond the scope of this note.

8. Implications for Emergent Gravity

The $\mathbb{CP}^n$ rewriting of $1/\alpha$ is not an isolated observation. It connects to a broader program in which both the gravitational constant and the cosmological constant are expressed in terms of α and the electron Yukawa coupling $y_e = \sqrt{2}\, m_e/v$ [Zhang 2026a,b]:

$$\alpha_G \equiv \frac{Gm_e^2}{\hbar c} = \alpha^8 \cdot y_e^5 \qquad (0.024\%) \tag{5}$$

$$\Lambda \cdot \ell_{\mathrm{Pl}}^2 = \frac{\alpha^4}{4\pi} \left(\frac{m_e}{m_{\mathrm{Pl}}}\right)^5 \qquad (1.9\%) \tag{6}$$

If α is determined by the geometry of $\mathbb{CP}^3$ (Eq. 4), then through Eqs. (5–6), Newton's constant and the cosmological constant would also be determined by twistor geometry:

$$\mathbb{CP}^3 \xrightarrow{\text{Eq. 4}} \alpha \xrightarrow{\text{Eq. 5}} G \xrightarrow{\text{Eq. 6}} \Lambda$$

This chain — from twistor space to the large-scale structure of the universe — is the motivation for the present note. The $\mathbb{CP}^n$ volume decomposition of $1/\alpha$ would be the *geometric foundation* of the emergent gravity program: the starting point from which G and Λ are derived.

Moreover, the non-perturbative character of Eq. (5) — the factor α^{-8} in $1/G$ cannot arise from any finite order of perturbation theory [Zhang 2026c] — is naturally accommodated by twistor methods, which produce all-order results from simple geometric calculations [Witten 2004].

We emphasize that this chain is currently speculative. Eqs. (5–6) are empirical formulas without first-principles derivations, and the α_G formula is in 11σ tension with CODATA 2018. The $\mathbb{CP}^n$ decomposition of $1/\alpha$ is approximate (2.2 ppm off). The chain would be falsified if any link is disproved. However, the existence of a coherent geometric framework connecting α, G, and Λ through a single mathematical structure ($\mathbb{CP}^3$) is sufficiently suggestive to warrant reporting.

9. Caveats

1. The formula is **ruled out as exact** at $> 27{,}000\sigma$. The geometric rewriting does not change this.
2. The weights $(2n-2)!$ were found by **reverse engineering** the known coefficients (1, 1, 4). Given three small integers, many patterns could be fitted.
3. The formula $1/\alpha = \pi + \pi^2 + 4\pi^3$ is **not new**. Our contribution is only the $\mathbb{CP}^n$ volume rewriting and the observation about the truncation.
4. Whether this is physics or numerology is **unknown**.

5. Atiyah's 2018 attempt to derive α from related geometric structures was not accepted, suggesting that extracting α from the geometry of $\mathbb{CP}^n$ is at best very difficult.

10. Conclusion

The approximation $1/\alpha \approx \pi + \pi^2 + 4\pi^3$ can be written as a weighted sum of Fubini-Study volumes of the subspaces $\mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \mathbb{CP}^3$, with weights equal to the factorial of the real dimension of each preceding subspace. The truncation at three terms matches the three complex dimensions of Penrose's twistor space. Combined with recently reported formulas expressing G and Λ in terms of α, this observation suggests a possible chain from twistor geometry to the large-scale structure of the universe. All links in this chain are currently empirical, and the α formula is ruled out as exact. Whether it reflects a connection between the fine-structure constant and twistor geometry, or is one of many possible factorial patterns fitting three small integers, remains open.

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