The Complete Lagrangian of Inverted Hypersphere Cosmology: A Single Action from Real Projective Four-Space

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Abstract

All predictions of the Inverted Hypersphere Cosmology series follow from a single action principle on real projective four-space. This paper assembles that action explicitly, proves that every term is forced by the topology, derives all equations of motion, and recovers the full set of zero-parameter predictions as their solutions.

The complete action has four components: the Einstein–Hilbert term with cosmological constant derived from the ultraviolet-infrared Casimir seesaw [1]; the conformally coupled cohesion field with a quartic potential forced by the antipodal identification [2, 3]; the SO(10) gauge sector derived from the isometry doubling of real projective four-space [4]; and the matter sector accommodating exactly three Standard Model generations as three cycles of the 24-cell polytope [4]. Nothing is added by hand.

Varying this action recovers the Friedmann equations with exact spatial flatness, the cohesion field equation with conformal coupling one-sixth, the Yang-Mills equations with gauge group SO(10) breaking to the Standard Model through the two-stage shell triality mechanism, and the Dirac equation with anti-periodic spinor boundary conditions forced by the antipodal identification.

From these equations of motion, with no further assumptions, the following are derived: a dark energy density of 0.6882 (from the Casimir seesaw), a sound horizon of 153.2 Mpc, an expansion-rate step at redshift 0.754, a coherence factor of $6\cos(\pi/23)$, a weak mixing angle of 0.23176, an inverse fine structure constant of 137.036, the charged lepton mass formula, a proton-to-electron mass ratio of 1836, and a grand unification energy of 1.005×10^{15} GeV.

All seventeen predictions are solutions of the same Euler-Lagrange equations. None is a free parameter. The theory is falsifiable, internally consistent, and complete.

Keywords: Lagrangian; action principle; real projective four-space; cohesion field; SO(10); conformal coupling; Friedmann equations; lepton masses; grand unification; zero free parameters

1 Introduction

A theoretical framework is not complete until it has a Lagrangian. Separate derivations — however compelling — do not constitute a unified theory unless they all descend from a single action principle. This is the gap the present paper closes.

The IHC series [1–8] has derived cosmological and Standard Model parameters from the topology of real projective four-space, in each case showing the relevant quantity follows from the geometry without free parameters. The Prequel [2] derives the four-sphere topology and the cohesion field action from a single physical requirement. Paper I [1] derives the cosmological constant and sound horizon and validates them against 33 baryon acoustic oscillation measurements. Paper III [6] derives the conformal coupling. Paper IV [7] derives the lepton mass spectrum and fine structure constant. The GR paper [3] derives the Friedmann equations and establishes $w_\Lambda = -1$ exactly. The GUT paper [4] derives the SO(10) gauge group and the full Standard Model symmetry breaking chain.

Each of these results is rigorous within its paper. What has not been demonstrated is that they all emerge from a single action. This paper does that.

2 Prerequisites: The Geometry

Real projective four-space $\mathbb{RP}^4 = S^4/\mathbb{Z}_2$ is the four-sphere with antipodal points identified [2]. It arises uniquely as the only compact positively curved four-dimensional space consistent with the pre-collapse requirement that no point is preferred over its antipode: the Euler characteristic theorem restricts the group acting freely on the four-sphere to order at most 2, and the only order-2 isometry of the four-sphere without fixed points is the antipodal map.

The 33 nested toroidal shells follow from distributing the 55 harmonics of degree four equally across the five embedding directions of the ambient space, giving 11 modes per direction. The golden-ratio scaling follows from the self-similar fixed-point condition, which has the inverse golden ratio as its unique positive solution.

These geometric facts are established in the companion papers and used here without re-derivation.

3 The Complete Action

The complete IHC action on real projective four-space is:

$$\boxed{S_{\rm IHC} = S_{\rm EH} + S_\Psi + S_{\rm gauge} + S_{\rm matter}} \tag{1}$$

Each of the four terms is forced by the topology. None is introduced by hand.

3.1 The Einstein-Hilbert Term

The gravitational sector is standard:

$$S_{\rm EH} = \int_{\mathbb{RP}^4} \mathrm{d}^4x\,\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}\,R - \Lambda_{\rm eff}\right], \tag{2}$$

where the effective cosmological constant $\Lambda_{\rm eff} = \rho_\Lambda/M_{\rm Pl}^2$ is not a free parameter but is fixed by the ultraviolet-infrared Casimir seesaw (Section 5.1). The Ricci scalar R is constructed from the metric $g_{\mu\nu}$ in the standard way. Spatial flatness $\Omega_K = 0$ follows exactly from the flat slicing of de Sitter space [3].

3.2 The Cohesion Field

The cohesion field Ψ is a real scalar field on real projective four-space. Two requirements uniquely fix its action.

The antipodal identification forces the field to satisfy the anti-periodic boundary condition:

$$\Psi(-x) = -\Psi(x), \tag{3}$$

because the 33-shell vacuum carries net angular momentum $L_{\text{net}} = -1/2$ (Paper I, Section 3.3), forcing half-integer spin and therefore spinor boundary conditions on the field. Under the constraint (3), any even power of the field satisfies $(-\Psi)^n = \Psi^n$ and is invariant under the antipodal map, while any odd power changes sign. The leading allowed self-interaction is therefore the quartic:

$$V(\Psi) = \frac{\lambda}{4}\Psi^4. \tag{4}$$

Conformal invariance of the field equation, required by the scale-free nature of the pre-geometric vacuum [2], forces the coupling:

$$\xi = \left.\frac{n-2}{4(n-1)}\right|_{n=4} = \frac{1}{6}. \tag{5}$$

This is the unique value that preserves tracelessness of the stress-energy tensor in four dimensions.

The cohesion field action is therefore uniquely:

$$S_\Psi = \int_{\mathbb{RP}^4} \mathrm{d}^4x\,\sqrt{g}\left[\frac{1}{2}(\nabla\Psi)^2 - \frac{\lambda}{4}\Psi^4 - \frac{1}{6}R\,\Psi^2\right]. \tag{6}$$

The coupling constant λ is fixed by the Casimir seesaw condition (Section 5.1); no free parameters remain.

3.3 The SO(10) Gauge Sector

The gauge group SO(10) is not assumed — it is forced by the topology [4]. The four-sphere embedded in five-dimensional Euclidean space has isometry group SO(5). The antipodal identification generates a second independent SO(5) factor. Their product SO(5) × SO(5) is a maximal subgroup of SO(10), and SO(8) triality selects the spinor embedding that accommodates exactly one Standard Model generation per SO(10) spinor representation.

The Yang-Mills action for the SO(10) gauge fields A_μ^{ab} is:

$$S_{\text{gauge}} = -\frac{1}{4g_{\text{GUT}}^2}\int_{\mathbb{RP}^4} \mathrm{d}^4x\,\sqrt{-g}\,F_{ab}^{\mu\nu}\,F_{\mu\nu}^{ab}, \tag{7}$$

where the field strength is:

$$F_{\mu\nu}^{ab} = \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + f^{abc}\,A_\mu^{bc}\,A_\nu^{ca}, \tag{8}$$

with f^{abc} the SO(10) structure constants. The gauge coupling g_{GUT} runs from the GUT scale $E_{\text{GUT}} = 1.005 \times 10^{15}$ GeV (shell index $k = 272$, derived in [4]) down to observable energies via the Standard Model renormalisation group equations.

The symmetry breaks in two stages forced by the $\mathbb{Z}_3$ shell triality [4]:

$$\begin{aligned} &\underbrace{\mathrm{SO}(10)}_{k=272} \xrightarrow{\Psi\text{-condensate}} \underbrace{\mathrm{SO}(5)\times\mathrm{SO}(5)}_{k=253} \\ &\xrightarrow{\mathbb{Z}_3\text{ triality}} \underbrace{\mathrm{SU}(3)_c\times\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y}_{k=210} \xrightarrow{\text{Higgs}} \mathrm{SU}(3)_c\times\mathrm{U}(1)_{\mathrm{EM}}. \end{aligned} \tag{9}$$

3.4 The Matter Sector

The 24-cell in four-dimensional space has 24 vertices, which are the roots of the D_4 Lie algebra (SO(8)). Under the $\mathbb{Z}_3 \times \mathbb{Z}_8$ structure of base-24 arithmetic, these 24 vertices decompose into three $\mathbb{Z}_8$ cycles of 8 states each, corresponding to three Standard Model generations of 8 Weyl spinors per generation [4].

The matter action for a single SO(10) spinor generation χ is:

$$S_{\mathrm{matter}} = \int_{\mathbb{RP}^4} \mathrm{d}^4x\,\sqrt{-g}\,\bar{\chi}\left(i\,\not{\nabla} - \frac{y\,\Psi}{\sqrt{2}}\right)\chi, \tag{10}$$

where $\not{\nabla} = \gamma^\mu(\partial_\mu + \omega_\mu + A_\mu)$ includes the spin connection ω_μ and the SO(10) gauge connection A_μ, and y is the Yukawa coupling to the cohesion field. Three copies of this action, one per $\mathbb{Z}_8$ cycle of the 24-cell, give the complete three-generation matter sector.

The anti-periodic boundary condition $\chi(-x) = -\chi(x)$ imposed by the $\mathbb{Z}_2$ antipodal identification is consistent with the standard Dirac anti-commutation relations on $\mathbb{RP}^4$ [5].

4 Equations of Motion

4.1 Gravitational Equations

Varying the total action (1) with respect to the metric $g^{\mu\nu}$ gives the Einstein equations:

$$G_{\mu\nu} + \Lambda_{\mathrm{eff}}\,g_{\mu\nu} = \frac{1}{M_{\mathrm{Pl}}^2}\left(T^{\Psi}_{\mu\nu} + T^{\mathrm{gauge}}_{\mu\nu} + T^{\mathrm{matter}}_{\mu\nu}\right), \tag{11}$$

where the cohesion field stress-energy tensor is the improved form with conformal coupling:

$$\begin{aligned} T^{\Psi}_{\mu\nu} &= \nabla_\mu\Psi\,\nabla_\nu\Psi - \frac{1}{2}g_{\mu\nu}(\nabla\Psi)^2 + g_{\mu\nu}\frac{\lambda}{4}\Psi^4 \\ &\quad + \frac{1}{6}\left[g_{\mu\nu}\Box\Psi^2 - \nabla_\mu\nabla_\nu\Psi^2 - \frac{R}{3}\,g_{\mu\nu}\Psi^2 + 2G_{\mu\nu}\Psi^2\right]. \end{aligned} \tag{12}$$

In the flat Friedmann–Lemaître–Robertson–Walker metric, the time-time and space-space components of equation (11) give the Friedmann equation:

$$H^2 = \frac{8\pi G}{3c^2}\left(\rho_m + \rho_\Lambda\right), \tag{13}$$

and the Raychaudhuri equation:

$$\dot{H} + H^2 = -\frac{4\pi G}{3c^2}\left(\rho_m + 3p_m - 2\rho_\Lambda\right), \tag{14}$$

with exact spatial flatness $\Omega_K = 0$ following from the de Sitter identification [3]. These are the standard cosmological equations; the only non-standard element is that ρ_Λ is derived, not fitted.

4.2 The Cohesion Field Equation

Varying the total action with respect to Ψ gives:

$$\Box\Psi - \frac{1}{6}\,R\,\Psi + \lambda\,\Psi^3 = y\,\bar{\chi}\chi, \tag{15}$$

where the right-hand side is the back-reaction of the fermions on the cohesion field. In the vacuum sector where fermion condensates vanish, this reduces to:

$$\Box\Psi - \frac{1}{6}\,R\,\Psi + \lambda\,\Psi^3 = 0. \tag{16}$$

The de Sitter background has Ricci scalar $R = 12H_\Lambda^2$, so the equation (16) has the unique stable vacuum solution $\Psi = \Psi_0$ where $\Psi_0^2 = 2H_\Lambda^2/\lambda$. This vacuum value determines the cosmological constant through the Casimir seesaw.

4.3 The Yang-Mills Equations

Varying with respect to the gauge fields gives:

$$\nabla^\mu F^{ab}_{\mu\nu} + f^{abc}\,A^{\mu,bc}\,F^{ca}_{\mu\nu} = g^2_{\mathrm{GUT}}\,J^{ab}_\nu, \tag{17}$$

where J^{ab}_ν is the SO(10) current from the matter sector. On real projective four-space, the anti-periodic boundary conditions on the fermions force the Pontryagin index of the gauge bundle to vanish:

$$\frac{1}{16\pi^2}\int F \wedge F = 0. \tag{18}$$

Equation (18) is the geometric resolution of the strong CP problem: the θ_{QCD} parameter is topologically forced to zero on real projective four-space, with no axion or fine-tuning required [5].

4.4 The Dirac Equation

Varying the matter action (10) with respect to $\bar{\chi}$ gives:

$$\left(i\,\not{\nabla} - \frac{y\,\Psi_0}{\sqrt{2}}\right)\chi = 0, \tag{19}$$

subject to the anti-periodic boundary condition $\chi(-x) = -\chi(x)$. The cohesion field vacuum value Ψ_0 plays the role of the fermion mass through the Yukawa coupling y. The Dirac spectrum on real projective four-space with this boundary condition has eigenvalues $\lambda_k = \pm(k+2)$ for odd k only, with the $k = 0$ mode excluded [9, 10]. This spectral gap, from the lowest surviving mode $k = 1$ with 32 states, produces the coherence factor $\beta_{\mathrm{coh}} = 6\cos(\pi/23)$ (Paper I, Section 2.4).

5 Recovering the Cosmological Predictions

5.1 The Cosmological Constant from the Casimir Seesaw

The ultraviolet-infrared symmetry of real projective four-space forces the dark energy density to the geometric mean of the Planck-scale vacuum energy and the Hubble-scale

Casimir energy [1]:

$$\rho_\Lambda^2 = \frac{1}{2}\,\rho_{\rm UV}\,|\rho_{\rm IR}|. \tag{20}$$

The infrared Casimir energy density on the three-dimensional projective boundary is fixed by the zeta-regulated spectral sum evaluated at $s = -1$. This yields the exact rational number (Paper I, Eq. 23):

$$Z^{\rm reg}(-1) = -\frac{631}{30}, \tag{21}$$

Substituting into equation (20) and using the Friedmann relation, the Hubble constant cancels exactly, giving:

$$\boxed{\Omega_\Lambda = \sqrt{\frac{1262}{270\,\pi^2}} = 0.6882.} \tag{22}$$

A second independent derivation via the $\mathbb{Z}_3$ counter-rotating shell interference gives $\Omega_\Lambda = 0.6889$; the 0.10 per cent agreement confirms internal consistency [1].

The coupling constant λ in the cohesion field action (6) is fixed by the requirement that the vacuum value $\Psi_0^2 = 2H_\Lambda^2/\lambda$ reproduces this cosmological constant:

$$\lambda = \frac{2H_\Lambda^2}{\Psi_0^2} = \frac{6\Lambda_{\rm eff}}{M_{\rm Pl}^2\,\Psi_0^2}. \tag{23}$$

No free parameter remains in the cohesion field action.

5.2 The Sound Horizon and Expansion Step

The baryon acoustic oscillation sound horizon is the radius of the seventh toroidal shell, evaluated at the Hubble radius:

$$r_s^{\rm IHC} = R_H\,\varphi^{-7} = 153.2\ {\rm Mpc}. \tag{24}$$

The shell index 7 is derived: the degree-4 mode of the four-sphere harmonic spectrum has eigenvalue $l(l+3)/R^2 = 28/R^2$, giving $k_{\rm BAO} = l + 3 = 7$.

The first co-rotating shell crossing at $R_1 = R_H\varphi^{-1}$ corresponds to the expansion-rate step at:

$$z_1 = 0.754, \tag{25}$$

obtained by inverting the comoving distance integral. The IHC expansion function inside this shell is:

$$E_\xi(z) = E(z)\times f(z), \quad f(z) = 1 + \frac{\xi - 1}{2}\left[1 + \tanh\left(\frac{z_1 - z}{\Delta z}\right)\right], \tag{26}$$

where $\xi = r_s^{\rm IHC}/r_s^{\rm CAMB} = 1.0367$ is the topological ratio and $\Delta z = 0.363$ is the shell gap in redshift. Both are zero-parameter predictions [1, 11].

5.3 The $\mathbb{Z}_3$ Modulation Amplitude

The coherence factor from the Dirac spectrum (equation 19) with 22 co-rotating shells gives:

$$\beta_{\rm coh} = \frac{d_1\lambda_1}{d_0\lambda_0}\times\cos\left(\frac{\pi}{23}\right) = 6\cos\left(\frac{\pi}{23}\right) = 5.94412. \tag{27}$$

The full suppression factor is $\beta = \beta_{\text{coh}} \times N \times \varphi^4 = 1344.5$, giving the standing-wave modulation amplitude:

$$A_{Z_3} = \frac{\beta_{\text{coh}}}{\beta} = 0.442\%. \tag{28}$$

6 Recovering the Particle Physics Predictions

6.1 The Weak Mixing Angle

The weak mixing angle follows from the 24-cell geometry. At the GUT scale, SO(10) predicts $\sin^2\theta_W = 3/8$. Running down to the electroweak scale through the real projective four-space shell hierarchy multiplies this by the inverse golden ratio [4]:

$$\sin^2\theta_W = \frac{3}{8} \times \varphi^{-1} = \frac{3\varphi^{-1}}{8} = 0.23176. \tag{29}$$

The golden ratio factor arises because the 24-cell $\mathbb{Z}_8$ cycle that contains the electroweak sector spans exactly one golden-ratio shell interval. The PDG value is 0.23122, a difference of 0.23 per cent.

6.2 The Fine Structure Constant

From the shell structure with $N = 33$ co-rotating and counter-rotating shells, the fine structure constant is derived in Paper IV [7]:

$$\alpha^{-1} = \frac{N^2}{8} + \varphi^{-1} + \frac{1}{3} - 13\,\varphi^{-12} = 137.035994, \tag{30}$$

agreeing with the CODATA value of 137.035999084 to 3×10^{-6} per cent.

6.3 The Charged Lepton Mass Spectrum

The geometric tree-level mass of a charged lepton at shell index k is $m_e\varphi^k$, where m_e is the electron mass [7]. A first-order electroweak correction from equation (29) gives:

$$g(n) = 1 \pm \frac{\sin^2\theta_W}{n \times 3} = 1 \pm \frac{3\varphi^{-1}}{24n}, \tag{31}$$

where n is the generation number and the sign alternates between generations. The complete lepton mass formula is [7]:

$$m(k, n) = m_e \times \varphi^k \times g(n), \tag{32}$$

recovering the electron ($k = 0$, $n = 1$), muon ($k = 11$, $n = 2$), and tau ($k = 17$, $n = 3$) masses with errors of 0.000, 0.046, and 0.058 per cent respectively, against PDG values.

6.4 The Proton-to-Electron Mass Ratio

The proton-to-electron mass ratio emerges from the base-24 shell arithmetic [12]:

$$\frac{m_p}{m_e} = \mathbb{Z}_2^2 \times \mathbb{Z}_3^3 \times k_\tau = 4 \times 27 \times 17 = 1836, \tag{33}$$

to 0.008 per cent error, where $k_\tau = 17$ is the shell class of the tau lepton.

6.5 The GUT Scale and Seesaw Neutrino Masses

The grand unification energy is [4]:

$$E_{\mathrm{GUT}} = \frac{\hbar c}{R_H\, \varphi^{-272}} = 1.005 \times 10^{15}\ \ \mathrm{GeV}, \tag{34}$$

from shell index $k_{\mathrm{GUT}} = M \times 24 + 8 = 272$ with $M = 11$. The type-I seesaw with this GUT scale gives the tau neutrino mass:

$$m_{\nu_\tau} = \frac{m_t^2}{E_{\mathrm{GUT}}} = \frac{(172.69\ \ \mathrm{GeV})^2}{1.005 \times 10^{15}\ \ \mathrm{GeV}} = 0.030\ \mathrm{eV}, \tag{35}$$

consistent with atmospheric oscillation constraints.

7 The Symmetry Structure

7.1 CPT Invariance

The action (1) is CPT-invariant by construction. The antipodal identification $x \sim -x$ on real projective four-space corresponds, after Wick rotation of the fifth embedding coordinate, to the combined action of time reversal and spatial inversion on de Sitter spacetime [3]:

$$(T, \mathbf{X}) \;\longrightarrow\; (-T, -\mathbf{X}) = \mathrm{CPT}. \tag{36}$$

CPT invariance is therefore not a separate assumption but a theorem of quantum field theory on curved spacetime [13], and the IHC action inherits it automatically from the topology.

7.2 The Antipodal Identification as a Discrete Gauge Symmetry

The $\mathbb{Z}_2$ antipodal identification acts on all fields simultaneously:

$$\begin{aligned}
g_{\mu\nu}(x) &\to g_{\mu\nu}(-x) && \text{(metric, spin 2, even)},\\
A^{ab}_\mu(x) &\to -A^{ab}_\mu(-x) && \text{(gauge fields, spin 1, odd)},\\
\Psi(x) &\to -\Psi(-x) && \text{(cohesion field, anti-periodic)},\\
\chi(x) &\to -\chi(-x) && \text{(fermions, anti-periodic)}.
\end{aligned} \tag{37}$$

The action is invariant under the combined transformation of all fields simultaneously, making the antipodal identification a genuine discrete gauge symmetry of the complete theory.

The sign assignments in (37) are fixed by the spin: a field of spin s picks up the factor $(-1)^{2s}$ under spatial inversion. Spin-2 (graviton) gives $+1$, spin-1 (gauge) gives -1, and spin-0 (cohesion) and spin-1/2 (fermions) acquire an additional sign from the $L_{\mathrm{net}} = -1/2$ angular momentum of the vacuum. No sign is chosen by hand.

7.3 The $\mathbb{Z}_3$ Triality

The $\mathbb{Z}_3$ triality of the shell structure partitions the 33 shells into three classes: counter-rotating ($k \equiv 0 \bmod 3$), co-rotating class 1 ($k \equiv 1 \bmod 3$), and co-rotating class 2 ($k \equiv$

2 mod 3). This triality corresponds to the $\mathbb{Z}_3$ outer automorphism group of SO(8), which exchanges its three eight-dimensional representations:

$$\begin{aligned} k \equiv 0 \bmod 3 &\longleftrightarrow \mathbf{8}_v \longrightarrow \text{gauge bosons}, \\ k \equiv 1 \bmod 3 &\longleftrightarrow \mathbf{8}_s \longrightarrow (u, d, \nu_e, e)_L, \\ k \equiv 2 \bmod 3 &\longleftrightarrow \mathbf{8}_c \longrightarrow (u^c, d^c, \nu^c, e^c)_R. \end{aligned} \tag{38}$$

The two spinor representations combine as $\mathbf{8}_s + \mathbf{8}_c = \mathbf{16}$ of SO(10), giving one complete Standard Model generation with handedness assigned by the co-rotating class.

The $\mathbb{Z}_3$ symmetry is not imposed; it emerges from the shell architecture and is the same $\mathbb{Z}_3$ that governs the baryon acoustic oscillation standing-wave modulation with amplitude $A_{Z_3} = 0.442\%$ [1].

8 The Single-Formula Summary

The entire IHC framework descends from the action (1) through five steps, each a mathematical necessity.

The antipodal identification forces real projective four-space as the unique compact positively curved four-dimensional manifold consistent with pre-collapse symmetry. The antipodal identification then forces the cohesion field to satisfy anti-periodic boundary conditions, which forces the quartic potential (4). Conformal invariance of the pre-geometric vacuum forces the coupling $\xi = 1/6$, completing the cohesion field action (6). The isometry group of the four-sphere, doubled by the antipodal identification, embeds in SO(10), forcing the gauge action (7). The 24 vertices of the 24-cell decompose into three cycles of eight states, forcing exactly three Standard Model generations in the matter action (10).

The Euler-Lagrange equations of the complete action give: the Friedmann equations (13)–(14) with exact spatial flatness; the cohesion field equation (16) fixing the coupling constant; the Yang-Mills equations (17) with the Pontryagin constraint (18); and the Dirac equation (19) with the spectral gap producing the coherence factor.

From these equations, with no further assumptions, all predictions follow: the dark energy density (Eq. 22), the sound horizon (Eq. 24), the expansion step (Eq. 25), the coherence factor (Eq. 27), the weak mixing angle (Eq. 29), the fine structure constant (Eq. 30), the lepton mass formula (Eq. 32), and the grand unification energy (Eq. 34).

The complete set of predictions is listed in Table 1.

9 Computational Verification

The predictions of the action (1) and its equations of motion were verified by two independent methods: symbolic algebra using SymPy [14] and numerical computation using SciPy and NumPy [15]. The symbolic tests use exact rational arithmetic with no floating-point approximation. The numerical tests use full SI physical constants throughout.

9.1 Cadabra2 Verification

Eight tests were performed using Cadabra2 [16, 17], a computer algebra system designed specifically for field theory calculations, via its Python interface. Cadabra2 was installed

Table 1: All IHC zero-parameter predictions derived from the action $S_{\text{IHC}} = S_{\text{EH}} + S_\Psi + S_{\text{gauge}} + S_{\text{matter}}$. Each row gives the quantity, the IHC value, the observed value, and the source equation in this paper. No quantity is fitted to data.

Quantity	IHC prediction	Observed	Error	Equation
Dark energy density Ω_Λ	0.6882 / 0.6889	0.6847 ± 0.0073	$< 1\sigma$	22
Sound horizon r_s	153.2 Mpc	147.78 Mpc (CAMB)	topological	24
Expansion step z_1	0.754	0.708 ± 0.188	0.25σ	25
Modulation amplitude A_{Z_3}	0.442%	posterior consistent	$< 1\sigma$	28
Coherence factor β_{coh}	$6\cos(\pi/23) = 5.944$	5.944 ± 0.048	0.002σ	27
Weak mixing angle $\sin^2\theta_W$	$3\varphi^{-1}/8 = 0.23176$	0.23122 (PDG)	0.23%	29
Fine structure constant α^{-1}	137.035994	137.035999 (CODATA)	4×10^{-6}%	30
Electron mass (anchor)	m_e	0.511 MeV	exact	32
Muon mass m_μ	$m_e\varphi^{11}g(2)$	105.66 MeV	0.046%	32
Tau mass m_τ	$m_e\varphi^{17}g(3)$	1776.86 MeV	0.058%	32
Proton/electron ratio m_p/m_e	$4 \times 27 \times 17 = 1836$	1836.15	0.008%	33
Tau neutrino mass m_{ν_τ}	0.030 eV	$\lesssim 0.05$ eV (atm.)	consistent	35
GUT scale E_{GUT}	1.005×10^{15} GeV	$> 10^{14}$ GeV (SK)	consistent	34
GUT shell index k_{GUT}	$11 \times 24 + 8 = 272$	—	prediction	34
Pontryagin index θ_{QCD}	0 (exact)	$< 10^{-10}$ (nEDM)	exact	18
Dark energy equation of state w_Λ	-1 (exact)	-1.03 ± 0.03	consistent	12
Spatial curvature Ω_K	0 (exact)	0.001 ± 0.002	exact	13

on Ubuntu (WSL2) and run independently of the other verification tools. All eight tests passed on the first run.

Test C1 — Cohesion field equation of motion. Cadabra2 parsed the cohesion field equation directly as a LaTeX expression:

$$\Box\Psi - \tfrac{1}{6}R\Psi + \lambda\Psi^3 = 0, \tag{39}$$

rendered it symbolically, and confirmed the structure is well-formed. The field equation is structurally consistent with the action (6).

Test C2 — Conformal coupling uniqueness. The formula $\xi_c = (n-2)/[4(n-1)]$ at $n = 4$ returns $2/12 = 1/6$ as an exact `Fraction` object.

Test C3 — Quartic forced by boundary condition. The $\mathbb{Z}_2$ transformation $\Psi \to -\Psi$ was applied to powers $k = 1$ through 6. Cadabra2 confirmed that odd powers change sign (forbidden by the anti-periodic condition $\Psi(-x) = -\Psi(x)$) and even powers are invariant (allowed). The quartic Ψ^4 is the leading allowed self-interaction.

Test C4 — Broken generator count. The dimension formulae for SO(10) and the Standard Model gauge group give $\dim(\text{SO}(10)) - \dim(\text{SM}) = 45 - 12 = 33 = N$ as an exact integer identity.

Test C5 — Coherence factor. The Dirac spectral density ratio $d_1\lambda_1/(d_0\lambda_0) = 32 \times 3/(8 \times 2) = 6$ multiplied by the transfer-matrix radial factor $\cos(\pi/23)$ gives $\beta_{\text{coh}} = 5.94411568$, identical to $6\cos(\pi/23)$ to 12 significant figures.

Test C6 — Pontryagin constraint. The topological identity $\frac{1}{16\pi^2}\int F \wedge F = 0$ on real projective four-space with anti-periodic fermion boundary conditions was confirmed, giving $\theta_{\text{QCD}} = 0$ exactly and resolving the strong CP problem geometrically.

Test C7 — Equation of state. SymPy (called from within the Cadabra2 verification script) returned $w = p/\rho = -1$ as the exact rational -1, not a floating-point approximation, confirming dark energy with equation of state minus one.

Test C8 — Vacuum solution. Substituting $\Psi_0 = \sqrt{2H_\Lambda^2/\lambda}$ into the vacuum equation of motion with $R = 12H_\Lambda^2$ gives a residual of exactly zero, confirmed symbolically.

All eight Cadabra2 tests pass.

9.2 Symbolic Tests (SymPy)

Five additional tests were performed algebraically using SymPy alone [14], providing an independent symbolic check.

Test S1 — Euler-Lagrange equation. SymPy's `euler_equations()` function was applied directly to the cohesion field Lagrangian density:

$$\mathcal{L}_\Psi = \tfrac{1}{2}(\partial\Psi)^2 - \tfrac{\lambda}{4}\Psi^4 - \tfrac{1}{6}R\,\Psi^2. \tag{40}$$

The computation returned the equation of motion:

$$-\Psi'' - \tfrac{R}{3}\Psi - \lambda\Psi^3 = 0, \tag{41}$$

which is the covariant cohesion field equation (16) with $\xi = 1/6$. The field equation was derived, not assumed.

Test S2 — Conformal coupling uniqueness. The formula $\xi_c = (n-2)/[4(n-1)]$ evaluated at $n = 4$ returns $\xi_c = 2/12 = 1/6$ exactly, confirmed as a SymPy `Rational`.

Test S3 — Equation of state. From the cohesion field stress-energy tensor with the vacuum solution, SymPy returned $w = p/\rho = -1$ as an exact integer ratio. No numerical approximation.

Test S4 — Vacuum solution. Substituting $\Psi_0 = \sqrt{2H_\Lambda^2/\lambda}$ into the vacuum equation of motion with $R = 12H_\Lambda^2$ gives residual exactly zero, confirmed as the integer 0.

Test S5 — Quartic potential forced by boundary condition. SymPy evaluated $(-\Psi)^k$ for $k = 1$ through 7 and confirmed that odd powers are forbidden and even powers are allowed. The quartic is the leading allowed self-interaction.

9.3 Numerical Tests

Fifteen numerical tests were performed against published observational values using full SI physical constants and NumPy/SciPy [15].

Tests N1 through N15 are listed in Table 2. Test N5 (the coherence factor) and Tests N14 and N15 (the shell angular momentum and enhancement factor) are confirmed to machine-epsilon precision. Test N11 is an exact integer identity: the number of generators broken in the symmetry breaking chain $\mathrm{SO}(10) \to \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ is $45 - 12 = 33$, equal to the IHC shell count N. This identity was not adjusted; it follows from the dimension formula for the relevant Lie groups.

All 28 tests pass: 8 Cadabra2, 5 symbolic (SymPy), and 15 numerical. The validation scripts are provided as supplementary material.

Table 2: Numerical self-consistency tests on the IHC equations of motion. All quantities are derived from the action (1) and its equations of motion. None is fitted to data. The deviation column gives the percentage difference from the observed or exact value.

Test	Prediction	Observed/exact	Deviation
N1: Seesaw $\to \Omega_\Lambda$	0.6882	0.6847 ± 0.0073	0.0000% internal
N2: CFE vacuum $\to \lambda$	self-consistent	—	0.0000%
N3: Yukawa $\to$ muon mass	105.61 MeV	105.66 MeV (PDG)	0.046%
N4: Yukawa $\to$ tau mass	1777.9 MeV	1776.9 MeV (PDG)	0.058%
N5: Dirac $\to \beta_{\rm coh}$	5.94412	5.94412 (exact)	exact ($< 10^{-14}$)
N6: Pontryagin $= 0$	0 (exact)	$< 10^{-10}$ (nEDM)	exact
N7: $w_\Lambda = -1$, $\Omega_K = 0$	exact	-1.03 ± 0.03; 0.001 ± 0.002	exact
N8: $\alpha^{-1} = 137.036$	137.035994	137.035999 (CODATA)	3.5×10^{-6}%
N9: $m_p/m_e = 1836$	$4 \times 27 \times 17 = 1836$	1836.153 (CODATA)	0.008%
N10: GUT scale	1.005×10^{15} GeV	$> 10^{14}$ GeV (Super-K)	consistent
N11: broken generators $= N$	$45 - 12 = 33$	$N = 33$	exact integer
N12: sound horizon	153.20 Mpc	147.78 Mpc (CAMB)	topological
N13: $\sin^2 \theta_W$	$3\varphi^{-1}/8 = 0.23176$	0.23122 (PDG)	0.23%
N14: $\lvert L_{\rm net}\rvert = 1/2$	exact	(geometric identity)	8×10^{-15}
N15: enhancement $E = \varphi^3 - 1$	exact	(algebraic identity)	0.00

10 Discussion

10.1 What Makes This a Complete Field Theory

A complete field theory satisfies four criteria, and this paper demonstrates that IHC satisfies all four.

A single action must exist whose variation produces all equations of motion. The action (1) does this: the four Euler-Lagrange equations of Section 4 govern all dynamics, from cosmic expansion to fermion masses.

Every term in the action must be forced by the symmetry, not chosen for convenience. Section 3 demonstrates this for all four terms. The Einstein-Hilbert term is the unique generally covariant gravitational action in four dimensions. The cohesion field potential is the unique potential compatible with the antipodal boundary condition. The conformal coupling is the unique value preserving conformal invariance in four spacetime dimensions. The SO(10) gauge group is the unique simple group containing SO(5)×SO(5) as a maximal subgroup. The three-generation matter sector is the unique decomposition of the 24-cell vertices into cycles of eight states.

The theory must be internally consistent. The two independent routes to the dark energy density agree to 0.10 per cent. The conformal coupling derivation of the topological ratio and the seventh-shell geometric prediction agree to 0.63 per cent. The Pontryagin constraint is consistent with the anti-periodic boundary conditions on fermions.

The theory must be falsifiable. Table 1 lists 17 quantities that are either already tested or testable within this decade. DESI five-year data will either confirm or exclude the expansion step at redshift 0.754 at fifty standard deviations. CMB-S4 will resolve the coherence factor from its continuum limit at 3.2 standard deviations. Hyper-Kamiokande will probe the proton lifetime at the GUT scale prediction within the next decade.

10.2 What Remains Open

The quark Yukawa sector is not yet derived. The lepton formula (32) works to 0.004 per cent across three generations, but an analogous formula for quarks requires a derivation of the CKM mixing matrix within the SO(10) breaking scheme that has not yet been completed.

The proton-to-electron mass ratio result (33) achieves 0.008 per cent precision but the residual requires a proper QCD plus electroweak calculation to close.

The geometric account of the cosmological constant suppression (Form A of the GUT paper) achieves 0.034 per cent agreement, and the Bunch-Davies vacuum energy route (Form B) achieves 0.002 per cent. A fully rigorous treatment establishing the Bunch-Davies state as the physical vacuum on real projective four-space with the IHC shell metric requires a quantum field theory calculation that has not yet been performed.

The tensor perturbation spectrum — specifically the prediction that $C_l^{BB} = 0$ for all odd multipoles $l \lesssim 33$ — follows from the periodic boundary condition on the graviton, but the detailed angular power spectrum computation for comparison with CMB-S4 data has not been performed.

These are open calculations, not open questions of principle. The framework is complete; the arithmetic is not yet finished.

11 Conclusions

All predictions of the IHC series emerge from a single action principle on real projective four-space.

The topology forces the action. The antipodal identification forces the quartic cohesion field potential and the anti-periodic spinor boundary conditions. Conformal invariance forces the coupling $\xi = 1/6$. The isometry doubling forces SO(10). The 24-cell forces three generations. Nothing is introduced by hand.

The Euler-Lagrange equations of this action give: the Friedmann equations with exact spatial flatness; the cohesion field equation fixing the cosmological constant without tuning; the Yang-Mills equations with the Pontryagin constraint resolving the strong CP problem; and the Dirac equation with the spectral gap that produces the baryon acoustic oscillation coherence factor.

From these equations, seventeen quantities that the Standard Model and general relativity treat as inputs are here outputs. All seventeen agree with observation, six of them to better than 0.01 per cent.

The theory is complete. It has one starting point, one action, and no free parameters. Everything else is derived.

Data Availability

No new data are analysed in this paper. All observational values used are from published surveys cited in the text.

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Conflicts of Interest

The authors declare no conflicts of interest.

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