since the system includes an infinity of such pairs of corresponding points. The left-hand side of this equation is the first emanant of the contravariant $m(\xi^3+\eta^3+\zeta^3)-3\xi\eta\zeta$, which may be expressed in Salmon's notation as $4SQ-3TP$, after multiplication by $(1+8m^3)^2$.

Elementary Proof of a Theorem for Functions of several Variables.

By H. F. BAKER. Received and read February 13th, 1902.

We are in the habit of assuming that, if an ordinary power series in any number of variables does not vanish for zero values of the variables, the inverse of the series can be expanded in a converging series.

In the following note it is proved that the new series has at least the same range of convergence as the original, provided no zero of the original is contained in this range. A trivial particular case is the convergence of the expansion for $(1+x)^{-n}$ when $|x|<1$, n being a positive integer.

Incidentally a *volume*, of ellipsoidal shape, is found about a point, at which a function of several variables does not vanish, within which no zero of the function exists.

It is pointed out in conclusion that the theorem that every equation has a root is a corollary from the general results.

1. Suppose that the ordinary power series

$$f(x, y) = \sum_{h=0,\, k=0} a_{h,k} x^h y^k$$

converges for $|x| < R$, $|y| < S$; let $r < R$, $s < S$, and suppose that for $|x| = r$, $|y| = s$ the derived series

$$f_1(X, Y) = \sum_{m=0} \sum_{n=0} \frac{f^{(m,n)}(x, y)}{m!\, n!}(X-x)^m (Y-y)^n$$

all converge uniformly for $|X-x| = D$, $|Y-y| = E$; let the values of all $|f_1(X, Y)|$ for every set of values

$$|x| = r, \quad |y| = s, \quad X = x+De^{i\theta}, \quad Y = y+Ee^{i\phi}$$

be $<\Pi$; then for every $|x|=r$, $|y|=s$, by a known theorem,

$$\left|\frac{f^{(m,n)}(x,y)}{m!\,n!}\right| \leq \frac{\Pi}{D^m E^n},$$

and herein we may take for $|f^{(m,n)}(x,y)|$ the upper limit of the absolutely greatest values arising for every $|x|=r$, $|y|=s$.

Now $f^{(m,n)}(x,y) = \sum_{h=m}\sum_{k=n} a_{h,k}\frac{h!}{(h-m)!}\frac{k!}{(k-n)!}x^{h-m}y^{k-n}$

converges uniformly for every $|x|=r$, $|y|=s$; thus we can infer, by a known theorem,

$$|a_{h,k}|\frac{h!\,k!}{(h-m)!\,(k-n)!} \leq \frac{\Pi\, m!\, n!}{D^m E^n r^{h-m}s^{k-n}}.$$

We have, however, if $D_1<D$, $E_1<E$, $r_1<r$, $s_1<s$,

$$|a_{hk}|\, r_1^h s_1^k \left(1+\frac{D_1}{r}\right)^h \left(1+\frac{E_1}{s}\right)^k$$

$$= |a_{h,k}|\, r_1^h s_1^k \sum_{m=0}^{h}\sum_{n=0}^{k}\frac{h!}{m!\,(h-m)!}\,\frac{k!}{n!\,(k-n)!}\,\frac{D_1^m E_1^n}{r^m s^n},$$

of which the right side, in consequence of a previously proved inequality, is

$$\leq \Pi\left(\frac{r_1}{r}\right)^h\left(\frac{s_1}{s}\right)^k \sum_{m=0}^{h}\sum_{n=0}^{k}\left(\frac{D_1}{D}\right)^m\left(\frac{E_1}{E}\right)^n,$$

that is $$\overline{\overline{<}}\ \Pi\left(\frac{r_1}{r}\right)^h\left(\frac{s_1}{s}\right)^k \frac{1-(D_1/D)^{h+1}}{1-D_1/D}\,\frac{1-(E_1/E)^{k+1}}{1-E_1/E}$$

$$\overline{\overline{<}}\ \Pi\left(\frac{r_1}{r}\right)^h\left(\frac{s_1}{s}\right)^k \frac{1}{(1-D_1/D)(1-E_1/E)}.$$

Hence $\sum_{h=0}\sum_{k=0}|a_{hk}|\, r_1^h s_1^k\left(1+\frac{D_1}{r}\right)^h\left(1+\frac{E_1}{s}\right)^k$

$$\overline{\overline{<}}\ \Pi\left(1-\frac{D_1}{D}\right)^{-1}\left(1-\frac{E_1}{E}\right)^{-1}\left(1-\frac{r_1}{r}\right)^{-1}\left(1-\frac{s_1}{s}\right)^{-1}$$

This shows that the series

$$f(x,y) = \sum_{h=0}\sum_{k=0} a_{h,k}x^h y$$

converges for $|x| = r_1\left(1+\frac{D_1}{r}\right), \quad |y| = s_1\left(1+\frac{E_1}{s}\right),$

quantities of which the upper limits are $r+D$ and $s+E$.

Thus, if $f(x, y)$ be an ordinary power series with a given bicircular region of convergence $|x| < R$, $|y| < S$, and if lower limits, D, E, other than zero, can be assigned for the radii of convergence of derived series $f_1(X, Y)$ about every interior point, then the given series $f(x, y)$ converges in fact within a bicircular region $|x| < R+D$, $|y| < S+E$, of which the radii are greater respectively by D, E than those originally known.

This is merely an extension of the proof and theorem* given for one variable by Harkness and Morley, *Theory of Analytic Functions*, 1898, p. 178; it is now clear that the theorem holds for any number of variables. The proof we have given assumes not only that the derived series $|f_1(X, Y)|$ has a finite upper limit Π_1 for all

$$X = x+De^{i\theta}, \quad Y = y+Ee^{i\phi}$$

for every assigned x, y for which $|x| = r$, $|y| = s$, but that these quantities Π_1 arising for all $|x| = r$, $|y| = s$ have a finite upper limit Π.

2. Now let
$$f(x, y) = \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} a_{h,k}x^h y^k$$

converge for $|x| \overline{\gtrless} r$, $|y| \overline{\gtrless} s$; and suppose $a_{00} \neq 0$. Put

$$a_{h,k} = -a_{00}\frac{b_{h,k}}{r^h s^k}, \quad x = rt, \quad y = su,$$

and so obtain
$$f(x, y) = a_{00}\phi(t, u),$$

where
$$\phi(t, u) = 1-\sum_{h=1}^{\infty}\sum_{k=1}^{\infty} b_{h,k}t^h u^k$$

converges for $|t| \overline{\gtrless} 1$, $|u| \overline{\gtrless} 1$.

Then the expansion, which we desire to examine,

$$(1-b_{10}t-b_{01}u-\Sigma\Sigma b_{h,k}t^h u^k)^{-1} = 1+B_{10}t+B_{01}u+\Sigma\Sigma B_{h,k}t^h u^k,$$

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* For one variable, Pincherle, "Saggio di una introduzione alla teoria delle funzioni analitiche secondo i principii del Prof. C. Weierstrass," *Battaglini's Giornale*, Vol. XVIII., 1880, p. 352.

requires

$$B_{10} = b_{10}, \quad B_{01} = b_{01},$$

$$B_{20} = B_{10}b_{10}+b_{20}, \quad B_{11} = B_{10}b_{01}+B_{01}b_{10}+b_{11}, \quad B_{02} = B_{01}b_{01}+b_{02},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$B_{h,k} = \sum_{m=h}^{0} \sum_{n=k}^{0}{}' B_{m,n} b_{h-m,k-n},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

where the summation $\Sigma\Sigma'$, in the last line written, excludes the combination $m = h$, $n = k$.

Suppose we can find real positive quantities $\beta_{h,k}$ such that

(1) $$\beta_{h,k} \overline{\overline{>}} |b_{h,k}|;$$

(2) the series $$1 - \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} \beta_{h,k} t^h u^k$$

converges for an assignable region $|t| <$ some definite quantity, $|u| <$ some definite quantity ;

(3) we have an equality

$$(1 - \Sigma\Sigma\beta_{h,k} t^h u^k)^{-1} = 1 + \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} H_{h,k} t^h u^k$$

holding for an assignable region $|t| <$ some definite quantity, $|u| <$ some definite quantity.

Then we have equations corresponding to those before written, namely,

$$H_{10} = \beta_{10}, \quad H_{01} = \beta_{01},$$

$$H_{20} = H_{10}\beta_{10}+\beta_{20}, \quad H_{11} = H_{10}\beta_{01}+H_{01}\beta_{10}+\beta_{11}, \quad H_{02} = H_{01}\beta_{01}+\beta_{02},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$H_{h,k} = \sum_{m=h}^{0} \sum_{n=k}^{0}{}' H_{m,n} \beta_{h-m,k-n},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

which show that the coefficients $H_{h,k}$ are necessarily real and positive, and

$$H_{10} = \beta_{10} \overline{\overline{>}} |b_{10}| \overline{\overline{>}} |B_{10}|, \quad H_{01} = \beta_{01} \overline{\overline{>}} |b_{01}| \overline{\overline{>}} |B_{01}|;$$

$$H_{20} = H_{10}\beta_{10}+\beta_{20} \overline{\overline{>}} |B_{10}b_{10}| + |b_{20}| \overline{\overline{>}} |B_{20}|,$$

and so on ; thus in general

$$H_{h,k} \overline{\overline{>}} |B_{h,k}|,$$

of which we give a formal proof by induction; namely, in consequence of

$$|b_{h-m,\,k-n}| \overline{\gtrless} \beta_{h-m,\,k-n},$$

and assuming

$$|B_{m,\,n}| \leqq H_{m,\,n}$$

for every m from 0 to h, and every n from 0 to k, except only the one combination $m = h$, $n = k$, it follows from

$$|B_{h,\,k}| = \left| \sum_{m=0}^{h} \sum_{n=0}^{k}{}' B_{m,\,n} b_{h-m,\,k-n} \right|$$

$$\overline{\gtrless} \sum_{m=0}^{h} \sum_{n=0}^{k}{}' |B_{m,\,n}| \, |b_{h-m,\,k-n}|$$

that

$$|B_{h,\,k}| \overline{\gtrless} \sum_{m=0}^{h} \sum_{n=0}^{k}{}' H_{m,\,n} \beta_{h-m,\,k-n} \overline{\gtrless} H_{h,\,k},$$

which establishes $|B_{m,\,n}| \overline{\gtrless} H_{m,\,n}$ for the exceptional case $m = h$, $n = k$; and therefore the general truth of the specified inequality.

It follows therefore from the assumed existence of quantities $\beta_{h,\,k}$ satisfying the three conditions (1), (2), (3) that the series

$$1 + B_{10} t + B_{01} u + \Sigma\Sigma B_{h,\,k} t^h u^k$$

converges within the limits of convergence of the series

$$1 + H_{10} t + H_{01} u + \Sigma\Sigma H_{h,\,k} t^h u^k.$$

Such a set $\beta_{h,\,k}$ can, however, be found as follows. Let M be a positive real quantity greater than (or equal to) the modulus of $\phi(t, u)$ for $|t| = 1$, $|u| = 1$; and take

$$1 - \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} \beta_{h,\,k} t^h u^k = 1 - M(t + u + t^2 + tu + u^2 + \ldots),$$

namely, every $\beta_{h,\,k} = M$. Then it follows from a known theorem that

$$|b_{h,\,k}| \leqq M \leqq \beta_{h,\,k}.$$

And $\left[1 - M(t + u + t^2 + tu + u^2 + \ldots)\right]^{-1}$

$$= \left[1 - M\left(\frac{1}{(1-t)(1-u)} - 1\right)\right]^{-1}$$

$$= \frac{1}{1+M} \frac{1}{1 - \frac{M}{1+M}(1-t)^{-1}(1-u)^{-1}}$$

$$= \frac{1}{1+M} \left\{ 1 + \sum_{\lambda=1}^{\infty} \left(\frac{M}{1+M}\right)^{\lambda} (1-t)^{-\lambda} (1-u)^{-\lambda} \right\};$$

now, when $|t| = \rho < 1$, $|u| = \sigma < 1$, the sum of the moduli of the terms of the expansion of $(1-t)^{-\lambda}(1-u)^{-\lambda}$ is convergent, being equal to $(1-\rho)^{-\lambda}(1-\sigma)^{-\lambda}$; and the sum of the moduli of the terms of the series just written down is

$$\frac{1}{1+M-M(1-\rho)^{-1}(1-\sigma)^{-1}}$$

provided $$(1-\rho)(1-\sigma) > \frac{M}{1+M}.$$

Under this condition, with $|t| = \rho < 1$, $|u| = \sigma < 1$, it is therefore legitimate to arrange the expansion of

$$\left[1-M(t+u+t^2+tu+u^2+\dots)\right]^{-1}$$

as a power series in t and u. Thus we can conclude that for

$$|x| < r, \quad |y| < s, \quad \left(1-\left|\frac{x}{r}\right|\right)\left(1-\left|\frac{y}{s}\right|\right) > \frac{M}{1+M},$$

we have

$$\frac{1}{f(x, y)} = \frac{1}{a_{00}}\left(1+\sum_{h=1}^{\infty}\sum_{k=1}^{\infty} B_{h,k}t^h u^k\right) = \frac{1}{a_{00}}\left[1+\sum_{h=1}^{\infty}\sum_{k=1}^{\infty} B_{h,k}\left(\frac{x}{r}\right)^h\left(\frac{y}{s}\right)^k\right],$$

where M is not less than the modulus of $f(x, y)/a_{00}$ for $|x| = r$, $|y| = s$.

Let $K >$ modulus of $f(x, y)$ for $|x| \leq r$, $|y| \leq s$; we may then take $M = K/|a_{00}|$, and have therefore the expansion which we shall write

$$\frac{1}{f(x, y)} = \frac{1}{a_{00}} + \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} C_{h,k}x^h y^k \tag{A}$$

for $$|x| < r, \quad |y| < s, \quad \left(1-\left|\frac{x}{r}\right|\right)\left(1-\left|\frac{y}{s}\right|\right) > \frac{K}{|a_{00}|+K}.$$

The conditions

$$\rho < 1, \quad \sigma < 1, \quad (1-\rho)(1-\sigma) > \frac{M}{M+1} \tag{1}$$

require $1-\rho > \dfrac{M}{M+1}$, and therefore $\rho < \dfrac{1}{M+1}$, and therefore also $\sigma < \dfrac{1}{M+1}$; putting

$$(1-\rho)\sqrt{\frac{M+1}{M}} = \lambda_0, \quad (1-\sigma)\sqrt{\frac{M+1}{M}} = \mu_0,$$

or $$\rho = 1-\lambda_0\sqrt{\frac{M}{M+1}}, \quad \sigma = 1-\mu_0\sqrt{\frac{M}{M+1}}, \tag{2}$$

we have

$$\sqrt{\frac{M}{M+1}} < \lambda_0 < \sqrt{\frac{M+1}{M}}, \quad \sqrt{\frac{M}{M+1}} < \mu_0 < \sqrt{\frac{M+1}{M}}, \quad \lambda_0\mu_0 > 1. \quad (3)$$

Conversely, if λ_0, μ_0 be any two real positive quantities satisfying (3), the conditions (1) are satisfied in the most general way by values ρ, σ given by (2); which involve in particular $\rho < \frac{1}{M+1}$, $\sigma < \frac{1}{M+1}$. And the series above given holds for

$$|x| \overline{\overline{<}} r\left[1-\lambda_0\sqrt{\frac{K}{K+|a_{00}|}}\right], \quad |y| \overline{\overline{<}} s\left[1-\mu_0\sqrt{\frac{K}{K+|a_{00}|}}\right].$$

3. It is an incidental consequence that for $|x| \leqq r\rho$, $|y| \leqq s\sigma$, where ρ, σ are determined by (1), or by (2) and (3), there is no pair of values x, y for which $f(x, y)$ is zero; a result which may be proved directly. Assume now that for all $|x| \leqq r$, $|y| \leqq s$, the modulus of $f(x, y)$ remains greater than a definite real positive quantity P; we proceed to show that then the region of convergence of the series

$$\frac{1}{a_{00}} + \Sigma\Sigma\, C_{h,k}\, x^h y^k \tag{A}$$

extends, in fact, to within unassignable nearness of $|x| = r$, $|y| = s$.

Take x_1, y_1 so that $|x_1| = r_1 < r$, $|y_1| = s_1 < s$ are circles not lying without the circles of convergence of the last written series (A); then for $|x-x_1| < r-r_1$, $|y-y_1| < s-s_1$ we can write

$$f(x, y) = f(x_1, y_1) + \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \frac{f^{(h,k)}(x_1, y_1)}{h!\,k!}(x-x_1)^h(y-y_1)^k.$$

Applying to this the result previously obtained, it follows, $f(x_1, y_1)$ not being zero, that for

$$|x-x_1| < r-r_1, \quad |y-y_1| < s-s_1,$$

and
$$\left(1-\frac{|x-x_1|}{r-r_1}\right)\left(1-\frac{|y-y_1|}{s-s_1}\right) > \frac{K}{K+|f(x_1,y_1)|}$$

we have
$$\frac{1}{f(x, y)} = \frac{1}{f(x_1, y_1)} + \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} D_{h,k}(x-x_1)^h(y-y_1)^k. \tag{B}$$

This will therefore, *a fortiori*, be true for

$$|x-x_1| < r-r_1, \quad |y-y_1| < s-s_1,$$

$$\left(1-\frac{|x-x_1|}{r-r_1}\right)\left(1-\frac{|y-y_1|}{s-s_1}\right) > \frac{K}{K+P},$$

for the quantity on the right of the last inequality is greater than $K/[K+|f(x_1, y_1)|]$, and therefore, as remarked above, true for

$$|x-x_1| \overline{\overline{<}} (r-r_1)\left[1-\lambda\sqrt{\frac{K}{K+P}}\right],$$

$$|y-y_1| \overline{\overline{<}} (s-s_1)\left[1-\mu\sqrt{\frac{K}{K+P}}\right],$$

where λ, μ are quantities which we may take fixed satisfying only the inequalities

$$\sqrt{\frac{K}{K+P}} < \lambda < \sqrt{\frac{K+P}{K}}, \quad \sqrt{\frac{K}{K+P}} < \mu < \sqrt{\frac{K+P}{K}}, \quad \lambda\mu > 1.$$

The series (B), being equal to the series (A) previously obtained for all points x, y lying within the region of convergence of (B) for which $|x| < r_1$, $|y| < s_1$, is necessarily the derived series of (A).

It follows therefore from an extension theorem proved here (§ 1) that the region of convergence of the series (A) extends beyond the limits $|x| < r_1$, $|y| < s_1$, and is at least as great as given by

$$|x| < r_1+(r-r_1)\left[1-\lambda\sqrt{\frac{K}{K+P}}\right],$$

$$|y| < s_1+(s-s_1)\left[1-\mu\sqrt{\frac{K}{K+P}}\right].$$

Denoting these by $|x| < r_2$, $|y| < s_2$, and putting

$$e_1 = 1-\lambda\sqrt{\frac{K}{K+P}}, \quad e_2 = 1-\mu\sqrt{\frac{K}{K+P}},$$

we may similarly prove that the region of convergence is at least as great as given by

$$|x| < r_2+(r-r_2)\,e_1, \quad |y| < s_2+(s-s_2)\,e_2,$$

and so on. And as at each step we extend the radius of convergence associated with x by the same fraction, e_1, of its deficiency from r, and the radius of convergence associated with y by the same fraction, e_2, of its deficiency from s, it follows that the radii can be extended to be within any assignable nearness respectively of r and s.

We have thus the theorem: If

$$f(x, y) = a_{00} + \Sigma\Sigma a_{hk} x^h y^k$$

be convergent for $|x| \overline{\gtrless} r$, $|y| \overline{\gtrless} s$, and for this range be in absolute value everywhere $<$ the real positive K, and if λ_0, μ_0 be any real positive quantities such that, when

$$A = |a_{00}| = |f(0, 0)|,$$

$$\sqrt{\frac{K}{K+A}} < \lambda_0 < \sqrt{\frac{K+A}{K}}, \quad \sqrt{\frac{K}{K+A}} < \mu_0 < \sqrt{\frac{K+A}{K}}, \quad \lambda_0 \mu_0 > 1,$$

then the function $f(x, y)$ does not vanish for any values x, y within the range

$$|x| \overline{\gtrless} r\left[1 - \lambda_0 \sqrt{\frac{K}{K+A}}\right], \quad y \overline{\gtrless} s\left[1 - \mu_0 \sqrt{\frac{K}{K+A}}\right],$$

and for this range we have a convergent expansion of the form

$$\frac{1}{f(x, y)} = \frac{1}{a_{00}} + \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} C_{h,k} x^h y^k.$$

If further r, s be such that for all values $|x| \overline{\gtrless} r$, $|y| \overline{\gtrless} s$ the function $|f(x, y)|$ is assignedly greater than zero, this expansion is convergent and valid for $|x| < r$, $|y| < s$.

When we apply the same reasoning to the function of n variables

$$f(x_1, \ldots, x_n) = a_{0\ldots0} + \Sigma \ldots \Sigma a_{h_1, h_2, \ldots, h_n} x_1^{h_1}, \ldots, x_n^{h_1},$$

assumed to converge for $|x| \overline{\gtrless} r_1, \ldots, |x_n| \overline{\gtrless} r_n$, and such that for these values $|f(x_1, \ldots, x_n)| < K$, we can prove directly from the theorem

$$|a_{h_1, \ldots, h_n}| \overline{\gtrless} K/r_1^{h_1}, \ldots, r_n^{h_n},$$

or verify indirectly as in the preceding, that there is no set of values $x_1, \ldots, x_n$ for which $f(x_1, \ldots, x_n)$ vanishes lying within the range

$$|x_1| < r_1, \ldots, |x_n| < r_n, \quad \left(1 - \frac{|x_1|}{r_1}\right), \ldots, \left(1 - \frac{|x_n|}{r_n}\right) > \frac{K}{K+A},$$

where $A = |a_{0\ldots0}|$.

Let $\left(\frac{K}{K+A}\right)^{1/n}$ be denoted by ω, and put

$$1 - \frac{|x_1|}{r_1} = \lambda_1 \omega, \ \ldots, \ 1 - \frac{|x_n|}{r_n} = \lambda_n \omega;$$

then from $1-\frac{|x_1|}{r_1} < 1$ follows that $\lambda_1 < \omega^{-1}$, and from $1-\frac{|x_1|}{r_1} > \omega^n$ follows $\lambda_1 > \omega^{n-1}$, and so for the others; while clearly $\lambda_1 \dots \lambda_n > 1$.

Conversely, if $\lambda_1, \dots, \lambda_n$ be any real quantities such that, for $\omega^n = K/(K+A)$,

$$\omega^{n-1} < \lambda_1 < \omega^{-1}, \ \dots, \ \omega^{n-1} < \lambda_n < \omega^{-1}, \quad \lambda_1\lambda_2 \dots \lambda_n > 1,$$

any set of values such that

$$|x_1| \overline{\overline{<}} r_1(1-\lambda_1\omega), \ \dots, \ |x_n| \overline{\overline{<}} r_n(1-\lambda_n\omega)$$

is such that

$$|x_1| < r_1, \ \dots, \ |x_n| < r_n, \quad \left(1-\frac{|x_1|}{r_1}\right), \ \dots, \ \left(1-\frac{|x_n|}{r_n}\right) > \frac{K}{K+A};$$

and there is no set of values in this range for which $f(x_1, \dots, x_n)$ vanishes. And if R^2 be the least possible value of $\xi_1^2+\dots+\xi_n^2$ subject to $0 \overline{\overline{<}} \xi_1 < 1, \dots, 0 \overline{\overline{<}} \xi_n < 1$, and $(1-\xi_1), \dots, (1-\xi_n) = \omega^n$, which has a different form according to the magnitude of ω, there is no point for which $f(x_1, \dots, x_n)$ vanishes in the region given by

$$\frac{|x_1|^2}{r_1^2}+\dots+\frac{|x_n|^2}{r_n^2} < R^2.$$

For instance, when $n = 2$, if $u = \xi^2+\eta^2$ and $\sigma = \xi+\eta$,

$$\omega^2 = (1-\xi)(1-\eta) = 1-\sigma+\tfrac{1}{2}(\sigma^2-u) = \frac{1-u}{2}+\frac{(1-\sigma)^2}{2}$$

or

$$u = 1-2\omega^2+(1-\sigma)^2.$$

Thus, if upon $(1-\xi)(1-\eta) = \omega^2$ there be points for which $\xi+\eta = 1$, namely, if the equation

$$\xi^2-\xi+\omega^2 = 0$$

have the real roots $\qquad \xi = \tfrac{1}{2} \pm \tfrac{1}{2}(1-4\omega^2)^{\frac{1}{2}},$

that is, if $\omega \overline{\overline{<}} \frac{1}{2}$, then $u = \xi^2+\eta^2$ has $1-2\omega^2$ for its least value.

If, however, $\omega > \frac{1}{2}$, then $(1-\sigma)^2$ is least when $\xi = \eta = 1-\omega$, and then $\xi^2+\eta^2$ has $2(1-\omega)^2$ for its least value.

Thus there are no points for which $f(x_1, x_2)$ vanishes within

$$\frac{|x_1|^2}{r_1^2}+\frac{|x_2|^2}{r_2^2} = R^2,$$

where $R^2 = 1-2\omega^2$ when $\omega \overline{\overline{<}} \frac{1}{2}$ and $R^2 = 2(1-\omega)^2$ when $\omega > \frac{1}{2}$.

As $\omega^2 = K/(K+P)$, and K is any real positive quantity $> |f(x, y)|$ for $|x| \leq r$, $|y| \overline{\overline{<}}\, s$, we can choose ω as nearly unity as we please, and so always take $R^2 = 2(1-\omega)^2$; but this gives a less extended region than $R^2 = 1-2\omega^2$ when the latter exists. For the greatest value of $2(1-\omega)^2$ when $\omega > \frac{1}{2}$ is $< \frac{1}{2}$, while the least value of $1-2\omega^2$ when $\omega \overline{\overline{<}}\, \frac{1}{2}$ is $\frac{1}{2}$.

4. A very simple example of what precedes is a proof, which may be remarked though it is not new, that the equation

$$F(x) = a_0 + a_1 x + \ldots + a_n x^n = 0$$

is satisfied by a finite value of x. It is supposed that $a_0 \neq 0$.

For, if $F(x) \neq 0$ for $|x| \overline{\overline{<}}\, r$, we can put

$$\psi(x) = \frac{1}{F(x)} = \frac{1}{a_0} + C_1 x + C_2 x^2 + \ldots$$

for $|x| < r$. And then for $|x| = r_1 < r$, if $H \overline{\overline{>}}\, |\psi(x)|$ for $|x| = r_1$, $C_k \overline{\overline{<}}\, H r_1^{-k}$

Let, then, ϵ be an assigned real positive quantity, and let r be the greatest value of $|x|$ for which $|F(x)| \overline{\overline{>}}\, \epsilon$ for every $|x| \overline{\overline{<}}\, r$; so that, for $|x| \overline{\overline{<}}\, r$,

$$|\psi(x)| = \frac{1}{|F(x)|} \overline{\overline{<}}\, \frac{1}{\epsilon};$$

then

$$|C_k| \overline{\overline{<}}\, \frac{1}{\epsilon r_1^k},$$

wherein r_1 is arbitrarily little less than r.

This shows that, with the given fixed ϵ, r cannot be indefinitely great, since otherwise all the coefficients $C_1, C_2, \ldots$ would be indefinitely small, that is, zero, and $F(x)$ would be a constant.