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A Way of Constructing Lattice Packings of Equal Spheres Corresponding to the Packing Density of the Lambda Series

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Abstract—The creation of cryptographic systems based on lattice theory is a promising line in the field of post-quantum cryptography. The aim of this work was to obtain new properties of lattices through related objects: dense packings of equal spheres. We propose a way of constructing lattice packings of equal spheres corresponding to the packing density of the Lambda series in dimensions 1–24, using a series of coefficients to the height of the fundamental parallelepiped of dimension $(n-1)$: $1/2, 1/3, 1/2, 0, 1/2, 1/3, 1/2, \sqrt{-1}, 1/2, 1/3, 1/2, 0, 1/2, 1/3, 1/2, \sqrt{-1}, 1/2, 1/3, 1/2, 0,$ $1/2, 1/3, 1/2$. The construction of lattice packings of equal spheres using this procedure was used up to dimension 11 inclusive.

Keywords: post-quantum cryptography, geometry of numbers, lattice theory, arithmetic minima of positive quadratic forms, lattice packings of equal spheres, Hermite constant

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1. INTRODUCTION

One line of post-quantum cryptography is lattice cryptography [1]. The main problem from lattice theory used to construct cryptographic systems is that of finding the shortest vector in a lattice [1, 2]. A related problem is that of constructing dense lattice packings of equal spheres [3]. Dense packings of spheres are also widely used in information encoding algorithms [4]. In [5], a proof is presented that no packing of unit spheres in Euclidean space R^8 has a density greater than the density of Korkin–Zolotarev lattice packing E_8. In [6], symmetric-group regularities are established in the distribution of the values of the minima of positive quadratic forms (PQFs) reduced according to Korkin–Zolotarev, and their correspondence to the values of the minimum heights of fundamental parallelepipeds of lattice packings of equal spheres is established. In [7], critical lattices up to dimension 8 are presented, and some of their metric properties are considered. In [8], a connection is established between algorithms for finding the shortest lattice vector and the Hermite constant. In [9], possible values of the Hermite constant are given in dimensions 9-23. This work presents a way of constructing lattice packings of equal spheres corresponding to the packing density of the Lambda series [3] in dimensions 1–24. Lattice packings of equal spheres are constructed using this procedure up to dimension 11 inclusive.

2. DENOTATIONS AND BASIC INFORMATION

In n-dimensional Euclidean space E^n, an orthonormal coordinate system is fixed and an n-dimensional basis is defined: $\varepsilon = \{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\}$, i.e. a system of n arbitrary independent vectors $\vec{e_1}, ..., \vec{e_n}$ with a common origin coinciding with the origin. Let $\xi_{1j}, \xi_{2j}, ..., \xi_{nj}$ be the column of coordinates of vector $\vec{e_j}$ of basis ε in this coordinate system. So basis ε is given coordinate matrix $\Xi = (\xi_{ij})$, and $|\det \Xi|$ is equal to volume V of a parallelepiped constructed on the vectors of basis ε.

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Any aspect of the geometry of positive quadratic forms can be considered, depending on the convenience of its study:

or as a geometric fact of space E^n with the lattice given in it;

or as some property on the set of PQFs;

or as a geometric fact in space E^N [10].

Let an n-dimensional lattice $\Gamma \in E^n$ and a set of n-dimensional balls of the same radius with centers at the lattice points be given, and these balls form a packing, i.e., pairwise have no common interior points. We will call the maximum radius of balls of such a packing the radius of the packing corresponding to lattice Γ and denote it by $r(\Gamma)$:

$$r(\Gamma) = \frac{1}{2} \min \Gamma, \tag{1}$$

where $\min \Gamma$ is the length of the minimum lattice vector. The density of the lattice packing, i.e. the packing of spheres of radius $r(\Gamma)$ with centers at lattice points Γ, is called quantity

$$d(\Gamma) = \Omega_n \frac{r^n(\Gamma)}{V(\Gamma)}, \tag{2}$$

where $V(\Gamma)$ is the volume of the fundamental parallelepiped of lattice Γ, and Ω_n is the volume of the n-dimensional unit ball.

Central density of lattice packing $\delta = 1/V(\Gamma)$.

The problem of densest lattice packings in space E^n consists in finding value

$$d_n = \sup d(\Gamma) \tag{3}$$

on the set of n-dimensional lattices and those lattices on which they are achieved. Since density $d(\Gamma)$ does not change under similarity transformations of space E^n, then when solving the problem of the densest lattice packings, the investigated set of lattices can be normalized by requiring, for example, $V(\Gamma) = 1$ or $r(\Gamma) = 1$.

Let $\{f\}$ be the equivalence class of PQFs corresponding to lattice Γ, and $f \in \{f\}$ be some representative of this class. We then have

$$d(\Gamma) = \frac{\Omega_n}{2^n} [\min f/(\det f)^{1/n}]^{n/2}. \tag{4}$$

The problem of the densest lattice packing in space E^n is equivalent to the problem of finding the upper bound of the ratio $\min f/(\det f)^{1/n}$ on the set of PQFs of n variables:

$$\gamma_n = \sup[\min f/(\det f)^{1/n}] = \sup[(\min \Gamma_f)^2/(V(\Gamma_f))^{2/n}]. \tag{5}$$

Quantity γ_n is called the Hermite constant. Constant γ_n and the density of the closest lattice packing are related by the formula

$$d_n = 2^{-n}\, \Omega_n\, \gamma^{n/2}. \tag{6}$$

When finding the Hermite constant, we can consider not the entire set of PQFs from n variables, but take one representative from each equivalence class and normalize it by setting $\min f = 1$ or $\det f = 1$.

The theory of reducing the PQFs of an arbitrary number of variables was presented in the work of Korkin and Zolotarev and was used there as a way of obtaining values and estimates of the Hermite constant, and thus of solving the problem of the densest packings [11].

Let us consider the Lagrange expansion for an arbitrary PQF:

$$f = A_1(x_1 - \sum_{k=2}^{n} \alpha_{1k} x_k)^2 + A_2(x_2 - \sum_{k=3}^{n} \alpha_{2k} x_k)^2 + \ldots + A_l(x_l - \sum_{k=l+1}^{n} \alpha_{lk} x_k)^2 \tag{7}$$
$$+ \ \ldots \ + A_n(x_{n-1} - \alpha_{n-1,n} x_n)^2 + A_n x_n^2$$

A positive quadratic form is called reduced according to Korkin–Zolotarev if in its expansion for $l = 1, 2, ..., n-1$ and $k = l+1, l+2, ..., n$ coefficients A_l are, respectively, values of minima of forms

$$\phi_1 = f(x_1, x_2, ..., x_n), \phi_{l+1}(x_{l+1}, x_{l+2}, ..., x_n) = \phi_l(x_l, x_{l+1}, ..., x_n) - A_l(x_l - \sum_{k=l+1}^{n} \alpha_{lk} x_k)^2$$

and coefficients $\alpha_{l,l+k}, \alpha_{lk}$ satisfy inequalities $0 \leqslant \alpha_{l,l+k} \leqslant 1/2, -1/2 \leqslant \alpha_{lk} \leqslant 1/2$. The following equality holds for each Korkin–Zolotarev decomposition:

$$\det f = A_1 A_2 ... A_n, \tag{8}$$

the first Korkin–Zolotarev inequality, $A_{k+1} \geqslant (3/4)A_k$, where $k = 1, 2, ..., n-1$, and the second Korkin–Zolotarev inequality $A_{i+1} \geqslant (2/3)A_i$, where $i = 1, 2, ..., n-2$.

According to the definition of Hermite constant γ_n and equality (7), representing each PQF equivalence class according to the Korkin–Zolotarev expansion, we have

$$\gamma_n = \sup A_1 (A_1 A_2 ... A_n)^{-1/n}. \tag{9}$$

An estimate of the Hermite constant was obtained in [12]:

$$\gamma_n \leqslant (4/3)^{(n-1)/2}. \tag{10}$$

Let us define the minimum vector of lattice $\min \Gamma = 2$. The values of the heights of the fundamental parallelepiped h_n of the lattice Γ then correspond to those of coefficients A_n of positive quadratic forms reduced according to Korkin–Zolotarev and are related by equality [6]:

$$h_n = 2\sqrt{A_n}. \tag{11}$$

A lattice for which equality

$$\min \Gamma = V(\Gamma)^{1/n} \gamma_n^{1/2}; \tag{12}$$

is satisfied is called extremal or limit. A limit lattice corresponds to a limit class of the PQFs. There are a finite number of them for each n. The problem of the densest lattice packing is solved after finding all limit forms. It is sufficient to choose from the limit forms the one for which, with the same value of the determinant for all forms, that of the minimum is the greatest [10].

The exact values of the Hermite constant are known for dimensions 1-8 and 24 [8], for which the corresponding extremal (limit) lattices have been obtained.

In general, inequality

$$\min \Gamma \leqslant V(\Gamma)^{1/n} \gamma_n^{1/2} \tag{13}$$

holds.

The exact value of the Hermite constant in each dimension thus allows us to reduce the upper limit of the value of $\min \Gamma$, which can be used as a criterion for verification in algorithms for finding the shortest vector in arbitrary lattices. For example, it is known that $\gamma_{24} = 4$, so for lattices of dimension $n \geqslant 24$,

$$\min \Gamma \leqslant V(\Gamma)^{1/n} * 2, \tag{14}$$

which is much better than using Hermite constant estimate (9) when

$$\min \Gamma \leqslant V(\Gamma)^{1/n} * (4/3)^{23/4} \approx V(\Gamma)^{1/n} * 5.22875. \tag{15}$$

This work has considered a way of constructing lattice packings of equal spheres in dimensions up to $n = 24$, corresponding to the packing density of the Lambda series [3], by applying a series of coefficients to the height of the fundamental parallelepiped, i.e., essentially allowing for (8) and (10) as ways of obtaining the values of the Hermite constant.

It is worth noting that in dimensions 11–13 the Lambda series packings are not the densest. In these dimensions the Kappa series packings are the densest. Both sequences are different ways of describing sections of highly symmetric Leech lattice Λ_{24} in dimension 24 [3].

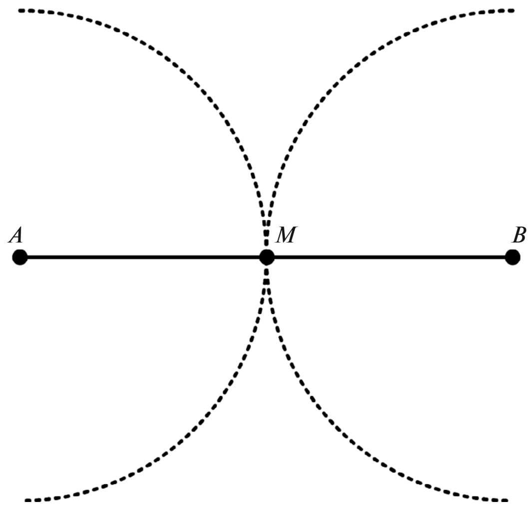

Fig. 1.

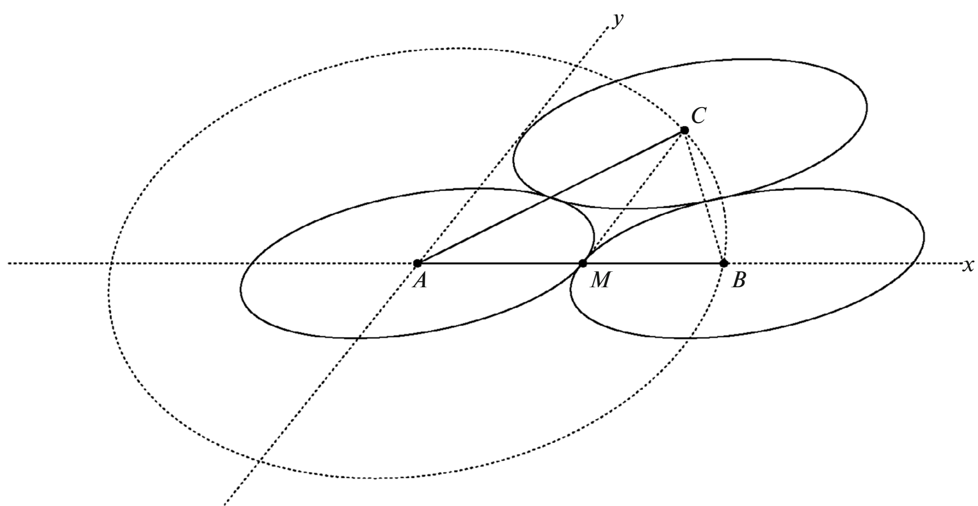

Fig. 2.

3. MAIN RESULTS

Proposition 1. *The height of the fundamental parallelepiped of the densest lattice packing of equal spheres in dimension $n = 2$ depends only on the choice of the radius of the packing and the base point of the height of the fundamental parallelepiped of dimension $n = 2$ by applying coefficient $1/2$ to the height of the fundamental parallelepiped of dimension $n = 1$.*

Let us consider a case where $n = 1$ (Fig. 1). We set the radius of packing, $r(\Gamma_1) = 1$, and the length of the minimal lattice vector $\min \Gamma_1 = 2$. Let us define point A as the origin and place the center of the one-dimensional unit ball in it. Let us place the center of the second one-dimensional unit ball at point B and distance $\min \Gamma_1 = 2$ from point A. We obtain a packing element of two one-dimensional unit balls with centers at points A and B. Tangency point $M \notin \Gamma_1$, $AM = MB = r(\Gamma_1) = 1$. Vector $\vec{a} = \overrightarrow{AB} = (0, 2)$ is the basis vector of lattice Γ_1, which corresponds to coordinate matrix $\Xi_1 = \left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right)$. The square of the volume of the fundamental parallelepiped of lattice $V(\Gamma_1)^2 = |\det \Xi_1|^2 = 2^2 = 4$. Contact number $K_N = 2$. The central density of lattice packing is $\delta_1 = 1/2$ and corresponds to the density of the closest lattice packing in $n = 1$, Λ_1 [13].

Let us consider a case where $n = 2$ (Fig. 2). To obtain the densest packing, we apply coefficient $1/2$ to the height of the fundamental parallelepiped from previous dimension $h_1 = AB = 2$, and find ourselves at point M, from which we draw a line perpendicular to point C on the circle formed by height $h_1 = AB$

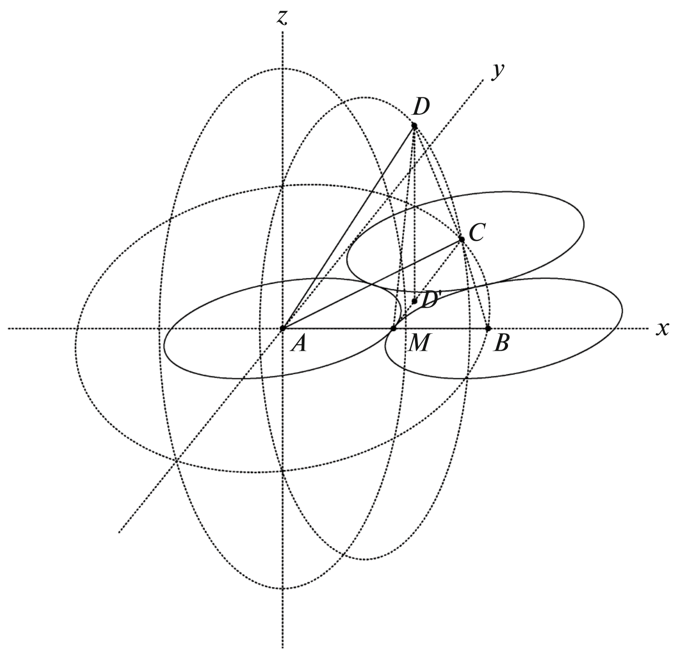

Fig. 3.

with its center at point A. The height of the fundamental parallelepiped in $n = 2$ can now be obtained using the Pythagorean theorem: $h_2 = CM = \sqrt{AC^2 - AM^2}$, where $AC = AB = h_1 = \min\Gamma_1 = 2$ and $AM = h_1/2 = r(\Gamma_1) = 1$. We obtain $h_2 = \sqrt{h_1^2 - (h_1/2)^2} = \sqrt{2^2 - (2/2)^2} = \sqrt{4-1} = \sqrt{3}$. The coordinate matrix of the lattice in $n = 2$, $\Xi_2 = \left(\begin{smallmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{smallmatrix}\right)$. The square of the volume of the fundamental parallelepiped of the lattice $V(\Gamma_2)^2 = |\det\Xi_2|^2 = (2 * \sqrt{3})^2 = 12$. Contact number $K_N = 6$. The central density of lattice packing is $\delta_2 = 1/(2\sqrt{3})$ and corresponds to the density of the closest lattice packing in $n = 2$, Λ_2 [13].

Proposition 2. *The height of the fundamental parallelepiped of the densest lattice packing of equal spheres in dimension $n = 3$ depends only on the choice of the radius of packing and the base point of the height of the fundamental parallel piped of dimension $n = 3$ by applying coefficient $1/3$ to the height of the fundamental parallelepiped of dimension $n = 2$.*

Let us consider a case where $n = 3$ (Fig. 3). To obtain the densest packing, we apply coefficient $1/3$ to the height of the fundamental parallelepiped from previous dimension $h_2 = MC = \sqrt{3}$. We obtain point D', from which we draw a perpendicular line to point D on the circle formed by height $h_2 = CM$ with the center at point M. The height of the fundamental parallelepiped in $n = 3$ can now be obtained using the Pythagorean theorem: $h_3 = DD' = \sqrt{MD^2 - MD'^2}$, where $MD = MC = h_2 = \sqrt{3}$ and $MD' = MC/3 = h_2/3 = \sqrt{3}/3$. We obtain $h_3 = \sqrt{h_2^2 - (h_2/3)^2} = \sqrt{\sqrt{3}^2 - (\sqrt{3}/3)^2} = \sqrt{3 - 1/3} = \sqrt{8/3}$.

The coordinate matrix of the lattice in $n = 3$ is

$$\Xi_3 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & \sqrt{3} & \sqrt{1/3} \\ 0 & 0 & \sqrt{8/3} \end{pmatrix}.$$

The square of the volume of the fundamental parallelepiped of lattice $V(\Gamma_3)^2 = |\det\Xi_3|^2 = (2 * \sqrt{3} * \sqrt{8/3})^2 = 32$. Contact number $K_N = 12$. The central density of lattice packing is $\delta_3 = 1/(4\sqrt{2})$ and corresponds to the density of the closest lattice packing in $n = 3$, Λ_3 [13].

Proposition 3. *The fundamental parallelepiped of the densest lattice packing of equal spheres in dimension $n = 4$ depends only on the choice of the radius of the packing and the base point of the height of the fundamental parallelepiped of dimension $n = 4$ by applying coefficient $1/2$ to the height of the fundamental parallelepiped of dimension $n = 3$.*

Let us consider a case where $n = 4$. To obtain the densest packing, we apply coefficient $1/2$ to the height of the fundamental parallelepiped from previous dimension $h_3 = DD' = \sqrt{8/3}$. We draw a perpendicular line from the resulting point to a point on the circle formed by height h_3 with the center at point D'. The height of the fundamental parallelepiped in $n = 4$ can now be obtained using the Pythagorean theorem: $h_4 = \sqrt{h_3^2 - (h_3/2)^2} = \sqrt{\sqrt{8/3}^2 - (\sqrt{8/12})^2} = \sqrt{8/3 - 2/3} = \sqrt{2}$. The coordinate matrix of the lattice in $n = 4$:

$$\Xi_4 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & \sqrt{3} & \sqrt{1/3} & -\sqrt{1/3} \\ 0 & 0 & \sqrt{8/3} & \sqrt{2/3} \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

The square of the volume of the fundamental parallelepiped of lattice $V(\Gamma_4)^2 = |\det \Xi_4|^2 = (2 * \sqrt{3} * \sqrt{8/3} * \sqrt{2})^2 = 64$. Contact number $K_N = 24$. The central density of lattice packing is $\delta_4 = 1/8$ and corresponds to the density of the closest lattice packing in $n = 4$, Λ_4 [13].

Proposition 4. *The height of the fundamental parallelepiped of the densest lattice packing of equal spheres in dimension $n = 5$ depends only on the choice of the radius of packing and the base point of the height of the fundamental parallelepiped of dimension $n = 5$ by applying coefficient 0 to the height of the fundamental parallelepiped of dimension $n = 4$.*

Let us consider a case when $n = 5$. To obtain the densest packing, we apply coefficient 0 to the height of the fundamental parallelepiped from previous dimension $h_4 = \sqrt{2}$. We draw a perpendicular line from the resulting point to a point on the circle formed by height h_4. The height of the fundamental parallelepiped in $n = 5$ can now be obtained using the Pythagorean theorem: $h_5 = \sqrt{h_4^2 - (h_4 * 0)^2} = \sqrt{\sqrt{2}^2 - 0^2} = \sqrt{2}$.

The coordinate matrix of the lattice in $n = 5$ is

$$\Xi_5 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 0 & \sqrt{3} & \sqrt{1/3} & -\sqrt{1/3} & -\sqrt{1/3} \\ 0 & 0 & \sqrt{8/3} & \sqrt{2/3} & \sqrt{2/3} \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

The square of the volume of the fundamental parallelepiped of lattice $V(\Gamma_5)^2 = |\det \Xi_5|^2 = (2 * \sqrt{3} * \sqrt{8/3} * \sqrt{2} * \sqrt{2})^2 = 128$. Contact number $K_N = 40$. The central density of lattice packing is $\delta_5 = 1/(8\sqrt{2})$ and corresponds to the density of the closest lattice packing in $n = 5$, Λ_5 [13].

Proposition 5. *The value of the height of the fundamental parallelepiped of the densest lattice packing of equal spheres in dimension $n = 6$ depends only on the choice of the radius of packing and the base point of the height of the fundamental parallelepiped of dimension $n = 6$ by applying coefficient $1/2$ to the height of the fundamental parallelepiped of dimension $n = 5$.*

Let us consider a case where $n = 6$. To obtain the densest packing, we apply coefficient $1/2$ to the height of the fundamental parallelepiped from previous dimension $h_5 = \sqrt{2}$. We draw a perpendicular line from the resulting point to a point on the circle formed by height h_5. The height of the fundamental parallelepiped in $n = 6$ can now be obtained using the Pythagorean theorem: $h_6 = \sqrt{h_5^2 - (h_5/2)^2} = \sqrt{\sqrt{2}^2 - (\sqrt{2}/2)^2} = \sqrt{3/2}$.

The coordinate matrix of the lattice in $n = 6$:

$$\Xi_6 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 0 & \sqrt{3} & \sqrt{1/3} & -\sqrt{1/3} & -\sqrt{1/3} & \sqrt{1/3} \\ 0 & 0 & \sqrt{8/3} & \sqrt{2/3} & \sqrt{2/3} & \sqrt{1/6} \\ 0 & 0 & 0 & \sqrt{2} & 0 & -\sqrt{1/2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{1/2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3/2} \end{pmatrix}.$$

The square of the volume of the fundamental parallelepiped of lattice $V(\Gamma_6)^2 = |\det \Xi_6|^2 = (2 * \sqrt{3} * \sqrt{8/3} * \sqrt{2} * \sqrt{2} * \sqrt{3/2})^2 = 192$. Contact number $K_N r =$. The central density of lattice packing is $\delta_6 = 1/(8\sqrt{3})$ and corresponds to the density of the closest lattice packing in $n = 6$, Λ_6 [13].

Proposition 6. *The height of the fundamental parallelepiped of the densest lattice packing of equal spheres in dimension $n = 7$ depends only on the choice of the radius of packing and the base point of the height of the fundamental parallelepiped of dimension $n = 7$ by applying coefficient $1/3$ to the height of the fundamental parallelepiped of dimension $n = 6$.*

Let us consider a case where $n = 7$. To obtain the densest packing, we apply coefficient $1/3$ to the height of the fundamental parallelepiped from previous dimension $h_6 = \sqrt{3/2}$. We draw a perpendicular line from the resulting point to a point on the circle formed by height h_6. The height of the fundamental parallelepiped in $n = 7$ can now be found using the Pythagorean theorem: $h_7 = \sqrt{h_6^2 - (h_6/3)^2} = \sqrt{\sqrt{3/2}^2 - (\sqrt{3/2}/3)^2} = \sqrt{4/3}$.

The coordinate matrix of the lattice in $n = 7$:

$$\Xi_7 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \sqrt{3} & \sqrt{1/3} & -\sqrt{1/3} & -\sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \\ 0 & 0 & \sqrt{8/3} & \sqrt{2/3} & \sqrt{2/3} & \sqrt{1/6} & \sqrt{1/6} \\ 0 & 0 & 0 & \sqrt{2} & 0 & -\sqrt{1/2} & \sqrt{1/2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{1/2} & -\sqrt{1/2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3/2} & -\sqrt{1/6} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{4/3} \end{pmatrix}.$$

The square of the volume of the fundamental parallelepiped of lattice $V(\Gamma_7)^2 = |\det \Xi_7|^2 = (2 * \sqrt{3} * \sqrt{8/3} * \sqrt{2} * \sqrt{2} * \sqrt{3/2} * \sqrt{4/3})^2 = 256$. Contact number $K_N = 126$. The central density of lattice packing $\delta_7 = 1/16$ and corresponds to the density of the closest lattice packing in $n = 7$, Λ_7 [13].

Proposition 7. *The height of the fundamental parallelepiped of the densest lattice packing of equal spheres in dimension $n = 8$ depends only on the choice of the radius of packing and the base point of the height of the fundamental parallelepiped of dimension $n = 8$ by applying coefficient $1/2$ to the height of the fundamental parallelepiped of dimension $n = 7$.*

Let us consider a case where $n = 8$. To obtain the densest packing, we apply the coefficient $1/2$ to the height of the fundamental parallelepiped from previous dimension $h_7 = \sqrt{4/3}$. We draw a perpendicular line from the resulting point to a point on the circle formed by height h_7. The height of the fundamental parallelepiped in $n = 8$ can now be obtained using the Pythagorean theorem: $h_8 = \sqrt{h_7^2 - (h_7/2)^2} = \sqrt{\sqrt{4/3}^2 - (\sqrt{4/3}/2)^2} = 1$.

The coordinate matrix of the lattice in $n = 8$ is

$$\Xi_8 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \sqrt{3} & \sqrt{1/3} & -\sqrt{1/3} & -\sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \\ 0 & 0 & \sqrt{8/3} & \sqrt{2/3} & \sqrt{2/3} & \sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} \\ 0 & 0 & 0 & \sqrt{2} & 0 & -\sqrt{1/2} & \sqrt{1/2} & \sqrt{1/2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{1/2} & -\sqrt{1/2} & \sqrt{1/2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3/2} & -\sqrt{1/6} & \sqrt{1/6} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{4/3} & \sqrt{1/3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The square of the volume of the fundamental parallelepiped of lattice $V(\Gamma_8)^2 = |\det \Xi_8|^2 = (2 * \sqrt{3} * \sqrt{8/3} * \sqrt{2} * \sqrt{2} * \sqrt{3/2} * \sqrt{4/3} * 1)^2 = 256$. Contact number $K_N = 240$. The central density of lattice packing $\delta_8 = 1/16$ and corresponds to the density of the closest lattice packing of $n = 8$, Λ_8 [13].

Proposition 8. *The height of the fundamental parallelepiped of the densest lattice packing of equal spheres in dimension $n = 9$ depends only on the choice of the radius of packing and the base point of the height of the fundamental parallelepiped of dimension $n = 9$ by applying coefficient $\sqrt{-1}$ to the height of the fundamental parallelepiped of dimension $n = 8$.*

Let us consider a case where $n = 9$. To obtain the densest packing, we apply coefficient $\sqrt{-1} = i$ to the height of the fundamental parallelepiped from previous dimension $h_8 = 1$. Since the real part of number i is zero, to obtain the base point for height h_9 in Euclidean space E^n, applying this coefficient is equivalent to multiplying by zero (see the case for $n = 5$). From the resulting point, we draw a perpendicular to a point on the circle formed by height h_8. The height of the fundamental parallelepiped in $n = 9$ can now be obtained using the Pythagorean theorem: $h_9 = \sqrt{h_8^2 - (h_8 * i)^2} = \sqrt{1^2 - (1 * \sqrt{-1})^2} = \sqrt{2}$.

The coordinate matrix of the lattice in $n = 9$:

$$\Xi_9 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & \sqrt{3} & \sqrt{1/3} & -\sqrt{1/3} & -\sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \\ 0 & 0 & \sqrt{8/3} & \sqrt{2/3} & \sqrt{2/3} & \sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} \\ 0 & 0 & 0 & \sqrt{2} & 0 & -\sqrt{1/2} & \sqrt{1/2} & \sqrt{1/2} & \sqrt{1/2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{1/2} & -\sqrt{1/2} & \sqrt{1/2} & -\sqrt{1/2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3/2} & -\sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{4/3} & \sqrt{1/3} & \sqrt{1/3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

The square of the volume of the fundamental parallelepiped of lattice $V(\Gamma_9)^2 = |\det \Xi_9|^2 = (2 * \sqrt{3} * \sqrt{8/3} * \sqrt{2} * \sqrt{2} * \sqrt{3/2} * \sqrt{4/3} * 1 * \sqrt{2})^2 = 512$. Contact number $K'_N = 2$. The central density of lattice packing is $\delta_9 = 1/(16\sqrt{2})$ and corresponds to the density of the densest lattice packing in $n = 9$, Λ_9 [13].

Proposition 9. *The height of the fundamental parallelepiped of the densest lattice packing of equal spheres in dimension $n = 10$ depends only on the choice of the radius of packing and the base point of the height of the fundamental parallelepiped of dimension $n = 10$ by applying coefficient $1/2$ to the height of the fundamental parallelepiped of dimension $n = 9$.*

Let us consider a case where $n = 10$. To obtain the densest packing, we apply coefficient $1/2$ to the height of the fundamental parallelepiped from previous dimension $h_9 = \sqrt{2}$. We draw a perpendicular line from the resulting point to a point on the circle formed by height h_9. The height of the fundamental parallelepiped in $n = 10$ can now be obtained using the Pythagorean theorem: $h_{10} = \sqrt{h_9^2 - (h_9 * 1/2)^2} = \sqrt{(\sqrt{2})^2 - (\sqrt{2} * 1/2)^2} = \sqrt{3/2}$. The coordinate matrix of the lattice in $n = 10$:

$$\Xi_{10} = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & \sqrt{3} & \sqrt{1/3} & -\sqrt{1/3} & -\sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \\ 0 & 0 & \sqrt{8/3} & \sqrt{2/3} & \sqrt{2/3} & \sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} \\ 0 & 0 & 0 & \sqrt{2} & 0 & -\sqrt{1/2} & \sqrt{1/2} & \sqrt{1/2} & \sqrt{1/2} & \sqrt{1/2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{1/2} & -\sqrt{1/2} & \sqrt{1/2} & -\sqrt{1/2} & \sqrt{1/2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3/2} & -\sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{4/3} & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{1/2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3/2} \end{pmatrix}.$$

The square of the volume of the fundamental parallelepiped of lattice $V(\Gamma_{10})^2 = |\det \Xi_{10}|^2 = (2 * \sqrt{3} * \sqrt{8/3} * \sqrt{2} * \sqrt{2} * \sqrt{3/2} * \sqrt{4/3} * 1 * \sqrt{2} * \sqrt{3/2})^2 = 768$. Contact number $K_N = 36$. The central density of lattice packing is $\delta_{10} = 1/(16\sqrt{3})$ and corresponds to the density of the closest lattice packing in $n = 10$, Λ_{10} [13].

Proposition 10. *The height of the fundamental parallelepiped of the second densest lattice packing of equal spheres in dimension $n = 11$ depends only on the choice of the radius of packing and the base point of the height of the fundamental parallelepiped of dimension $n = 11$ by applying coefficient $1/3$ to the height of the fundamental parallelepiped of dimension $n = 10$.*

Let us consider a case where $n = 11$. To obtain second-density packing, we apply coefficient $1/3$ to the height of the fundamental parallelepiped from previous dimension $h_{10} = \sqrt{3/2}$. We draw a perpendicular line from the resulting point to a point on the circle formed by height h_{10}. The height of the fundamental parallelepiped in $n = 11$ can now be obtained using the Pythagorean theorem: $h_{11} = \sqrt{h_{10}^2 - (h_{10} * 1/3)^2} = \sqrt{(\sqrt{3/2})^2 - (\sqrt{3/2} * 1/3)^2} = \sqrt{4/3}$.

The coordinate matrix of the lattice in $n = 11$:

$$\Xi_{11} = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \sqrt{3} & \sqrt{1/3} & -\sqrt{1/3} & -\sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \\ 0 & 0 & \sqrt{8/3} & \sqrt{2/3} & \sqrt{2/3} & \sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} \\ 0 & 0 & 0 & \sqrt{2} & 0 & -\sqrt{1/2} & \sqrt{1/2} & \sqrt{1/2} & \sqrt{1/2} & \sqrt{1/2} & \sqrt{1/2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{1/2} & -\sqrt{1/2} & \sqrt{1/2} & -\sqrt{1/2} & \sqrt{1/2} & \sqrt{1/2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3/2} & -\sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} & \sqrt{1/6} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{4/3} & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{1/2} & \sqrt{1/2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3/2} & -\sqrt{1/6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{4/3} \end{pmatrix}.$$

The square of the volume of the fundamental parallelepiped of lattice $V(\Gamma_{11})^2 = |\det \Xi_{11}|^2 = (2 * \sqrt{3} * \sqrt{8/3} * \sqrt{2} * \sqrt{2} * \sqrt{3/2} * \sqrt{4/3} * 1 * \sqrt{2} * \sqrt{3/2} * \sqrt{4/3})^2 = 1024$. Contact number $K_N = 438$. The central density of lattice packing $\delta_{11} = 1/32$ and corresponds to the density of the second densest lattice packing in $n = 11$, $\Lambda_{11}^{\max}$ [14].

Proposition 11. *The heights of the fundamental parallelepipeds of second-density lattice packings of equal spheres in dimensions* $n = 12, 13$ *depend only on the choice of the radius of packing and the base points of the heights of the fundamental parallelepipeds of dimensions* $n = 12, 13$ *by applying coefficients* $1/2$ *and* 0 *to the heights of the fundamental parallelepipeds of dimensions* $n = 11$ *and* $n = 12$*, respectively.*

Proposition 12. *The heights of the fundamental parallelepipeds of the densest lattice packings of equal spheres in dimensions of* $n = 14$ *to* $n = 24$ *depend only on the choice of the radius of packing and the base points of the heights of the fundamental parallelepipeds in dimensions from* $n = 14$ *to* $n = 24$ *by applying coefficients* $1/2, 1/3, 1/2, \sqrt{-1}, 1/2, 1/3, 1/2, 0, 1/2, 1/3, 1/2$ *to the heights of the fundamental parallelepipeds in dimensions from* $n = 13$ *to* $n = 23$*, respectively.*

4. CONCLUSIONS

We obtained new properties of lattices and related objects and proposed a way of constructing lattice packings of equal spheres corresponding to the packing density of the Lambda series in dimensions 1–24, using a series of coefficients to the height of the fundamental parallelepiped of dimension $(n - 1)$: 1/2, 1/3, 1/2, 0, 1/2, 1/3, 1/2, $\sqrt{-1}$, 1/2, 1/3, 1/2, 0, 1/2, 1/3, 1/2, $\sqrt{-1}$, 1/2, 1/3, 1/2, 0, 1/2, 1/3, 1/2.

We constructed lattice packings of equal spheres in this way up to dimension 11 inclusive. The use of the obtained properties of lattice packings of equal spheres in dimensions 1–24 is a promising direction for studying the following groups of dimensions with the aim of extending the obtained patterns to the case of higher dimensions.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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