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Variational derivation and compatible discretizations of the Maxwell-GLM system

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We present a novel variational derivation of the Maxwell-GLM system, which augments the original vacuum Maxwell equations via a generalized Lagrangian multiplier (GLM) approach by adding two supplementary acoustic subsystems and which was originally introduced by Munz *et al.* for purely numerical purposes to treat the divergence constraints of the magnetic and the electric field in the vacuum Maxwell equations within general purpose and non-structure-preserving numerical schemes for hyperbolic partial differential equations (PDEs). The mathematical properties of the model are: (i) its symmetric hyperbolicity, (ii) the extra conservation law for the total energy density and, most importantly, (iii) the very peculiar combination of the basic differential operators, since both curl-curl and div-grad combinations are mixed within this kind of system. A similar mixture of Maxwell-type and acoustic-type subsystems has recently been also forwarded by Buchman *et al.* in the context of a reformulation of the Einstein field equations of general relativity in terms of tetrads. This motivates our interest in this class of PDE, since the system is by itself very interesting from a mathematical point of view and can therefore serve as useful prototype system for the development of new structure-preserving numerical methods. Up to now, to the best of our knowledge, there exists neither a rigorous variational derivation of this class of hyperbolic PDE systems, nor do exactly energy-conserving and asymptotic-preserving (AP) schemes exist for them. The objectives of this paper are to derive the Maxwell-GLM system from an underlying variational principle, show its consistency with Hamiltonian mechanics and special relativity, extend it to the general nonlinear case and to develop new exactly energy-conserving and AP finite volume schemes for its discretization.

1. Introduction

Owing to their mathematical elegance, since their discovery, the Maxwell equations of electromagnetism [1] have inspired a vast amount of research in theoretical physics as well as in pure and applied mathematics. The first compatible numerical scheme for the discretization of the Maxwell equations in the time domain is due to Yee [2], who introduced a suitable staggered mesh for the definition of the discrete electric and magnetic field in combination with compatible differential operators that guarantee that the discrete divergence of the discrete magnetic field is always zero and that the discrete divergence of the electric field vanishes in the absence of charges. Starting from the Yee scheme several different and more general numerical frameworks have been developed to guarantee the divergence-free property exactly at the discrete level, that is, via the use of so-called mimetic finite differences [3–5], compatible finite volume schemes [6–8] and compatible finite elements [9–14]. Many fewer schemes have been developed that preserve the curl-free property of a vector field, see e.g. [15–19]. Instead of imposing the divergence constraints on the electric and the magnetic field exactly, Munz *et al.* in [20] introduced a new approach called generalized Lagrangian multiplier (GLM) divergence cleaning, which resembles conceptually the concept of artificial compressibility applied in the numerical solution of the incompressible Navier–Stokes equations [21]. The idea of GLM divergence cleaning is to admit divergence errors and to augment the original Maxwell equations by two additional acoustic subsystems so that the numerical divergence errors in the magnetic and electric field, which are generated by not exactly compatible general purpose schemes, do not accumulate locally but are rather transported away with an artificial cleaning speed c_h via the acoustic subsystems that are coupled to the original Maxwell equations. A similar approach has recently also been forwarded for the numerical treatment of curl errors in curl-free vector fields, see [22–26], leading to a similar combination of acoustic and Maxwell-type systems. The mathematical structure of the augmented Maxwell-GLM system is very intriguing due to its particular combination of curl-curl and div-grad operators, which are usually not present in classical partial differential equation (PDE) systems of continuum physics.

Based on the groundbreaking work of Godunov [27,28], which established for the first time a rigorous connection between underlying variational principle, thermodynamic compatibility and symmetric hyperbolicity in the sense of Friedrichs [29], a vast class of PDE systems of continuum physics has been subsequently studied and developed by Godunov and Romenski and collaborators in the last decades, the so-called class of symmetric hyperbolic and thermodynamically compatible (SHTC) systems, see [30–39] and references therein. The systems which fall into this class range from the compressible Euler and MHD equations over the equations of nonlinear hyperelasticity to the equations describing compressible multi-phase flows, superfluid helium and continuum mechanics with torsion. However, so far, in this very large class of systems no mixed coupling of curl-curl and div-grad operators was present, which is typical for the Maxwell-GLM system studied in this paper. Yet, as will be shown later, the Maxwell-GLM system actually falls into the class of SHTC systems and can be derived from an underlying variational principle. This paper therefore extends the so far known class of SHTC systems.

Another system that indeed has a similar structure to the Maxwell-GLM system studied in this paper is the tetrad formulation of Buchman *et al.* of the Einstein field equations of general relativity, see [40,41]. For a certain gauge choices, the principal part of their PDE system reduces to a large set of Maxwell-type equations that are coupled with acoustic subsystems. Therefore, we believe that the Maxwell-GLM system studied in this paper is a useful mathematical prototype system for other, more complex, field theories and is also suitable for the development of new structure-preserving numerical methods. The main objectives of this paper are therefore the following:

- — provide a new derivation of the Maxwell-GLM system from an underlying variational principle;
- — show the symmetric hyperbolicity of the system and the related additional extra conservation law for the total energy density;
- — study the asymptotic behaviour of the Maxwell-GLM system in the case when the cleaning speed c_h tends to infinity;
- — extend the Maxwell-GLM system to a more general nonlinear Lagrangian densities; this together with the variational principle and the extra conservation law allows us to conclude that the Maxwell-GLM system falls indeed into the wider class of SHTC systems established by Godunov and Romenski and thus enlarges the known class of SHTC systems by a new one with different structure;
- — show the connection to Hamiltonian mechanics by providing a Poisson bracket, from which the system can also be derived;
- — show the compatibility of the new variational principle with special relativity;
- — develop new structure-preserving schemes for the Maxwell-GLM system and study their behaviour.

The rest of this paper is therefore organized as follows. In §2, we present the augmented Maxwell-GLM system together with its extra conservation law for the total energy density, we show its symmetric hyperbolicity, study the behaviour of the model in the limit when the cleaning speed tends to infinity and present a novel and rigorous variational derivation of this kind of PDE system. The only similar attempt we are aware of was documented in [42] but it is rather different from the variational approach presented in this paper. In §3, we show the compatibility of the presented variational principle with special relativity, showing the connection with the Maxwell tensor and the Faraday tensor and that the Lagrangian density is a relativistic scalar invariant. In §4, we present two compatible discretizations. The first scheme is a semi-discrete finite volume method on collocated grids that preserves the total energy exactly and which also trivially extends to the nonlinear case. The second scheme is a vertex-based staggered semi-implicit scheme that preserves the basic vector calculus identities $\nabla \cdot \nabla \times \mathbf{A} = 0$ and $\nabla \times \nabla \phi = 0$ exactly on the discrete level and which is also exactly totally energy conservative. Some numerical results are presented in §5. The paper closes with some concluding remarks and an outlook to future research in §6.

2. The augmented Maxwell-GLM system

(a) Governing equations and extra conservation law

The Maxwell-GLM system of Munz *et al.* [20], which consists of the vacuum Maxwell equations augmented by two acoustic subsystems reads

$$\partial_t \mathbf{B} + c_0 \nabla \times \mathbf{E} + c_h \nabla \phi = 0, \tag{2.1}$$

$$\partial_t \phi + c_h \nabla \cdot \mathbf{B} = 0, \tag{2.2}$$

$$\partial_t \mathbf{E} - c_0 \nabla \times \mathbf{B} + c_h \nabla \psi = 0 \tag{2.3}$$

and

$$\partial_t \psi + c_h \nabla \cdot \mathbf{E} = 0. \tag{2.4}$$

In the following, we will refer to equations (2.1)–(2.4) also as the *Maxwell–Munz* system. Here, $c_0 \in \mathbb{R}^+$ is the constant vacuum light speed of the original Maxwell equations and $c_h \in \mathbb{R}^+$ is the constant speed associated with the two acoustic subsystems. For Maxwell-compatible initial data $\phi = \psi = 0$ and $\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{E} = 0$ at $t = 0$ the system preserves the involutions $\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{E} = 0$ also for all times $t > 0$. Note that in the above system we have replaced the electric field of the original Maxwell equations by a rescaled electric field $c_0\mathbf{E}$, to make the symmetric hyperbolicity more evident. Throughout this paper, we assume c_0 and c_h to be constant in space and time. The above system contains a mixed combination of curl-curl and div-grad operators that go beyond the standard Maxwell equations or those for linear acoustics. The same structure is present in the tetrad formulation of the Einstein field equations of Buchman *et al.* [40,41] for certain gauge choices, which make the above system an interesting mathematical prototype system. In the following, we show that the above system admits an extra conservation law for the total energy density:

$$\mathcal{E} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \frac{1}{2}(\psi^2 + \phi^2). \tag{2.5}$$

We follow the pioneering work of Godunov [27] and Romenski [34] and introduce the state vector $\mathbf{q} = (\mathbf{B}, \phi, \mathbf{E}, \psi)$, as well as the associated vector of thermodynamic *dual* variables, the so-called *main field* of Ruggeri *et al.* [43], defined as $\mathbf{p} = \partial\mathcal{E}/\partial\mathbf{q}$. Furthermore, we introduce the so-called *generating potential* L, [34], which is the Legendre transform of $\mathcal{E}$ and is defined as

$$L = \mathbf{p} \cdot \mathbf{q} - \mathcal{E}. \tag{2.6}$$

For the simple quadratic energy potential (equation (2.5)), we immediately obtain $\mathbf{p} = \mathbf{q} = (\mathbf{B}, \phi, \mathbf{E}, \psi)$ and $L = \mathcal{E}$. Multiplying equations (2.1)–(2.4) with the dual variables $\mathbf{p}$ and summing

$$\mathbf{B} \cdot (2.1) + \phi \cdot (2.2) + \mathbf{E} \cdot (2.3) + \psi \cdot (2.4) \tag{2.7}$$

yields, after simple calculations, the following total energy conservation law

$$\partial_t \mathcal{E} + \nabla \cdot (c_0 \mathbf{E} \times \mathbf{B} + c_h(\psi \mathbf{E} + \phi \mathbf{B})) = 0, \tag{2.8}$$

where $\mathbf{E} \times \mathbf{B}$ is the Poynting vector. It is obvious that for initial data that are compatible with the original vacuum Maxwell equations, i.e. that satisfy $\nabla \cdot \mathbf{B} = 0$, $\nabla \cdot \mathbf{E} = 0$ and $\phi = const$, $\psi = const$ at the initial time $t = 0$ the divergence of the magnetic and of the electric field remain zero for all times.

(b) Symmetric hyperbolicity

The system (equations (2.1)–(2.4)) is *symmetric hyperbolic* in the sense of Friedrichs [29], since it can be written in the form (summation over repeated space indices is assumed over the entire paper)

$$\mathbf{M}\partial_t \mathbf{p} + \mathbf{H}_k \partial_k \mathbf{p} = 0, \quad k \in \{1, 2, 3\}, \tag{2.9}$$

with $\partial_k = \partial/\partial x_k$, $\mathbf{M} = L_{\mathbf{pp}} = \frac{\partial^2 L}{\partial \mathbf{p} \partial \mathbf{p}} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix and the matrices $\mathbf{H}_k$ are given by

$$\mathbf{H}_1 = \begin{pmatrix} 0 & 0 & 0 & c_h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_0 & 0 & 0 \\ c_h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_h \\ 0 & 0 & c_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_h & 0 & 0 & 0 \end{pmatrix}, \tag{2.10}$$

$$\mathbf{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & c_0 & 0 \\ 0 & 0 & 0 & c_h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_0 & 0 & 0 & 0 \\ 0 & c_h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_h \\ c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_h & 0 & 0 \end{pmatrix} \tag{2.11}$$

and
$$\mathbf{H}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -c_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_h & 0 & 0 & 0 & 0 \\ 0 & 0 & c_h & 0 & 0 & 0 & 0 & 0 \\ 0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_h \\ 0 & 0 & 0 & 0 & 0 & 0 & c_h & 0 \end{pmatrix}. \tag{2.12}$$

The matrices $\mathbf{H}_i$ are obviously symmetric and the matrix $\mathbf{M} = \mathbf{I}$ is clearly symmetric and strictly positive definite, which is an immediate consequence of the strict convexity of the quadratic total energy potential. We therefore conclude that system (equations (2.1)–(2.4)) is symmetric hyperbolic in the sense of Friedrichs [29]. For the sake of completeness, we also give the eigenvalues and eigenvectors of the system. The eigenvalues of all matrices $\mathbf{H}_i$ are

$$\lambda_{1,2} = -c_h, \quad \lambda_{3,4} = -c_0, \quad \lambda_{5,6} = +c_0 \quad \text{and} \quad \lambda_{7,8} = +c_h. \tag{2.13}$$

It is important to realize that the eigenvalues associated with c_h do not correspond to the propagation of any physical signal, which allows us to analyse the case when $c_h > c_0$. The associated matrix of right eigenvectors for $\mathbf{H}_1$ reads

$$\mathbf{R} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.14}$$

According to [44,45] the symmetric hyperbolicity of the system guarantees well-posedness of the initial value problem, at least for short times.

(c) Formal asymptotic limit for $c_h \to \infty$

In this section, we carry out a formal asymptotic analysis of the Maxwell–Munz system in the case of $c_h \to \infty$ for constant c_0, i.e. when the dimensionless parameter $\epsilon = c_0/c_h \to 0$. We follow the procedure outlined in [46,47] for the compressible Navier–Stokes equations, while a mathematically rigorous analysis of the incompressible limit of the compressible Euler, Navier–Stokes and MHD equations was provided in [48,49]. Similar to [48,49], we assume well-prepared initial data. In particular, we require that $\nabla \cdot \mathbf{B}(\mathbf{x}, 0) = \mathcal{O}(\epsilon^2)$ and $\nabla \cdot \mathbf{E}(\mathbf{x}, 0) = \mathcal{O}(\epsilon^2)$ hold, as well as $\phi(\mathbf{x}, 0) = \phi_0 = const$ and $\psi(\mathbf{x}, 0) = \psi_0 = const$. Expanding all variables in a power series of ϵ, the so-called Hilbert expansion, yields

$$\mathbf{B} = \mathbf{B}_0 + \epsilon \mathbf{B}_1 + \epsilon^2 \mathbf{B}_2 + \mathcal{O}(\epsilon^3), \tag{2.15}$$

$$\phi = \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \mathcal{O}(\epsilon^3), \tag{2.16}$$

$$\mathbf{E} = \mathbf{E}_0 + \epsilon \mathbf{E}_1 + \epsilon^2 \mathbf{E}_2 + \mathcal{O}(\epsilon^3) \tag{2.17}$$

and
$$\psi = \psi_0 + \epsilon \psi_1 + \epsilon^2 \psi_2 + \mathcal{O}(\epsilon^3). \tag{2.18}$$

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Inserting the above series expansions into all equations (2.1)–(2.4) dividing by c_0 and collecting terms with equal powers in $\epsilon = c_0/c_h$ leads to

$$\epsilon^{-1}\nabla\phi_0 + \epsilon^0(c_0^{-1}\partial_t\mathbf{B}_0 + \nabla\times\mathbf{E}_0 + \nabla\phi_1) + \mathcal{O}(\epsilon) = 0, \tag{2.19}$$

$$\epsilon^{-1}\nabla\cdot\mathbf{B}_0 + \epsilon^0(c_0^{-1}\partial_t\phi_0 + \nabla\cdot\mathbf{B}_1) + \mathcal{O}(\epsilon) = 0, \tag{2.20}$$

$$\epsilon^{-1}\nabla\psi_0 + \epsilon^0(c_0^{-1}\partial_t\mathbf{E}_0 - \nabla\times\mathbf{B}_0 + \nabla\psi_1) + \mathcal{O}(\epsilon) = 0 \tag{2.21}$$

and

$$\epsilon^{-1}\nabla\cdot\mathbf{E}_0 + \epsilon^0(c_0^{-1}\partial_t\psi_0 + \nabla\cdot\mathbf{E}_1) + \mathcal{O}(\epsilon) = 0. \tag{2.22}$$

Note that all quantities $\mathbf{B}$, ϕ, $\mathbf{E}$ and ψ must be *bounded* due to the energy conservation principle (equation (2.8)). Equations (2.20) and (2.22) immediately provide the result that the leading-order magnetic field and the leading-order electric field must become divergence-free for $\epsilon \to 0$, i.e.

$$\nabla\cdot\mathbf{B}_0 = 0, \quad \text{and} \quad \nabla\cdot\mathbf{E}_0 = 0. \tag{2.23}$$

Furthermore, from equations (2.19) and (2.21), we obtain

$$\nabla\phi_0 = 0, \quad \text{and} \quad \nabla\psi_0 = 0, \tag{2.24}$$

which means that for $\epsilon \to 0$ the leading-order terms of the two scalar fields become constant in space and thus depend only on time, i.e. $\phi_0 = \phi_0(t)$ and $\psi_0 = \psi_0(t)$. Taking the terms of order $\epsilon^0 = 1$ in equations (2.20) and (2.22) yields

$$c_0^{-1}\partial_t\phi_0 + \nabla\cdot\mathbf{B}_1 = 0 \quad \text{and} \quad c_0^{-1}\partial_t\psi_0 + \nabla\cdot\mathbf{E}_1 = 0. \tag{2.25}$$

Since ϕ_0 and ψ_0 are constant in space and only functions in time, application of the Gauss theorem provides the time rate of change of both ϕ_0 and ψ_0 as

$$\frac{\mathrm{d}\phi_0}{\mathrm{d}t} = -c_0\int_{\partial\Omega}\mathbf{B}_1\cdot\mathbf{n}\,\mathrm{d}S, \quad \text{and} \quad \frac{\mathrm{d}\psi_0}{\mathrm{d}t} = -c_0\int_{\partial\Omega}\mathbf{E}_1\cdot\mathbf{n}\,\mathrm{d}S. \tag{2.26}$$

Assuming that the following *boundary conditions* hold for both, the magnetic and the electric field

$$\mathbf{B}\cdot\mathbf{n} = 0, \quad \forall\,\mathbf{x}\in\partial\Omega \quad \text{and} \quad \mathbf{E}\cdot\mathbf{n} = 0, \quad \forall\,\mathbf{x}\in\partial\Omega \tag{2.27}$$

we obtain from equation (2.26) that $\mathrm{d}\phi_0/\mathrm{d}t = \mathrm{d}\psi_0/\mathrm{d}t = 0$. Hence, from the terms of order ϵ^0 of equations (2.20) and (2.22) we find that with the boundary conditions (equation (2.27)) we also have

$$\nabla\cdot\mathbf{B}_1 = 0 \quad \text{and} \quad \nabla\cdot\mathbf{E}_1 = 0. \tag{2.28}$$

Thus, from equations (2.15) and (2.17) we, therefore, can conclude that for $\epsilon \to 0$ the divergence errors of the magnetic and of the electric field scale quadratically in ϵ, i.e.

$$\nabla\cdot\mathbf{B} = \mathcal{O}(\epsilon^2) \quad \text{and} \quad \nabla\cdot\mathbf{E} = \mathcal{O}(\epsilon^2). \tag{2.29}$$

Note that this is not a mathematically rigorous proof, but just a formal asymptotic analysis. A rigorous mathematical analysis is out of scope of the present paper and is left to future work. We also emphasize the importance of the assumption of well-prepared initial data.

(d) Underlying variational principle

To derive equations (2.1)–(2.4) from an underlying variational principle we introduce a vector potential $\mathbf{A}$ and a scalar potential Z. We then make the following *definitions*:

$$\mathbf{E} = -\frac{1}{c_0}\partial_t\mathbf{A}, \quad \phi = -\frac{1}{c_h}\partial_t Z \tag{2.30}$$

and

$$\mathbf{B} = \nabla\times\mathbf{A} + \nabla Z, \quad \psi = \frac{c_h}{c_0}\nabla\cdot\mathbf{A}. \tag{2.31}$$

We also introduce the additional quantity

$$\mathbf{Y} = \nabla Z. \tag{2.32}$$

Since c_0 and c_h are constant, an *immediate consequence* of the above definitions combined with classical vector calculus identities are the following relations:

$$\nabla \cdot \mathbf{B} = \nabla^2 Z = \nabla \cdot \mathbf{Y} \tag{2.33}$$

and

$$\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A}, \tag{2.34}$$

the first one of which is similar in structure to the pressure-correction equation in the context of numerical methods for the incompressible Navier–Stokes equations [50–52], while the second one is classical in electrodynamics. Furthermore, applying the time derivative to equation (2.31) and using the definitions (equation (2.30)) we immediately obtain the time evolution equation for the magnetic field as

$$\partial_t \mathbf{B} - \nabla \times \partial_t \mathbf{A} - \nabla \partial_t Z = 0, \quad \text{hence} \quad \partial_t \mathbf{B} + c_0 \nabla \times \mathbf{E} + c_h \nabla \phi = 0, \tag{2.35}$$

and the time evolution equation for the scalar field ψ as

$$\partial_t \psi - \frac{c_h}{c_0} \nabla \cdot \partial_t \mathbf{A} = 0, \quad \text{hence} \quad \partial_t \psi + c_h \nabla \cdot \mathbf{E} = 0. \tag{2.36}$$

It is obvious that equations (2.35) and (2.36) are identical with equations (2.1) and (2.4), respectively. A further immediate consequence of the above definitions is the PDE

$$\partial_t \mathbf{Y} + \nabla \phi = 0, \tag{2.37}$$

but for which we have so far found no use. To obtain the missing governing PDE for $\mathbf{E}$ and ϕ, we now postulate the following Lagrangian density in terms of $\mathbf{A}$ and Z:

$$\Lambda = \frac{1}{2} \left[\left(\frac{1}{c_0} \partial_t \mathbf{A} \right)^2 - (\nabla \times \mathbf{A})^2 - \left(\frac{c_h}{c_0} \nabla \cdot \mathbf{A} \right)^2 + \left(\frac{1}{c_h} \partial_t Z \right)^2 - \nabla Z \cdot \nabla Z \right] \tag{2.38}$$

or, in index notation,

$$\Lambda = \frac{1}{2} \left(\frac{1}{c_0^2} \partial_t A_i \partial_t A_i - \epsilon_{lpq} \partial_p A_q \epsilon_{lrs} \partial_r A_s - \frac{c_h^2}{c_0^2} \partial_r A_r \delta_{pq} \partial_p A_q + \left(\frac{1}{c_h} \partial_t Z \right)^2 - \partial_i Z \partial_i Z \right), \tag{2.39}$$

where δ_{ik} is the usual Kronecker delta, while ϵ_{ijk} is the three-dimensional Levi-Civita tensor. We now state the following *variational principle*: Find $\mathbf{A}$ and Z so that the action

$$S[A_k, Z] = \int_\Omega \Lambda \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \tag{2.40}$$

is minimized. The first Euler–Lagrange equation related to equation (2.40) and the scalar potential Z reads ($Z_{,t} := \partial_t Z$, $Z_{,k} := \partial_k Z$):

$$-\frac{\partial}{\partial t} \frac{\partial \Lambda}{\partial Z_{,t}} - \frac{\partial}{\partial x_k} \frac{\partial \Lambda}{\partial Z_{,k}} + \frac{\partial \Lambda}{\partial Z} = 0, \tag{2.41}$$

which immediately becomes

$$-\frac{1}{c_h^2} \frac{\partial Z_{,t}}{\partial t} + \delta_{ik} \frac{\partial Z_{,i}}{\partial x_k} = 0 \tag{2.42}$$

and

$$\frac{1}{c_h} \partial_t \phi + \nabla^2 Z = 0, \tag{2.43}$$

which, thanks to equation (2.33) becomes the final evolution equation of ϕ, i.e. equation (2.2):

$$\partial_t \phi + c_h \nabla \cdot \mathbf{B} = 0. \tag{2.44}$$

Similarly, the Euler–Lagrange equation related to equation (2.40) and the vector potential **A** reads ($A_{i,t} := \partial_t A_i$, $A_{i,k} := \partial_k A_i$)

$$-\frac{\partial}{\partial t}\frac{\partial \Lambda}{\partial A_{i,t}} - \frac{\partial}{\partial x_k}\frac{\partial \Lambda}{\partial A_{i,k}} + \frac{\partial \Lambda}{\partial A_i} = 0, \tag{2.45}$$

which translates into

$$-\frac{1}{c_0^2}\frac{\partial A_{i,t}}{\partial t} + \frac{\partial}{\partial x_k}\epsilon_{lpq}A_{q,p}\epsilon_{lrs}\delta_{kr}\delta_{is} + \frac{c_h^2}{c_0^2}\frac{\partial}{\partial x_k}A_{r,r}\delta_{pq}\delta_{pk}\delta_{qi} = 0 \tag{2.46}$$

or, equivalently,

$$-\frac{1}{c_0^2}\frac{\partial A_{i,t}}{\partial t} + \frac{\partial}{\partial x_k}\epsilon_{lpq}A_{q,p}\epsilon_{lki} + \frac{c_h^2}{c_0^2}\frac{\partial}{\partial x_k}A_{r,r}\delta_{ki} = 0. \tag{2.47}$$

In standard vector notation and using the identities (equations (2.30) and (2.34)), the above equation becomes

$$\partial_t \mathbf{E} - c_0 \nabla \times \mathbf{B} + c_h \nabla \psi = 0, \tag{2.48}$$

which is identical to equation (2.3). We therefore have established the underlying variational principle from which the Maxwell–Munz system can be derived. We stress that the procedure shown in this paper is different from the one introduced in [42]. We note that the PDE for the magnetic field **B** (equation (2.1)) and for the scalar field ψ are immediate consequences of the definitions, while the PDE for the electric field **E** and for the scalar quantity ϕ are the Euler–Lagrange equations associated with the variational principle (equation (2.39)). As usual in the framework of SHTC equations [34], the equations come in pairs. One being an Euler–Lagrange equation, the other one being a mere consequence of the definitions. We recall, for completeness and for comparison, that the original Lagrangian of the classical Maxwell equations reads

$$\Lambda_{\mathrm{Max}} = \frac{1}{2}\left[\left(\frac{1}{c_0}\partial_t \mathbf{A}\right)^2 - (\nabla \times \mathbf{A})^2\right], \tag{2.49}$$

with the definitions $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\partial_t \mathbf{A}/c_0$, see [53].

(e) Variational formulation for arbitrary nonlinear Lagrangian

Let us show that the variational derivation described in the previous section can be generalized to an arbitrary Lagrangian, which might be important for establishing connections between the Maxwell-GLM system and other nonlinear systems with a similar structure, e.g. [40,41]. Let us consider an action integral for a vector A_k and a scalar potential Z:

$$S[A_k, Z] = \int \Lambda(A_k, Z; \partial_t A_k, \partial_t Z, \partial_k A_j, \partial_k Z)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t. \tag{2.50}$$

We are interested in reparametrizing the Lagrangian Λ in the following variables:

$$E_k = -\partial_t A_k = -A_{k,t}, \quad \varphi = -\partial_t Z = -Z_{,t} \tag{2.51}$$

and

$$B_k = \epsilon_{kij}\partial_i A_j + \partial_k Z = \epsilon_{kij}A_{j,i} + Z_{,k}, \quad \psi = \partial_k A_k = A_{k,k}, \tag{2.52}$$

i.e.

$$\mathcal{L}(E_k, B_k, \varphi, \psi) = \Lambda(A_k, Z; \partial_t A_k, \partial_t Z, \partial_k A_j, \partial_k Z). \tag{2.53}$$

Note that the definitions (equations (2.51) and (2.52)) differ from equations (2.30) and (2.31) only by the scaling factors, which, without the loss of generality, are absorbed into the fields E_k, φ, B_k

and ψ in this section. The relations between the derivatives of the parametrizations $\mathcal{L}$ and Λ read

$$\frac{\partial \Lambda}{\partial A_{k,t}} = -\frac{\partial \mathcal{L}}{\partial E_k}, \quad \frac{\partial \Lambda}{\partial Z_t} = -\frac{\partial \mathcal{L}}{\partial \varphi} \tag{2.54}$$

and

$$\frac{\partial \Lambda}{\partial A_{k,j}} = \epsilon_{jki}\frac{\partial \mathcal{L}}{\partial B_i} + \delta_{kj}\frac{\partial \mathcal{L}}{\partial \psi}, \quad \frac{\partial \Lambda}{\partial Z_k} = \frac{\partial \mathcal{L}}{\partial B_k}. \tag{2.55}$$

Therefore, assuming $\partial \Lambda/\partial A_k = 0$ and $\partial \Lambda/\partial Z = 0$, the variation with respect to A_k and Z gives the Euler–Lagrange equations (for brevity, we adopt the notations $\mathcal{L}_{E_k} = \partial \mathcal{L}/\partial E_k$, etc.):

$$\frac{\partial \mathcal{L}_{E_k}}{\partial t} + \epsilon_{kij}\partial_i \mathcal{L}_{B_j} - \partial_k \mathcal{L}_\psi = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}_\varphi}{\partial t} - \partial_k \mathcal{L}_{B_k} = 0, \tag{2.56}$$

that must be accompanied by the following integrability conditions, which are direct consequences of the definitions (equations (2.51) and (2.52)):

$$\frac{\partial B_k}{\partial t} + \epsilon_{kij}\partial_i E_j + \partial_k \varphi = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial t} + \partial_k E_k = 0. \tag{2.57}$$

After a partial Legendre transform to $\mathcal{L}(E_k, B_k, \varphi, \psi)$ and introducing the new state variables

$$D_k := \mathcal{L}_{E_k}, \qquad \phi := \mathcal{L}_\varphi \quad \text{and} \quad \mathcal{E}(D_k, B_k, \phi, \psi) := E_k \mathcal{L}_{E_k} + \varphi \mathcal{L}_\varphi - \mathcal{L}, \tag{2.58}$$

with the following relations between the old and new parametrization

$$\mathcal{E}_{B_k} = -\mathcal{L}_{B_k}, \quad \mathcal{E}_\psi = -\mathcal{L}_\psi, \quad E_k = \mathcal{E}_{D_k}, \quad \varphi = \mathcal{E}_\phi, \tag{2.59}$$

equations (2.56) and (2.57) become

$$\frac{\partial B_k}{\partial t} + \epsilon_{kij}\partial_i \mathcal{E}_{D_j} + \partial_k \mathcal{E}_\phi = 0, \tag{2.60}$$

$$\frac{\partial \phi}{\partial t} + \partial_k \mathcal{E}_{B_k} = 0, \tag{2.61}$$

$$\frac{\partial D_k}{\partial t} - \epsilon_{kij}\partial_i \mathcal{E}_{B_j} + \partial_k \mathcal{E}_\psi = 0 \tag{2.62}$$

and

$$\frac{\partial \psi}{\partial t} + \partial_k \mathcal{E}_{D_k} = 0. \tag{2.63}$$

This nonlinear system can be immediately symmetrized in the dual variables $\mathbf{p} = (\mathcal{E}_{B_k}, \mathcal{E}_\phi, \mathcal{E}_{D_k}, \mathcal{E}_\psi)$ and in the new potential (full Legendre transform):

$$L(\mathbf{p}) = D_k \mathcal{E}_{D_k} + B_k \mathcal{E}_{B_k} + \phi \mathcal{E}_\phi + \psi \mathcal{E}_\psi - \mathcal{E}, \tag{2.64}$$

because it has exactly the same symmetric structure of the spatial part as in equations (2.10)–(2.12), and therefore, it can be called a symmetric hyperbolic system in the case the potential $\mathcal{E}$ is convex, e.g. see [34]. Moreover, note that system (equations (2.60)–(2.63)) has an extra conservation law (it can be trivially obtained by multiplying each equation by the corresponding dual variable and summing them exactly as in equation (2.7)):

$$\frac{\partial \mathcal{E}}{\partial t} + \partial_k(\epsilon_{kij}\mathcal{E}_{D_i}\mathcal{E}_{B_j} + \mathcal{E}_\psi \mathcal{E}_{D_k} + \mathcal{E}_\phi \mathcal{E}_{B_k}) = 0. \tag{2.65}$$

It, therefore, can be categorized as a SHTC system [34]. Such a system may be in principle applied to the context of continuum mechanics with torsion [38] or to the reformulation of the Einstein field equations in terms of tetrads [40,54], as well as to classical electrodynamics in nonlinear media. Let us show how this system can be reduced to the Maxwell-GLM system. Consider the following energy potential:

$$\mathcal{E} = \frac{1}{2}\left(\frac{1}{\varepsilon}||\mathbf{D}||^2 + \frac{1}{\mu}||\mathbf{B}||^2\right) + \frac{1}{2}(\alpha^2\phi^2 + \beta^2\psi^2), \quad c_0^2 = \frac{1}{\varepsilon\mu}, \tag{2.66}$$

assuming that $\alpha^2 = \mu c_h^2$, $\beta^2 = \varepsilon c_h^2$, and let us also rescale the fields as

$$\hat{B}_k := B_k, \quad \hat{E}_k := \frac{1}{c_0}E_k, \quad \hat{\phi} := c_h\mu\phi \quad \text{and} \quad \hat{\psi} := \frac{c_h}{c_0}\psi. \tag{2.67}$$

After this, system (equations (2.60)–(2.63)) becomes the Maxwell-GLM system (equations (2.1)–(2.4)).

(f) Hamiltonian nature of the Maxwell-GLM system

An intimate connection between the Lagrangian and Hamiltonian formulations of mechanics is well established, see e.g. [55]. It is also known that in addition to the Lagrangian formulation, the Maxwell equations possess a Hamiltonian formulation, i.e. their time evolution PDEs can be generated by a Poisson bracket, see e.g. [56, Sec.3.10.1] or [39]. The Hamiltonian formulation of mechanics and electromagnetism is a powerful theoretical tool as it is well known from the history of theoretical physics, as well as it is important for designing numerical methods, e.g. see [57]. It is therefore interesting to investigate if the Maxwell system with the GLM divergence cleaning still possesses a Hamiltonian formulation. The answer to this question is positive. Indeed, using the ‘reverse engineering’ approach (i.e. starting from the PDEs), see [56, sec. A.5] or [39, eqs. (68)–(69)], the following Poisson bracket for system (equations (2.60)–(2.63)), and subsequently for the Maxwell–Munz system (equations (2.1)–(2.4)), can be obtained:

$$\begin{aligned}\{\mathcal{A},\mathcal{B}\}^{(B_k,D_k,\psi,\phi)} = \int &([\mathcal{A}_{D_k}(\varepsilon_{kij}\partial_i\mathcal{B}_{B_j} - \partial_k\mathcal{B}_\psi) - (\varepsilon_{kij}\partial_i\mathcal{A}_{B_j} - \partial_k\mathcal{A}_\psi)\mathcal{B}_{D_k}] \\ &+ [\mathcal{A}_\phi(-\partial_k\mathcal{B}_{B_k}) - (-\partial_k\mathcal{A}_{B_k})\mathcal{B}_\phi])\,\mathrm{d}\mathbf{x},\end{aligned} \tag{2.68}$$

where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary sufficiently smooth functionals of the variables (B_k, D_k, ψ, ϕ) and possibly of their first spatial gradients, and $\mathcal{A}_{B_k}$, $\mathcal{B}_{B_k}$, etc. are the functional derivatives with respect to B_k, D_k, ψ, ϕ, respectively, i.e. $\mathcal{A}_{B_k} = \delta\mathcal{A}/\delta B_k$, etc. However, as it follows from the previous discussion, we assume that the functionals do not depend on the gradients of the fields, and therefore their functional derivatives are equal to the partial derivatives, i.e. $\mathcal{A}_{B_k} = \partial\mathcal{A}/\partial B_k$, etc.

The Poisson bracket (equation (2.68)), however, is not canonical, and hence the Jacobi identity must be checked for it. The Jacobi identity was checked with the help of the symbolic computation software Mathematica, and in particular with the package [58], and it was found that the Jacobi identity holds for the bracket (equation (2.68)). To see how equations (2.60)–(2.63) unfold from the Poisson bracket $\{\mathcal{A},\mathcal{B}\}^{(B_k,D_k,\psi,\phi)}$, one can apply integration by parts in (equation (2.68)), and rewrite the bracket as

$$\begin{aligned}\{\mathcal{A},\mathcal{B}\}^{(B_k,D_k,\psi,\phi)} = \int &(\mathcal{A}_{B_k}(-\varepsilon_{kij}\partial_i\mathcal{B}_{D_j} - \partial_k\mathcal{B}_\phi) + \mathcal{A}_\phi(-\partial_k\mathcal{B}_\phi) \\ &+ \mathcal{A}_{D_k}(\varepsilon_{kij}\partial_i\mathcal{B}_{B_j} - \partial_k\mathcal{B}_\psi) + \mathcal{A}_\psi(-\partial_k\mathcal{B}_{D_k}))\mathrm{d}\mathbf{x}.\end{aligned} \tag{2.69}$$

On the other hand, in the Hamiltonian formalism, the reversible time evolution of an arbitrary functional $\mathcal{A}(B_k, D_k, \psi, \phi) = \int A\mathrm{d}\mathbf{x}$ is given as

$$\frac{\partial\mathcal{A}}{\partial t} = \{\mathcal{A},\mathcal{E}\}^{(B_k,D_k,\psi,\phi)}, \tag{2.70}$$

with $\mathcal{E}$ being the energy of the system. Therefore, after noting that the left-hand side of equation (2.70) can also be formally written as

$$\frac{\partial\mathcal{A}}{\partial t} = \int\left(\frac{\delta\mathcal{A}}{\delta B_k}\frac{\partial B_k}{\partial t} + \frac{\delta\mathcal{A}}{\delta\phi}\frac{\partial\phi}{\partial t} + \frac{\delta\mathcal{A}}{\delta D_k}\frac{\partial D_k}{\partial t} + \frac{\delta\mathcal{A}}{\delta\psi}\frac{\partial\psi}{\partial t}\right)\mathrm{d}\mathbf{x}, \tag{2.71}$$

and due to the arbitrariness of the functional $\mathcal{A}$, one can directly read the expressions for the time evolutions of the corresponding variables in the parentheses in equation (2.69), e.g. $\partial B_k/\partial t = -\varepsilon_{kij}\partial_i \mathcal{E}_{D_j} - \partial_k \mathcal{E}_\phi$, which is exactly equation (2.60), etc. Recall that the energy conservation law (equation (2.65)) can be trivially obtained in the Hamiltonian formalism as

$$\frac{\partial \mathcal{E}}{\partial t} = \{\mathcal{E}, \mathcal{E}\}^{(B_k, D_k, \psi, \phi)} = 0, \tag{2.72}$$

and equation (2.71) is another manifestation of how the energy conservation laws (equations (2.8) and (2.65)) were derived. Finally, we remark that if one takes the pairs (A_k, D_k) and (Z, ϕ) as the canonical variables, and hence admits that the functionals may depend on their spatial gradients, then there exists a canonical Poisson bracket

$$\{\mathcal{A}, \mathcal{B}\}^{(A_k, D_k, Z, \phi)} = \int ((\mathcal{A}_{D_k}\mathcal{B}_{A_k} - \mathcal{A}_{A_k}\mathcal{B}_{D_k}) + (\mathcal{A}_\phi \mathcal{B}_Z - \mathcal{A}_Z \mathcal{B}_\phi))\, \mathrm{d}\mathbf{x}, \tag{2.73}$$

where (keeping in mind definitions (equations (2.51) and (2.52)) and assuming the functionals do not depend explicitly on A_k and Z), the functional derivatives w.r.t. the variables A_k and Z read

$$\mathcal{A}_{A_k} = \frac{\delta \mathcal{A}}{\delta A_k} = \frac{\partial \mathcal{A}}{\partial A_k} - \partial_i \left(\frac{\partial \mathcal{A}}{\partial A_{k,i}} \right) = \varepsilon_{kij}\partial_i \mathcal{A}_{B_j} - \partial_k \mathcal{A}_\psi \tag{2.74}$$

and

$$\mathcal{A}_Z = \frac{\delta \mathcal{A}}{\delta Z} = \frac{\partial \mathcal{A}}{\partial Z} - \partial_i \left(\frac{\partial \mathcal{A}}{\partial Z_i} \right) = -\partial_i \mathcal{A}_{B_i}. \tag{2.75}$$

Using these formulae, one can transform the bracket $\{\mathcal{A}, \mathcal{B}\}^{(A_k, D_k, Z, \phi)}$ to the bracket $\{\mathcal{A}, \mathcal{B}\}^{(B_k, D_k, \psi, \phi)}$. This transformation, however, is not canonical. Of course, the time evolution equations corresponding to $\{\mathcal{A}, \mathcal{B}\}^{(A_k, D_k, Z, \phi)}$ are second-order PDEs in (A_k, D_k, Z, ϕ), which is not the scope of this paper, since we focus on first-order PDE only.

3. Extension to special relativity

In this section, we use the Einstein summation convention of repeated indices. Greek indices run from 0 to 3, Latin indices run from 1 to 3. Moreover, we set $c_0 = c_h = 1$ and make use of the Lorentz–Heaviside notation, which is equivalent to the Gauss notation where all 4π factors in the original Maxwell equations disappear. In the following, we present the extension to special relativity of the equations discussed so far. The space–time is, therefore, a flat Minkowski one, $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and $g^{\mu\nu}$ its inverse, and in Cartesian coordinates, the space–time lane element is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{3.1}$$

It is convenient to introduce the so-called laboratory observer, defined by the four velocity

$$n^\mu = (1, 0, 0, 0) \quad \text{and} \quad n_\mu = (-1, 0, 0, 0). \tag{3.2}$$

We now briefly recall the usual relativistic tensor formalism of the Maxwell equations in special relativity, before showing the necessary changes to obtain its GLM version. Even if we are assuming a flat space–time in Cartesian coordinates, with Christoffel symbols that are identically zero, we nevertheless write the equations by adopting covariant derivatives, in view of possible future extensions to intrinsically curved space–times. At the end of this section, we reach the conclusion that the system (equations (2.1)–(2.4)) is already consistent with special relativity. However, this must be shown through a proper discussion.

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(a) Pure Maxwell

We write the Maxwell tensor $F^{\mu\nu}$ and its dual, the Faraday tensor ${}^*\!F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$, as

$$F^{\mu\nu} = n^\mu E^\nu - E^\mu n^\nu + \epsilon^{\mu\nu\lambda\kappa} B_\lambda n_\kappa \tag{3.3}$$

and

$${}^*\!F^{\mu\nu} = n^\mu B^\nu - B^\mu n^\nu - \epsilon^{\mu\nu\lambda\kappa} E_\lambda n_\kappa, \tag{3.4}$$

where $\epsilon^{\mu\nu\lambda\kappa}$ is the usual fully anti-symmetric rank four Levi-Civita tensor. We recall that the general rank n Levi-Civita tensor is defined as follows:

$$\epsilon^{\mu_1\cdots\mu_n} = \prod_{1\le p<q\le n} \frac{\mu_p - \mu_q}{p-q}.$$

Note that E^μ and B^μ, are both spatial four-vectors, with zero temporal components, and they express the electric field and the magnetic field as measured by the laboratory observer. The Maxwell tensor can also be written in terms of the four-vector potential A_μ as

$$F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]} := \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.5}$$

The first couple of Maxwell equations is a purely formal consequence of the definitions introduced so far. In fact, if we contract equation (3.5) with the Levi-Civita, we get

$$\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu} = 2\epsilon^{\mu\nu\alpha\beta}\nabla_{[\mu}A_{\nu]} \Longrightarrow {}^*\!F^{\alpha\beta} = \epsilon^{\mu\nu\alpha\beta}\nabla_\mu A_\nu \Longrightarrow \nabla_\beta {}^*\!F^{\alpha\beta} = 0. \tag{3.6}$$

It is easy to show (see [53]) that the four equations expressed by equation (3.6) correspond to the standard divergence-free condition for the magnetic field plus the evolution of the magnetic field. The Lagrangian density must be a relativistic invariant, and it is given by

$$\Lambda = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(E^2 - B^2). \tag{3.7}$$

The second couple of Maxwell equations is obtained as an Euler–Lagrange equation with respect to the vector potential A_μ, that is,

$$\partial_\mu\left(\frac{\partial\Lambda}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial\Lambda}{\partial A_\nu} = 0. \tag{3.8}$$

After a few calculations, this provides $\nabla_\nu F^{\mu\nu} = 0$, which corresponds to the standard divergence-free condition for the electric field in vacuum, plus the evolution of the electric field.

(b) The relativistic Maxwell-GLM

In the new framework, the Maxwell and the Faraday tensors are modified as

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = n_\mu E_\nu - E_\mu n_\nu + \epsilon_{\mu\nu\lambda\kappa}B^\lambda n^\kappa + \epsilon_{\mu\nu\alpha\beta}n^\alpha g^{\beta\gamma}\nabla_\gamma Z, \tag{3.9}$$

$${}^*\!F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} = n_\mu B_\nu - n_\nu B_\mu + \epsilon_{\mu\nu\alpha\beta}n^\alpha E^\beta - (n_\mu\nabla_\nu Z - n_\nu\nabla_\mu Z), \tag{3.10}$$

from which it follows that the purely spatial magnetic and electric fields can be obtained as

$$B_i = {}^*\!F_{i\nu}n^\nu + \nabla_i Z = (\nabla\times\mathbf{A})_i + \nabla_i Z \tag{3.11}$$

and

$$E_i = F_{i\nu}n^\nu. \tag{3.12}$$

We stress that, according to equations (3.11)–(3.12), only the magnetic field is affected by the extra terms in the electromagnetic tensors. The modified Lagrangian density is

$$\begin{aligned}\Lambda &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\nabla_\mu A^\mu)^2 - \frac{1}{2}g^{\mu\nu}\nabla_\mu Z\nabla_\nu Z\\ &= \frac{1}{2}(E^2 - (\nabla\times\mathbf{A})^2) - \frac{1}{2}(\nabla_\mu A^\mu)^2 - \frac{1}{2}g^{\mu\nu}\nabla_\mu Z\nabla_\nu Z.\end{aligned} \tag{3.13}$$

(i) First couple of Maxwell-GLM

The first couple of Maxwell equations is obtained as the sum of two formal identities plus one Euler–Lagrange equation. The first formal identity is

$$\nabla_\nu {}^*\!F^{\mu\nu} = 0. \tag{3.14}$$

The second formal identity is

$$\nabla_\nu (n^\nu \nabla^\mu Z + g^{\mu\nu}\phi) = 0, \tag{3.15}$$

which follows directly from the definition $\phi = -\partial_t Z$ and the fact that $\nabla_\mu n^\mu = 0$. Finally, we need the Euler–Lagrange equation with respect to the field Z, that is,

$$\partial_\mu \left(\frac{\partial \Lambda}{\partial(\partial_\mu Z)} \right) - \frac{\partial \Lambda}{\partial Z} = 0, \tag{3.16}$$

from which we obtain a wave equation for Z:

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta Z = 0. \tag{3.17}$$

Now, by subtracting equation (3.15) from equation (3.15), we obtain

$$\nabla_\nu ({}^*\!F^{\mu\nu} - n^\nu \nabla^\mu Z - g^{\mu\nu}\phi) = 0. \tag{3.18}$$

It is easy to show that equation (3.18) is the covariant relativistic version of equations (2.1)–(2.2), where the wave equation (3.17) must be used. In fact, by first considering the temporal component of equation (3.18) and using equation (3.17), we can find

$$\begin{aligned} &\partial_i {}^*\!F^{ti} - \partial_\nu (n^\nu \partial^t Z) - g^{t\nu}\partial_\nu \phi = 0, \Longrightarrow \partial_i (n^t B^i - n^t \partial^i Z) + \partial_t^2 Z + \partial_t \phi = 0, \Longrightarrow \\ &\partial_i B^i - \partial_i \partial^i Z + \partial_t^2 Z + \partial_t \phi = 0 \Longrightarrow \partial_i B^i + \partial_t \phi = 0, \end{aligned} \tag{3.19}$$

which is equation (2.2) with $c_h = 1$. On the other hand, by considering the spatial component of equation (3.18), we obtain

$$\begin{aligned} &\partial_\nu {}^*\!F^{i\nu} + \partial_\nu (n^\nu \partial^i Z) - g^{ij}\partial_j \phi = 0 \\ &\quad \Longrightarrow \partial_t (-n^t B^i + n^t \partial^i Z) + \partial_j (\epsilon^{ij\alpha\beta} n_\alpha E_\beta) - \partial_t (n^t \partial^i Z) - g^{ij}\partial_j \phi = 0 \\ &\quad \Longrightarrow -\partial_t B^i + \partial_t \partial^i Z - \epsilon^{ijk}\partial_j E_k - \partial_t \partial^i Z - g^{ij}\partial_j \phi = 0 \\ &\quad \Longrightarrow \partial_t B^i + \epsilon^{ijk}\partial_j E_k + g^{ij}\partial_j \phi = 0, \end{aligned} \tag{3.20}$$

which is equation (2.1) with $c_0 = c_h = 1$.

(ii) Second couple of Maxwell-GLM

The second couple of Maxwell equations is obtained as an Euler–Lagrange equation with respect to the fields A_μ, that is,

$$\partial_\mu \left(\frac{\partial \Lambda}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \Lambda}{\partial A_\nu} = 0. \tag{3.21}$$

After a few calculations this provides

$$\nabla_\nu (F^{\mu\nu} - g^{\mu\nu}\psi) = 0, \tag{3.22}$$

which is the covariant relativistic version of equations (2.3)–(2.4), where $\psi = \nabla_\mu A^\mu$. In fact, the temporal component of equation (3.22) provides

$$\partial_\nu F^{t\nu} - g^{t\nu}\partial_\nu \psi = 0, \Longrightarrow \partial_i F^{ti} - g^{tt}\partial_t \psi = 0, \Longrightarrow \partial_i E^i + \partial_t \psi = 0, \tag{3.23}$$

which is equation (2.4) with $c_h = 1$. Finally, the spatial component of equation (3.22) gives

$$\begin{aligned}
&\partial_\nu F^{i\nu} - g^{i\nu}\partial_\nu \psi = 0 \\
&\quad \Longrightarrow \partial_t(-n^t E^i) + \partial_j(\epsilon^{ij\alpha\beta} B_\alpha n_\beta + \epsilon^{ij\alpha\beta} n_\alpha \partial_\beta Z) - g^{ij}\partial_j\psi = 0 \\
&\quad \Longrightarrow -\partial_t E^i + \epsilon^{ijk}\partial_j B_k - \epsilon^{ijk}\partial_j\partial_k Z - g^{ij}\partial_j\psi = 0 \\
&\quad \Longrightarrow \partial_t E^i - \epsilon^{ijk}\partial_j B_k + g^{ij}\partial_j\psi = 0,
\end{aligned} \tag{3.24}$$

which is equation (2.3) with $c_0 = c_h = 1$.

4. Compatible numerical discretizations

Here we present two structure-preserving discretizations of the Maxwell–Munz system. The first is an explicit semi-discrete finite volume scheme on *collocated* meshes that is *very simple* to implement and which preserves total energy exactly at the semi-discrete level, both in the linear as well as in the general *nonlinear* case. But this scheme in general does *not* respect the involution constraints yet. The second scheme is a *staggered* semi-implicit discretization that conserves total energy exactly and respects the involution constraints, but which is available only for linear systems so far. We clearly emphasize the limitations of both schemes presented in this paper and combining all desirable features, i.e. total energy conservation and involution preservation in the linear and nonlinear case, will be subject to future research that goes beyond the scope of this work.

(a) Exactly energy-conserving semi-discrete numerical schemes on collocated meshes

Throughout this section, we will employ the Einstein summation convention and use the following compact notation for a general nonlinear SHTC PDE system (equations (2.1)–(2.4)):

$$\partial_t \mathbf{q} + \partial_k \mathbf{f}_k(\mathbf{q}) = 0, \tag{4.1}$$

where $\mathbf{q} = (\mathbf{B}, \phi, \mathbf{E}, \psi)$ is the state vector and $\mathbf{f}_k$ is the flux tensor. Thanks to the compatibility of equation (4.1) with the extra conservation law for the total energy (equation (2.8)), which can be written more compactly with the total energy flux F_k according to equation (2.8) as

$$\partial_t \mathcal{E} + \partial_k F_k = 0, \tag{4.2}$$

the following identity holds at the continuous level (recall that $\mathbf{p} = \partial\mathcal{E}/\partial\mathbf{q}$):

$$\mathbf{p} \cdot \partial_k \mathbf{f}_k(\mathbf{q}) = \partial_k F_k. \tag{4.3}$$

Based on the ideas outlined in [59–63] in the following, we show how to achieve this compatibility property also exactly at the discrete level. The computational domain $\Omega \subset \mathbb{R}^d$ in d space dimensions could in principle be paved by a fairly general mesh of *orthogonal* polygonal / polyhedral control volumes denoted by Ω^ℓ in the following. However, in this paper we limit ourselves to the simple Cartesian case. The common edge / face of two neighbouring polygons / polyhedra Ω^ℓ and $\Omega^\imath$ is $\partial\Omega^{\ell\imath} = \Omega^\ell \cap \Omega^\imath$ and $\mathcal{N}_\ell$ is the set of neighbours of the element Ω^ℓ. The compatible semi-discrete finite volume scheme for the discretization of the hyperbolic system of conservation laws (equation (4.1)) with extra conservation law (equation (4.2)), equation (4.3), reads for each control volume Ω^ℓ as follows:

$$\frac{\partial \mathbf{q}^\ell}{\partial t} = -\sum_{\Omega^\imath \in \mathcal{N}_\ell} \frac{|\partial\Omega^{\ell\imath}|}{|\Omega^\ell|} \mathbf{f}^{\ell\imath}. \tag{4.4}$$

Throughout this paper the scheme (equation (4.4)) is integrated in time via a high-order Runge–Kutta method of suitable order greater or equal than four. The compatible numerical flux in normal direction is denoted by $\mathbf{f}^{\ell r}$ and must satisfy the following discrete compatibility condition, which is a discrete analogy to the continuous identity (equation (4.3)):

$$\mathbf{p}^{\ell} \cdot (\mathbf{f}^{\ell r} - \mathbf{f}_k^{\ell} n_k^{\ell r}) + \mathbf{p}^{r} \cdot (\mathbf{f}_k^{r} n_k^{\ell r} - \mathbf{f}^{\ell r}) = (F_k^{r} - F_k^{\ell}) n_k^{\ell r}, \tag{4.5}$$

with $\mathbf{n}^{\ell r} = \{n_k^{\ell r}\}$ the unit normal vector pointing from element Ω^{ℓ} to its neighbour Ω^{r}. Therefore, the following identity is obviously true: $\mathbf{n}^{r\ell} = -\mathbf{n}^{\ell r}$. If a numerical flux $\mathbf{f}^{\ell r}$ satisfies equation (4.5) then it is easy to prove total energy conservation. We thus have the following:

Theorem 4.1 (Total energy conservation). *The finite volume scheme (equation (4.4)) with a numerical flux $\mathbf{f}^{\ell r}$ that satisfies equation (4.5) conserves total energy in the sense that, for vanishing boundary fluxes, we have*

$$\int_{\Omega} \frac{\partial \mathcal{E}}{\partial t} \, \mathrm{d}\mathbf{x} = 0. \tag{4.6}$$

Proof. We take the dot product of the semi-discrete scheme (equation (4.4)) with the discrete main field variables $\mathbf{p}^{\ell}$, multiply by the cell volume $|\Omega^{\ell}|$ and obtain

$$|\Omega^{\ell}| \mathbf{p}^{\ell} \cdot \frac{\partial \mathbf{q}^{\ell}}{\partial t} = - \sum_{\Omega^{r} \in \mathcal{N}_{\ell}} |\partial \Omega^{\ell r}| \mathbf{p}^{\ell} \cdot \mathbf{f}^{\ell r}.$$

Adding and subtracting $\sum_{\Omega^{r} \in \mathcal{N}_{\ell}} |\partial \Omega^{\ell r}| \frac{1}{2} \mathbf{p}^{r} \cdot \mathbf{f}^{\ell r}$ leads to

$$|\Omega^{\ell}| \frac{\partial \mathcal{E}^{\ell}}{\partial t} = - \sum_{\Omega^{r} \in \mathcal{N}_{\ell}} |\partial \Omega^{\ell r}| \frac{1}{2} ((\mathbf{p}^{\ell} + \mathbf{p}^{r}) \cdot \mathbf{f}^{\ell r} + (\mathbf{p}^{\ell} - \mathbf{p}^{r}) \cdot \mathbf{f}^{\ell r}).$$

Using the compatibility condition (equation (4.5)) leads to

$$|\Omega^{\ell}| \frac{\partial \mathcal{E}^{\ell}}{\partial t} = - \sum_{\Omega^{r} \in \mathcal{N}_{\ell}} |\partial \Omega^{\ell r}| \frac{1}{2} ((F_k^{r} - F_k^{\ell}) n_k^{\ell r} - (\mathbf{p}^{r} \cdot \mathbf{f}_k^{r} - \mathbf{p}^{\ell} \cdot \mathbf{f}_k^{\ell}) n_k^{\ell r} + (\mathbf{p}^{\ell} + \mathbf{p}^{r}) \cdot \mathbf{f}^{\ell r}).$$

Since the integral of the normal vector over a closed surface vanishes, the following identity holds:

$$\sum_{\Omega^{r} \in \mathcal{N}_{\ell}} |\partial \Omega^{\ell r}| \mathbf{n}^{\ell r} = 0. \tag{4.7}$$

Adding $\mathbf{p}^{\ell} \cdot \mathbf{f}_k^{\ell} + F_k^{\ell}$ multiplied by equation (4.7) one obtains

$$\begin{aligned} |\Omega^{\ell}| \frac{\partial \mathcal{E}^{\ell}}{\partial t} &= - \sum_{\Omega^{r} \in \mathcal{N}_{\ell}} |\partial \Omega^{\ell r}| \frac{1}{2} ((F_k^{r} + F_k^{\ell}) n_k^{\ell r} - (\mathbf{p}^{r} \cdot \mathbf{f}_k^{r} + \mathbf{p}^{\ell} \cdot \mathbf{f}_k^{\ell}) n_k^{\ell r} + (\mathbf{p}^{\ell} + \mathbf{p}^{r}) \cdot \mathbf{f}^{\ell r}) \\ &= - \sum_{\Omega^{r} \in \mathcal{N}_{\ell}} |\partial \Omega^{\ell r}| F^{\ell r}, \end{aligned}$$

with the numerical total energy flux in the normal direction $\mathbf{n}^{\ell r}$ given by

$$F^{\ell r} = \frac{1}{2} ((F_k^{r} + F_k^{\ell}) n_k^{\ell r} - (\mathbf{p}^{r} \cdot \mathbf{f}_k^{r} + \mathbf{p}^{\ell} \cdot \mathbf{f}_k^{\ell}) n_k^{\ell r} + (\mathbf{p}^{r} + \mathbf{p}^{\ell}) \cdot \mathbf{f}^{\ell r}). \tag{4.8}$$

The total energy conservation is then finally obtained by summing up over all elements Ω^{ℓ}, assuming that the fluxes are zero at the boundary of the domain and by employing the telescopic sum property, since the sum of all numerical total energy fluxes $F^{\ell r}$ at the internal interfaces cancel

$$\int_{\Omega} \frac{\partial \mathcal{E}}{\partial t} \, \mathrm{d}\mathbf{x} = \sum_{\ell} |\Omega^{\ell}| \frac{\partial \mathcal{E}^{\ell}}{\partial t} = 0.$$

■

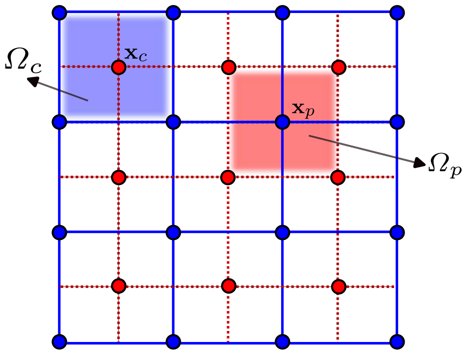

Figure 1. Schematic representation of the primary Ω_c and of the dual Ω_p mesh.

Compatible Abgrall-type scheme. As in [61–64] the *thermodynamically compatible* **Abgrall flux** reads

$$\mathbf{f}^{\ell r} = \frac{1}{2}(\mathbf{f}_k^{\ell} + \mathbf{f}_k^{r})n_k^{\ell r} - \alpha^{\ell r}(\mathbf{p}^{r} - \mathbf{p}^{\ell}), \tag{4.9}$$

with the scalar correction factor $\alpha^{\ell r}$ that is directly obtained by imposing the discrete compatibility condition (equation (4.5)) on the numerical flux and which is given by

$$\alpha^{\ell r} = \frac{(F_k^{r} - F_k^{\ell})n_k^{\ell r} + \frac{1}{2}(\mathbf{p}^{r} + \mathbf{p}^{\ell}) \cdot (\mathbf{f}_k^{\ell} - \mathbf{f}_k^{r})n_k^{\ell r}}{(\mathbf{p}^{r} - \mathbf{p}^{\ell})^2}. \tag{4.10}$$

One can easily verify that the Abgrall flux (equation (4.9)) with the correction factor (equation (4.10)) satisfies the discrete compatibility condition (equation (4.5)) *by construction*, since the correction factor $\alpha^{\ell r}$ is obtained by imposing equation (4.5) on equation (4.9). For a vanishing denominator, i.e. for $\mathbf{p}^{r} = \mathbf{p}^{\ell}$ both states at the interface coincide and thus the numerical flux becomes trivial and hence no correction is needed. To avoid division by zero, in this case we can simply set $\alpha^{\ell r} = 0$. For more details, see the provided MATLAB code.

(b) Fully discrete structure-preserving staggered semi-implicit scheme

Here we present a completely different compatible discretization compared to the previous one. The method is fully discrete, globally exactly energy-conserving and uses a *staggered mesh* with *mimetic* discrete differential operators that preserve the essential vector calculus identities exactly at the discrete level. The computational domain is denoted by Ω. Furthermore, we denote the control volumes of the primary mesh by Ω_c with cell centres $\mathbf{x}_c$ and vertices $\mathbf{x}_p$, while the control volumes of the dual mesh are denoted by Ω_p, with cell centres $\mathbf{x}_p$ (see figure 1). When needed, later we also use the summation indices a and q for cells and vertices, respectively. For all implementation details, see the provided MATLAB code.

In the following, we introduce two mimetic discrete nabla operators, see [3,65–67]:

$$\nabla_c^p \circ = \frac{1}{|\Omega_c|} \sum_{p \in \Omega_c} l_{pc} \mathbf{n}_{pc} \circ, \tag{4.11}$$

and its dual

$$\nabla_p^c \circ = \frac{1}{|\Omega_p|} \sum_{c \in \Omega_p} l_{pc} \mathbf{n}_{cp} \circ, \tag{4.12}$$

with $\mathbf{n}_{cp} = -\mathbf{n}_{pc}$. Note that in equations (4.11) and (4.12), the discrete nabla operators have not been applied to any field yet. Their concrete application to scalar and discrete vector fields will be introduced below. Owing to the Gauss theorem, the following identities are obviously true:

$$\sum_{p\in\Omega_c} l_{pc}\mathbf{n}_{pc} = 0 \quad \text{and} \quad \sum_{c\in\Omega_p} l_{pc}\mathbf{n}_{cp} = 0. \tag{4.13}$$

For scalar fields $\phi_p = \phi(\mathbf{x}_p)$ and vector fields $\mathbf{A}_p = \mathbf{A}(\mathbf{x}_p)$ defined on the vertices of the primal grid and for scalar and vector fields defined on the centres of the primal grid (the vertices of the dual grid) $\phi_c = \phi(\mathbf{x}_c)$ and $\mathbf{A}_c = \mathbf{A}(\mathbf{x}_c)$ the following discrete gradient, divergence and curl operators and their duals are naturally defined via equations (4.11) and (4.12):

$$\nabla_c^p \phi_p = \frac{1}{|\Omega_c|}\sum_{p\in\Omega_c} l_{pc}\mathbf{n}_{pc}\phi_p, \quad \nabla_p^c \phi_c = \frac{1}{|\Omega_p|}\sum_{c\in\Omega_p} l_{pc}\mathbf{n}_{cp}\phi_c, \tag{4.14}$$

$$\nabla_c^p \cdot \mathbf{A}_p = \frac{1}{|\Omega_c|}\sum_{p\in\Omega_c} l_{pc}\mathbf{n}_{pc}\cdot \mathbf{A}_p, \quad \nabla_p^c \cdot \mathbf{A}_c = \frac{1}{|\Omega_p|}\sum_{c\in\Omega_p} l_{pc}\mathbf{n}_{cp}\cdot \mathbf{A}_c \tag{4.15}$$

and

$$\nabla_c^p \times \mathbf{A}_p = \frac{1}{|\Omega_c|}\sum_{p\in\Omega_c} l_{pc}\mathbf{n}_{pc}\times \mathbf{A}_p, \quad \nabla_p^c \times \mathbf{A}_c = \frac{1}{|\Omega_p|}\sum_{c\in\Omega_p} l_{pc}\mathbf{n}_{cp}\times \mathbf{A}_c. \tag{4.16}$$

Furthermore, after some calculations one obtains the following discrete vector calculus identities, which are the fundamental pillar of the proposed numerical scheme: the discrete curl applied to the discrete gradient of a discrete scalar field ϕ vanishes identically, i.e.

$$\nabla_c^p \times \nabla_p^a \phi_a = 0 \quad \text{and} \quad \nabla_p^c \times \nabla_c^q \phi_q = 0, \tag{4.17}$$

where repeated indices denote summation over cells and vertices, respectively, according to the usual Einstein summation convention. This is the discrete analogue of the continuous identity $\nabla \times \nabla\phi = 0$. Furthermore, the discrete divergence applied to the discrete curl of a vector field $\mathbf{A}$ vanishes:

$$\nabla_c^p \cdot \nabla_p^a \times \mathbf{A}_a = 0 \quad \text{and} \quad \nabla_p^c \cdot \nabla_c^q \times \mathbf{A}_q = 0, \tag{4.18}$$

which reflects the continuous identity $\nabla\cdot\nabla\times\mathbf{A} = 0$ at the discrete level. The extended calculations are shown in the appendix.

The discrete magnetic field and the discrete cleaning scalar ψ are governed by the differential identities (equations (2.35) and (2.36)) and are located at the cell centres of the primal mesh and are denoted by $\mathbf{B}_c^n = \mathbf{B}(\mathbf{x}_c, t^n)$ and $\psi_c^n = \psi(\mathbf{x}_c, t^n)$, respectively, while the discrete electric field and the discrete cleaning scalar ϕ are governed by the Euler–Lagrange equations (equations (2.44) and (2.48)) and are located in the vertices of the primal mesh, i.e. the cell centres of the dual mesh, and are denoted by $\mathbf{E}_p^n = \mathbf{E}(\mathbf{x}_p, t^n)$ and $\phi_p^n = \phi(\mathbf{x}_p, t^n)$, respectively. Furthermore, we use the notations

$$\mathbf{B}_c^{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{B}_c^n + \mathbf{B}_c^{n+1}), \quad \mathbf{E}_p^{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{E}_p^n + \mathbf{E}_p^{n+1})$$

and

$$\psi_c^{n+\frac{1}{2}} = \frac{1}{2}(\psi_c^n + \psi_c^{n+1}), \quad \phi_p^{n+\frac{1}{2}} = \frac{1}{2}(\phi_p^n + \phi_p^{n+1}).$$

We now propose the following compatible discretization of equations (2.1)–(2.4):

$$\mathbf{B}_c^{n+1} = \mathbf{B}_c^n - \Delta t c_0 \nabla_c^p \times \mathbf{E}_p^{n+\frac{1}{2}} - \Delta t c_h \nabla_c^p \phi_p^{n+\frac{1}{2}}, \tag{4.19}$$

$$\phi_p^{n+1} = \phi_p^n - \Delta t c_h \nabla_p^c \cdot \mathbf{B}_c^{n+\frac{1}{2}}, \tag{4.20}$$

$$\mathbf{E}_p^{n+1} = \mathbf{E}_p^n + \Delta t c_0 \nabla_p^c \times \mathbf{B}_c^{n+\frac{1}{2}} - \Delta t c_h \nabla_p^c \psi_c^{n+\frac{1}{2}} \tag{4.21}$$

and

$$\psi_c^{n+1} = \psi_c^n - \Delta t c_h \nabla_c^p \cdot \mathbf{E}_p^{n+\frac{1}{2}}. \tag{4.22}$$

Inserting equation (4.19) into equation (4.20) and using the discrete vector identity (equation (4.18)), one obtains the following discrete wave equation for the cleaning scalar ϕ:

$$\phi_p^{n+1} - \frac{1}{4}\Delta t^2 c_h^2 \nabla_p^c \cdot \nabla_c^q \phi_q^{n+1} = \phi_p^n - \Delta t c_h \nabla_p^c \cdot \mathbf{B}_c^n + \frac{1}{4}\Delta t^2 c_h^2 \nabla_p^c \cdot \nabla_c^q \phi_q^n. \tag{4.23}$$

Inserting equations (4.19) and (4.22) into equation (4.21), we obtain a discrete vector wave equation for the electric field:

$$\mathbf{E}_p^{n+1} + \frac{1}{4}\Delta t^2 c_0^2 \nabla_p^c \times \nabla_c^q \times \mathbf{E}_q^{n+1} - \frac{1}{4}\Delta t^2 c_h^2 \nabla_p^c \nabla_c^q \cdot \mathbf{E}_q^{n+1} = \mathbf{r}_p, \tag{4.24}$$

with the known right-hand side

$$\mathbf{r}_p = \mathbf{E}_p^n + \Delta t c_0 \nabla_p^c \times \mathbf{B}_c^n - \frac{1}{4}\Delta t^2 c_0^2 \nabla_p^c \times \nabla_c^q \times \mathbf{E}_q^n - \Delta t c_h \nabla_p^c \psi_c^n + \frac{1}{4}\Delta t^2 c_h^2 \nabla_p^c \nabla_c^q \cdot \mathbf{E}_q^n. \tag{4.25}$$

The above substitutions correspond to the use of the Schur complement in the linear system (equations (4.19)–(4.22)). Once the new cleaning scalar ϕ and the new electric field have been obtained, the magnetic field and the cleaning scalar ψ can be directly updated via equations (4.19) and (4.22). This procedure is analogous to the post-projection stage in semi-implicit schemes for the incompressible Navier–Stokes equations [50,51,68] or in semi-implicit schemes for shallow water flows [69,70]. We emphasize that the final equations to be solved are discrete second-order wave equations for the electric field and for the cleaning variable ϕ, respectively, and thus reflect the solution of the original Euler–Lagrange equations (2.47) and (2.42) on a discrete level; not for the original potentials $\mathbf{A}$ and Z, but for their time derivatives $\mathbf{E}$ and ϕ.

In the following, we study the total energy conservation and the asymptotic-preserving (AP) property of the proposed structure-preserving semi-implicit scheme.

Theorem 4.2. *In the case of periodic boundaries the scheme (equations (4.19)–(4.22)) conserves the global discrete total energy*

$$\mathcal{E}^n = \sum_{c\in\Omega} |\Omega_c| \frac{1}{2} (\boldsymbol{B}_c^n)^2 + \sum_{p\in\Omega} |\Omega_p| \frac{1}{2} (\phi_p^n)^2 + \sum_{p\in\Omega} |\Omega_p| \frac{1}{2} (\boldsymbol{E}_p^n)^2 + \sum_{c\in\Omega} |\Omega_c| \frac{1}{2} (\psi_c^n)^2$$

exactly, i.e.

$$\mathcal{E}^{n+1} = \mathcal{E}^n.$$

Proof. Computing the dot product of equation (4.19) with $\mathbf{B}_c^{n+\frac{1}{2}}$ and of equation (4.21) with $\mathbf{E}_p^{n+\frac{1}{2}}$, multiplying equation (4.20) with $\phi_p^{n+\frac{1}{2}}$ and equation (4.22) with $\psi_c^{n+\frac{1}{2}}$ yields

$$|\Omega_c| \mathbf{B}_c^{n+\frac{1}{2}} \cdot (\mathbf{B}_c^{n+1} - \mathbf{B}_c^n) = -\Delta t \mathbf{B}_c^{n+\frac{1}{2}} \cdot \left(c_0 \sum_{p\in\Omega_c} l_{pc} \mathbf{n}_{pc} \times \mathbf{E}_p^{n+\frac{1}{2}} + c_h \sum_{p\in\Omega_c} l_{pc} \mathbf{n}_{pc} \phi_p^{n+\frac{1}{2}} \right),$$

$$|\Omega_p| \phi_p^{n+\frac{1}{2}} (\phi_p^{n+1} - \phi_p^n) = -\Delta t \phi_p^{n+\frac{1}{2}} c_h \sum_{c\in\Omega_p} l_{pc} \mathbf{n}_{cp} \cdot \mathbf{B}_c^{n+\frac{1}{2}},$$

$$|\Omega_p| \mathbf{E}_p^{n+\frac{1}{2}} \cdot (\mathbf{E}_p^{n+1} - \mathbf{E}_p^n) = +\Delta t \mathbf{E}_p^{n+\frac{1}{2}} \cdot \left(c_0 \sum_{c\in\Omega_p} l_{pc} \mathbf{n}_{cp} \times \mathbf{B}_c^{n+\frac{1}{2}} - c_h \sum_{c\in\Omega_p} l_{pc} \mathbf{n}_{cp} \psi_c^{n+\frac{1}{2}} \right)$$

and

$$|\Omega_c| \psi_c^{n+\frac{1}{2}} (\psi_c^{n+1} - \psi_c^n) = -\Delta t \psi_c^{n+\frac{1}{2}} c_h \sum_{p\in\Omega_c} l_{pc} \mathbf{n}_{pc} \cdot \mathbf{E}_p^{n+\frac{1}{2}}.$$

Since the quantities $\mathbf{B}_c^{n+\frac{1}{2}}$ and $\psi_c^{n+\frac{1}{2}}$ do not depend on the index p and since $\phi_p^{n+\frac{1}{2}}$ and $\mathbf{E}_p^{n+\frac{1}{2}}$ do not depend on c we can take these terms on the right-hand side into the sums. Furthermore, we have that

$$\mathbf{B}_c^{n+\frac{1}{2}} \cdot (\mathbf{B}_c^{n+1} - \mathbf{B}_c^{n}) = \frac{1}{2}(\mathbf{B}_c^{n+1})^2 - \frac{1}{2}(\mathbf{B}_c^{n})^2,$$

$$\phi_p^{n+\frac{1}{2}} (\phi_p^{n+1} - \phi_p^{n}) = \frac{1}{2}(\phi_p^{n+1})^2 - \frac{1}{2}(\phi_p^{n})^2,$$

$$\mathbf{E}_p^{n+\frac{1}{2}} \cdot (\mathbf{E}_p^{n+1} - \mathbf{E}_p^{n}) = \frac{1}{2}(\mathbf{E}_p^{n+1})^2 - \frac{1}{2}(\mathbf{E}_p^{n})^2$$

and

$$\psi_c^{n+\frac{1}{2}} (\psi_c^{n+1} - \psi_c^{n}) = \frac{1}{2}(\psi_c^{n+1})^2 - \frac{1}{2}(\psi_c^{n})^2.$$

We thus obtain

$$|\Omega_c| \left(\frac{1}{2}(\mathbf{B}_c^{n+1})^2 - \frac{1}{2}(\mathbf{B}_c^{n})^2\right) = -\Delta t \left(c_0 \sum_{p\in\Omega_c} l_{pc} \mathbf{n}_{pc} \cdot \mathbf{E}_p^{n+\frac{1}{2}} \times \mathbf{B}_c^{n+\frac{1}{2}} + c_h \sum_{p\in\Omega_c} l_{pc} \mathbf{n}_{pc} \cdot \phi_p^{n+\frac{1}{2}} \mathbf{B}_c^{n+\frac{1}{2}} \right),$$

$$|\Omega_p| \left(\frac{1}{2}(\phi_p^{n+1})^2 - \frac{1}{2}(\phi_p^{n})^2\right) = -\Delta t c_h \sum_{c\in\Omega_p} l_{pc} \mathbf{n}_{cp} \cdot \mathbf{B}_c^{n+\frac{1}{2}} \phi_p^{n+\frac{1}{2}},$$

$$|\Omega_p| \left(\frac{1}{2}(\mathbf{E}_p^{n+1})^2 - \frac{1}{2}(\mathbf{E}_p^{n})^2\right) = +\Delta t \left(c_0 \sum_{c\in\Omega_p} l_{pc} \mathbf{n}_{cp} \cdot \mathbf{B}_c^{n+\frac{1}{2}} \times \mathbf{E}_p^{n+\frac{1}{2}} - c_h \sum_{c\in\Omega_p} l_{pc} \mathbf{n}_{cp} \psi_c^{n+\frac{1}{2}} \mathbf{E}_p^{n+\frac{1}{2}} \right)$$

and

$$|\Omega_c| \left(\frac{1}{2}(\psi_c^{n+1})^2 - \frac{1}{2}(\psi_c^{n})^2\right) = -\Delta t \psi_c^{n+\frac{1}{2}} c_h \sum_{p\in\Omega_c} l_{pc} \mathbf{n}_{pc} \cdot \mathbf{E}_p^{n+\frac{1}{2}}.$$

Summing each equation over all elements and summing up all equations yields

$$\begin{aligned} \mathcal{E}^{n+1} - \mathcal{E}^{n} &= -\Delta t c_0 \sum_c \sum_{p\in\Omega_c} l_{pc} \mathbf{n}_{pc} \cdot \mathbf{E}_p^{n+\frac{1}{2}} \times \mathbf{B}_c^{n+\frac{1}{2}} + \Delta t c_0 \sum_p \sum_{c\in\Omega_p} l_{pc} \mathbf{n}_{cp} \cdot \mathbf{B}_c^{n+\frac{1}{2}} \times \mathbf{E}_p^{n+\frac{1}{2}} \\ &\quad - \Delta t c_h \sum_c \sum_{p\in\Omega_c} l_{pc} \mathbf{n}_{pc} \cdot \phi_p^{n+\frac{1}{2}} \mathbf{B}_c^{n+\frac{1}{2}} - \Delta t c_h \sum_p \sum_{c\in\Omega_p} l_{pc} \mathbf{n}_{cp} \cdot \mathbf{B}_c^{n+\frac{1}{2}} \phi_p^{n+\frac{1}{2}} \\ &\quad - \Delta t c_h \sum_p \sum_{c\in\Omega_c} l_{pc} \mathbf{n}_{cp} \cdot \psi_c^{n+\frac{1}{2}} \mathbf{E}_p^{n+\frac{1}{2}} - \Delta t c_h \sum_c \sum_{p\in\Omega_p} l_{pc} \mathbf{n}_{pc} \cdot \mathbf{E}_p^{n+\frac{1}{2}} \psi_c^{n+\frac{1}{2}} = 0, \end{aligned} \tag{4.26}$$

since $\mathbf{B}_c^{n+\frac{1}{2}} \times \mathbf{E}_p^{n+\frac{1}{2}} = -\mathbf{E}_p^{n+\frac{1}{2}} \times \mathbf{B}_c^{n+\frac{1}{2}}$ and $\mathbf{n}_{cp} = -\mathbf{n}_{pc}$ and thus all pairwise interactions between p and c cancel. This proves total energy conservation and thus energy stability of the proposed numerical scheme. ∎

In the following, we analyse the properties of the new scheme (equations (4.19)–(4.22)) for fairly general initial data, Maxwell-compatible initial data and well-prepared initial data.

Theorem 4.3. *In the case of periodic boundaries and general initial data that satisfy the condition* $\phi_p^0 = \phi^0 = const$ *and* $\psi_c^0 = \psi^0 = const$ *the scheme (equations (4.19)–(4.22)) is AP in the sense that the discrete divergence* $\nabla_c^p \cdot \mathbf{E}_p^{n+\frac{1}{2}} \to \mathcal{O}(\epsilon^2)$ *and* $\nabla_p^c \cdot \mathbf{B}_c^{n+\frac{1}{2}} \to \mathcal{O}(\epsilon^2)$ *when* $c_h \to \infty$ *for fixed* c_0, *i.e. when* $\epsilon = c_0/c_h \to 0$.

Proof. Formal asymptotic expansion of the discrete solution in terms of $\epsilon = c_0/c_h$ yields

$$\left.\begin{aligned}
\mathbf{B}_c^n &= \mathbf{B}_{c,0}^n + \epsilon \mathbf{B}_{c,1}^n + \epsilon^2 \mathbf{B}_{c,2}^n + \mathcal{O}(\epsilon^3),\\
\phi_p^n &= \phi_{p,0}^n + \epsilon \phi_{p,1}^n + \epsilon^2 \phi_{p,2}^n + \mathcal{O}(\epsilon^3),\\
\mathbf{E}_p^n &= \mathbf{E}_{p,0}^n + \epsilon \mathbf{E}_{p,1}^n + \epsilon^2 \mathbf{E}_{p,2}^n + \mathcal{O}(\epsilon^3)\\
\text{and}\qquad \psi_c^n &= \psi_{c,0}^n + \epsilon \psi_{c,1}^n + \epsilon^2 \psi_{c,2}^n + \mathcal{O}(\epsilon^3).
\end{aligned}\right\} \tag{4.27}$$

Insertion into the semi-implicit scheme (equations (4.19)–(4.22)) provides the following relations up to order ϵ^0:

$$\epsilon^{-1} \nabla_c^p \phi_{p,0}^{n+\frac{1}{2}} + \epsilon^0 \left(c_0^{-1} \frac{\mathbf{B}_{c,0}^{n+1} - \mathbf{B}_{c,0}^n}{\Delta t} + \nabla_c^p \times \mathbf{E}_{p,0}^{n+\frac{1}{2}} + \nabla_c^p \phi_{p,1}^{n+\frac{1}{2}} \right) = 0, \tag{4.28}$$

$$\epsilon^{-1} \nabla_p^c \cdot \mathbf{B}_{c,0}^{n+\frac{1}{2}} + \epsilon^0 \left(c_0^{-1} \frac{\phi_{p,0}^{n+1} + \phi_{p,0}^n}{\Delta t} + \nabla_p^c \cdot \mathbf{B}_{c,1}^{n+\frac{1}{2}} \right) = 0, \tag{4.29}$$

$$\epsilon^{-1} \nabla_p^c \psi_{c,0}^{n+\frac{1}{2}} + \epsilon^0 \left(c_0^{-1} \frac{\mathbf{E}_{p,0}^{n+1} - \mathbf{E}_{p,0}^n}{\Delta t} - \nabla_p^c \times \mathbf{B}_{c,0}^{n+\frac{1}{2}} + \nabla_p^c \psi_{c,1}^{n+\frac{1}{2}} \right) = 0 \tag{4.30}$$

and

$$\epsilon^{-1} \nabla_c^p \cdot \mathbf{E}_{p,0}^{n+\frac{1}{2}} + \epsilon^0 \left(c_0^{-1} \frac{\psi_{c,0}^{n+1} - \psi_{c,0}^n}{\Delta t} + \nabla_c^p \cdot \mathbf{E}_{p,1}^{n+\frac{1}{2}} \right) = 0. \tag{4.31}$$

From the leading-order terms that scale with ϵ^{-1} in equations (4.29) and (4.31), we immediately obtain

$$\nabla_p^c \cdot \mathbf{B}_{c,0}^{n+\frac{1}{2}} = 0 \quad \text{and} \quad \nabla_c^p \cdot \mathbf{E}_{p,0}^{n+\frac{1}{2}} = 0. \tag{4.32}$$

From the leading-order terms in equations (4.28) and (4.30) one has $\nabla_c^p \phi_{p,0}^{n+\frac{1}{2}} = 0$ and $\nabla_p^c \psi_{c,0}^{n+\frac{1}{2}} = 0$. Since the initial data are assumed to satisfy the hypothesis of the theorem, i.e. $\phi_p^0 = \phi^0$ and $\psi_c^0 = \psi^0$, i.e. $\nabla_c^p \phi_{p,0}^n = 0$ and $\nabla_p^c \psi_{c,0}^n = 0$ for $n = 0$, we thus obtain that at leading order we have $\nabla_c^p \phi_{p,0}^{n+1} = 0$ and $\nabla_p^c \psi_{c,0}^{n+1} = 0$ for $n = 0$. Via induction it is obvious that this holds also for all other $n > 0$. This means that $\phi_{p,0}^n = \phi_0^n$ and $\psi_{c,0}^n = \psi_0^n$ are both constant in space and are only functions of time. Inserting these results into equations (4.29) and (4.31) and summing up over all elements yields

$$c_0^{-1} \sum_p |\Omega_p| \frac{\phi_0^{n+1} - \phi_0^n}{\Delta t} = -\sum_p |\Omega_p| \nabla_p^c \cdot \mathbf{B}_{c,1}^{n+\frac{1}{2}} = -\sum_p \sum_{c \in \Omega_p} l_{pc} \mathbf{n}_{cp} \cdot \mathbf{B}_{c,1}^{n+\frac{1}{2}} = 0 \tag{4.33}$$

and

$$c_0^{-1} \sum_c |\Omega_c| \frac{\psi_0^{n+1} - \psi_0^n}{\Delta t} = -\sum_c |\Omega_c| \nabla_c^p \cdot \mathbf{E}_{p,1}^{n+\frac{1}{2}} = -\sum_c \sum_{p \in \Omega_c} l_{pc} \mathbf{n}_{pc} \cdot \mathbf{E}_{p,1}^{n+\frac{1}{2}} = 0, \tag{4.34}$$

where the last identities in the above equations follow from the fact that the order of the indices in the double sums can be rearranged thanks to the periodic boundary conditions and where equation (4.13) has been used:

$$\sum_p \sum_{c \in \Omega_p} l_{pc} \mathbf{n}_{cp} \cdot \mathbf{B}_{c,1}^{n+\frac{1}{2}} = \sum_c \sum_{p \in \Omega_c} l_{pc} \mathbf{n}_{cp} \cdot \mathbf{B}_{c,1}^{n+\frac{1}{2}} = \sum_c \mathbf{B}_{c,1}^{n+\frac{1}{2}} \cdot \sum_{p \in \Omega_c} l_{pc} \mathbf{n}_{cp} = 0 \tag{4.35}$$

and

$$\sum_c \sum_{p \in \Omega_c} l_{pc} \mathbf{n}_{pc} \cdot \mathbf{E}_{p,1}^{n+\frac{1}{2}} = \sum_p \sum_{c \in \Omega_p} l_{pc} \mathbf{n}_{pc} \cdot \mathbf{E}_{p,1}^{n+\frac{1}{2}} = \sum_p \mathbf{E}_{p,1}^{n+\frac{1}{2}} \cdot \sum_{c \in \Omega_p} l_{pc} \mathbf{n}_{pc} = 0. \tag{4.36}$$

We thus have from equations (4.33) and (4.34) that

$$c_0^{-1}|\Omega|\frac{\phi_0^{n+1}-\phi_0^n}{\Delta t}=0 \quad \text{and} \quad c_0^{-1}|\Omega|\frac{\psi_0^{n+1}-\psi_0^n}{\Delta t}=0, \tag{4.37}$$

with $|\Omega|=\sum_p|\Omega_p|=\sum_c|\Omega_c|$ the area of the computational domain. Hence, at leading order the discrete time derivatives of the discrete scalar fields ϕ and ψ vanish. Substituting this result with $\phi_{p,0}^n=\phi_0^n$ and $\psi_{c,0}^n=\psi_0^n$ back into equations (4.29) and (4.31) yields

$$\nabla_p^c\cdot\mathbf{B}_{c,1}^{n+\frac{1}{2}}=0 \quad \text{and} \quad \nabla_c^p\cdot\mathbf{E}_{p,1}^{n+\frac{1}{2}}=0. \tag{4.38}$$

With the expansion (equation (4.27)) we thus obtain the sought result:

$$\nabla_p^c\cdot\mathbf{B}_{c}^{n+\frac{1}{2}}=\mathcal{O}(\epsilon^2) \quad \text{and} \quad \nabla_c^p\cdot\mathbf{E}_{p}^{n+\frac{1}{2}}=\mathcal{O}(\epsilon^2) \tag{4.39}$$

for $\epsilon\to 0$. ■

Property 4.1. For discrete initial data that are compatible with the vacuum Maxwell equations (Maxwell-compatible initial data), i.e. that satisfy $\nabla_p^c\cdot\mathbf{B}_c^0=0$, $\nabla_c^p\cdot\mathbf{E}_p^0=0$, $\phi_p^0=0$, $\psi_c^0=0$ at time $t=0$, i.e. for $n=0$, we have $\nabla_p^c\cdot\mathbf{B}_c^n=0$ and $\nabla_c^p\cdot\mathbf{E}_p^n=0$ for all times $t>0$, i.e. for all $n>0$.

Proof. Using the discrete identities contained in equation (4.18) one obtains the following two discrete wave equations from the scheme (equations (4.19)–(4.22)):

$$\phi_p^{n+1}-\frac{1}{4}\Delta t^2c_h^2\nabla_p^c\cdot\nabla_c^q\phi_q^{n+1}=\phi_p^n-\Delta tc_h\nabla_p^c\cdot\mathbf{B}_c^n+\frac{1}{4}\Delta t^2c_h^2\nabla_p^c\cdot\nabla_c^q\phi_q^n \tag{4.40}$$

and

$$\psi_c^{n+1}-\frac{1}{4}\Delta t^2c_h^2\nabla_c^p\cdot\nabla_p^a\psi_a^{n+1}=\psi_c^n-\Delta tc_h\nabla_c^p\cdot\mathbf{E}_p^n+\frac{1}{4}\Delta t^2c_h^2\nabla_c^p\cdot\nabla_p^a\psi_a^n. \tag{4.41}$$

From equations (4.40) and (4.41) and $\nabla_p^c\cdot\mathbf{B}_c^0=0$, $\nabla_c^p\cdot\mathbf{E}_p^0=0$, $\phi_p^0=0$, $\psi_c^0=0$ it immediately follows that $\phi_p^n=0$ and $\psi_c^n=0$ for all $n>0$. Using this result and applying the discrete divergence operators to equations (4.19) and (4.21) yields $\nabla_p^c\cdot\mathbf{B}_c^{n+1}=\nabla_p^c\cdot\mathbf{B}_c^n=0$ and $\nabla_p^c\cdot\mathbf{E}_c^{n+1}=\nabla_p^c\cdot\mathbf{E}_c^n=0$, if $\nabla_p^c\cdot\mathbf{B}_c^0=0$ and $\nabla_p^c\cdot\mathbf{E}_c^0=0$, i.e. for Maxwell-compatible initial data the scheme preserves the divergence-free condition of the magnetic and electric field exactly at the discrete level for all times. ■

Property 4.2. For *well-prepared* initial data in the sense $\nabla_p^c\cdot\mathbf{B}_c^0=\mathcal{O}(\epsilon^2)$, $\nabla_c^p\cdot\mathbf{E}_p^0=\mathcal{O}(\epsilon^2)$, $\phi_p^0=0$, $\psi_c^0=0$ at time $t=0$, i.e. for $n=0$, we have $\nabla_p^c\cdot\mathbf{B}_c^n=\mathcal{O}(\epsilon^2)$ and $\nabla_c^p\cdot\mathbf{E}_p^n=\mathcal{O}(\epsilon^2)$ for all times $t>0$, i.e. for all $n>0$.

Proof. The proof follows the same lines as the one for the previous proposition. ■

5. Numerical results

In all of the following tests we set $c_0=1$ and use the nine-stage seventh-order accurate Runge–Kutta scheme of Butcher [71] to integrate the semi-discrete hyperbolic thermodynamically compatible (HTC) scheme (equation (4.4)) in time. Furthermore, in all the tests, we use periodic boundary conditions everywhere.

(a) Numerical convergence study

We first carry out a systematic numerical convergence study of the two numerical schemes presented in this paper. For this purpose, we consider the following initial data of a planar wave

Table 1. L^2 error norms and corresponding empirical convergence rates computed on two subsequent meshes for the planar wave travelling in the direction n = (1, 1, 0) obtained with the semi-discrete HTC scheme on uniform grids composed of $N \times N$ elements.

	L^2 errors, e_i				numerical convergence			
N_i	20	40	80	160	orders, p_{i+1}			
B_1	2.57×10^{-2}	6.45×10^{-3}	1.61×10^{-3}	4.04×10^{-4}	—	1.99	2.00	2.00
B_2	2.57×10^{-2}	6.45×10^{-3}	1.61×10^{-3}	4.04×10^{-4}	—	1.99	2.00	2.00
B_3	1.45×10^{-1}	3.65×10^{-2}	9.13×10^{-3}	2.28×10^{-3}	—	1.99	2.00	2.00
ϕ	3.63×10^{-2}	9.12×10^{-3}	2.28×10^{-3}	5.71×10^{-4}	—	1.99	2.00	2.00
E_1	1.54×10^{-1}	3.87×10^{-2}	9.69×10^{-3}	2.42×10^{-3}	—	1.99	2.00	2.00
E_2	5.14×10^{-2}	1.29×10^{-2}	3.23×10^{-3}	8.07×10^{-4}	—	1.99	2.00	2.00
ψ	7.27×10^{-2}	1.82×10^{-2}	4.57×10^{-3}	1.14×10^{-3}	—	1.99	2.00	2.00

Table 2. L^2 error norms and corresponding empirical convergence rates computed on two subsequent meshes for the planar wave travelling in the direction n = (1, 1, 0) obtained with the fully discrete semi-implicit scheme on uniform grids composed of $N \times N$ elements.

	L^2 errors, e_i				numerical convergence			
N_i	20	40	80	160	order, p_{i+1}			
B_1	3.06×10^{-2}	7.74×10^{-3}	1.94×10^{-3}	4.85×10^{-4}	—	1.98	2.00	2.00
B_2	3.06×10^{-2}	7.74×10^{-3}	1.94×10^{-3}	4.85×10^{-4}	—	1.98	2.00	2.00
B_3	1.73×10^{-1}	4.38×10^{-2}	1.10×10^{-2}	2.75×10^{-3}	—	1.98	2.00	2.00
ϕ	4.33×10^{-2}	1.09×10^{-2}	2.74×10^{-3}	6.86×10^{-4}	—	1.98	2.00	2.00
E_1	1.84×10^{-1}	4.64×10^{-2}	1.16×10^{-2}	2.91×10^{-3}	—	1.98	2.00	2.00
E_2	6.12×10^{-2}	1.55×10^{-2}	3.88×10^{-3}	9.71×10^{-4}	—	1.98	2.00	2.00
ψ	8.65×10^{-2}	2.19×10^{-2}	5.49×10^{-3}	1.37×10^{-3}	—	1.98	2.00	2.00

travelling in the direction $\mathbf{n} = (1, 1, 0)$. The initial condition is given by

$$\mathbf{B}(x, y) = \mathbf{B}_0 \sin(\pi(x - y)), \quad \phi(x, y) = 0.25 \sin(\pi(x - y)) \tag{5.1}$$

and

$$\mathbf{E}(x, y) = \mathbf{E}_0 \sin(\pi(x - y)), \quad \psi(x, y) = 0.5 \sin(\pi(x - y)), \tag{5.2}$$

with $\mathbf{B}_0 = (0.25b, -0.25b, 1)^T$, $\mathbf{E}_0 = (1.5b, 0.5b, 0)^T$ and $b = \sqrt{2}/2$. The parameters of the model are chosen to be $c_h = 1.0$ and $c_0 = 1.0$. The computational domain $\Omega = [-1, 1]^2$ is discretized with a sequence of successively refined uniform grids composed of $N \times N$ elements and simulations are run for one period until a final time of $t = \sqrt{2}$, when the exact solution of the problem coincides again with the initial condition. The semi-discrete HTC scheme employs a seventh-order accurate Runge–Kutta method for time discretization [71]. In both schemes, the time step is forced to obey the Courant-Friedrichs-Lewy (CFL) stability condition with CFL = 0.9, i.e. $\Delta t = \mathrm{CFL}/(c_0/\Delta x + c_0/\Delta y)$. Tables 1 and 2 list the L^2 errors e_i and the associated numerical convergence rates (experimental order of accuracy) p_{i+1} computed as $p_{i+1} = \log(e_{i+1}/e_i)/\log(N_i/N_{i+1})$ of all relevant quantities at the final time depending on the chosen grid spacing. As one can observe, both methods converge with second order of accuracy in both space and time, as expected.

In addition figure 2 shows the temporal evolution of the relative total energy error $\mathcal{E}^n/\mathcal{E}^0 - 1$ for both schemes and for all meshes considered in the convergence study. As one can observe, and as expected, total energy is preserved up to machine precision in all cases.

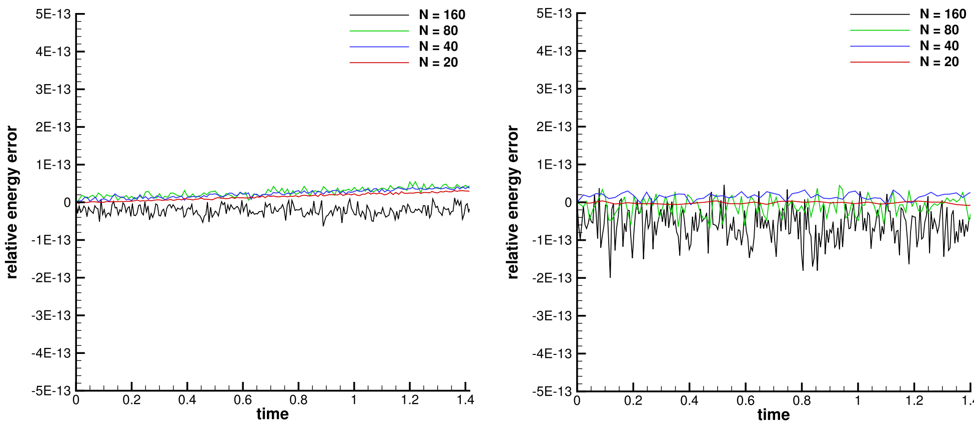

Figure 2. Temporal evolution of the relative total energy error $\mathcal{E}^n/\mathcal{E}^0 - 1$ for a sequence of successively refined uniform grids composed of $N \times N$ elements obtained with the semi-discrete HTC scheme (left panel) as well as the fully discrete semi-implicit scheme (right panel).

(b) Discrete total energy conservation and exact divergence preservation

Both the semi-discrete HTC scheme and the fully discrete semi-implicit scheme are exactly energy conservative, see theorems 4.1 and 4.2, respectively. Furthermore, the structure-preserving staggered semi-implicit scheme is also exactly compatible with the divergence-free condition of the electric and the magnetic field provided the initial data are compatible with those of the vacuum Maxwell equations, see property 4.1. To test these properties of the schemes, we therefore solve the following test problem, given by the initial data:

$$\left.\begin{aligned} &\mathbf{B}(x,y) = \mathbf{B_0}\exp\left(-\frac{1}{2}\mathbf{x}^2/\sigma^2\right), \quad \phi(x,y) = \phi_0\exp\left(-\frac{1}{2}\mathbf{x}^2/\sigma^2\right)\\ \text{and}\qquad &\mathbf{E}(x,y) = \mathbf{E_0}\exp\left(-\frac{1}{2}\mathbf{x}^2/\sigma^2\right), \quad \psi(x,y) = \psi_0\exp\left(-\frac{1}{2}\mathbf{x}^2/\sigma^2\right), \end{aligned}\right\} \tag{5.3}$$

Simulations are run for all schemes in the domain $\Omega = [-1, 1]^2$ until a final time of $t = 10$, which is substantially larger than the final time of the previous test. We propose two different test cases, T1 and T2. In the first test (T1) we choose initial data that are compatible with the original vacuum Maxwell equations. We therefore set $\mathbf{B_0} = \mathbf{E_0} = (0, 0, 10^{-2})$ and $\phi_0 = \psi_0 = 0$. In the second test (T2) we choose initial data that are deliberately not compatible with the original vacuum Maxwell equations, setting $\mathbf{B_0} = \mathbf{E_0} = (0.25 \times 10^{-2}, 0, 10^{-2})$ and $\phi_0 = \psi_0 = 0.5 \times 10^{-2}$. In both tests the half width is chosen as $\sigma = 0.2$. As in the previous test case, we set $c_0 = 1$, $c_h = 1$.

The left panel of figure 3 shows the temporal evolution of the relative total energy error $\mathcal{E}^n/\mathcal{E}^0 - 1$ provided by the fully discrete semi-implicit scheme (SIMM) on a fixed mesh composed of 50×50 elements for both tests T1 and T2, and time step $\Delta t = \mathrm{CFL}/(c_0/\Delta x + c_0/\Delta y)$, where $\mathrm{CFL} = 0.9$. In addition, test T2 has also been solved employing the semi-discrete HTC scheme on a fixed mesh composed of 160×160 elements evolved in time according to a seventh-order accurate Runge–Kutta method with a time step restricted by the CFL stability condition with $\mathrm{CFL} = 0.6$. As one can observe, the total energy is preserved up to machine precision in all the cases. The right-hand panel of figure 3 shows the time evolution of the divergence errors of the magnetic and the electric field obtained with the SIMM scheme for test T1 . As expected, the divergence errors remain of the order of machine precision for all times. We now repeat test case T2 also for a nonlinear total energy potential. We choose a total energy density given by

$$\mathcal{E} = c_0\exp\left(\frac{1}{2}\mathbf{B}^2\right) + c_0\exp\left(\frac{1}{2}\mathbf{E}^2\right) + \frac{c_h^2}{c_0}\exp\left(\frac{1}{2}\phi^2\right) + \frac{c_h^2}{c_0}\exp\left(\frac{1}{2}\psi^2\right). \tag{5.4}$$

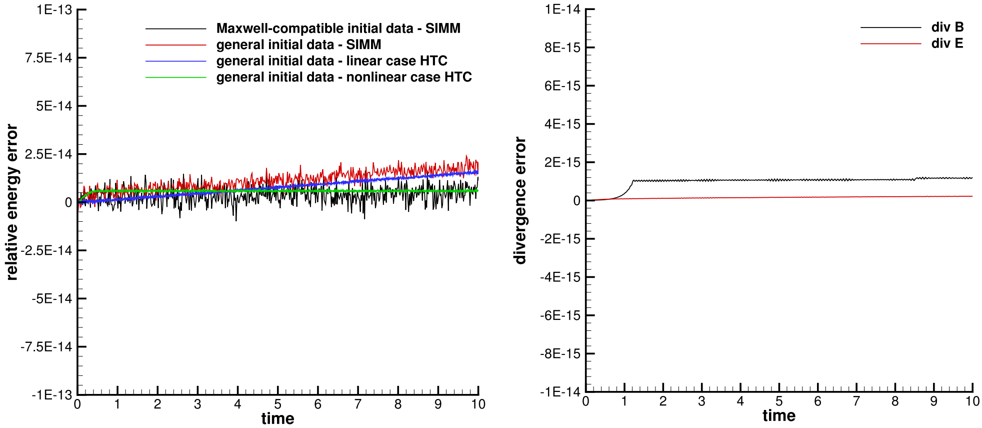

Figure 3. Left panel: Temporal evolution of the relative total energy error $\mathcal{E}^n/\mathcal{E}^0 - 1$ for initial data that is compatible with the vacuum Maxwell equations (black line), for general initial data employing the semi-implicit scheme (red line) as well as the HTC scheme (blue line) and for general initial data in case the nonlinear total energy potential (equation (5.4)) is used and the problem solved with the HTC scheme (green line). Right panel: time series of the divergence errors of the magnetic and electric field for the SIMM scheme and for initial data that is compatible with the vacuum Maxwell equations.

The simulation is run for the semi-discrete HTC scheme employing a nine stages seventh-order accurate Runge–Kutta method for time discretization, where the time step is restricted by a CFL stability condition with CFL$=0.9$. The temporal evolution of the relative total energy error $\mathcal{E}^n/\mathcal{E}^0 - 1$ on a fixed mesh composed of 160×160 elements is reported in the left-hand panel of figure 3. As expected, total energy is conserved up to machine precision, with only a very slight linear growth in time due to the natural accumulation of round-off errors in the computer.

(c) Discrete AP property for $c_h \to \infty$

In this section we show numerical evidence for the fact that the staggered semi-implicit structure-preserving scheme is also AP in the limit $c_h \to \infty$, i.e. $\epsilon = c_0/c_h \to 0$. For this purpose, we use again the same initial condition (equation (5.3)) as in the previous section, with perturbation amplitudes in which the magnetic and electric field are initially *not* divergence-free any more but with $\phi_0 = \psi_0 = 0$, to satisfy the conditions of theorem 4.3 and property 4.2. We run two sets of simulations on the domain $\Omega = [-1,+1]^2$ discretized with 40×40 elements until a final time of $t = 0.1$ with a fixed time step of size $\Delta t = 10^{-2}$. We choose different cleaning speeds c_h in the range between 10^2 and 10^5 and compute the discrete divergence of the magnetic and electric fields in terms of c_h.

(i) Test 1: well-prepared initial data

Here we choose an initial perturbation that scales with ϵ^2 by setting the parameters $\mathbf{B_0} = \mathbf{E_0} = \epsilon^2 \cdot (1,0,1)$. Table 3 lists that the divergence errors remain of the order ϵ^2, as expected and predicted by property 4.2.

(ii) Test 2: not well-prepared initial data

In this test we choose a constant perturbation that does not scale with ϵ by setting $\mathbf{B_0} = \mathbf{E_0} = (10^{-4}, 0, 10^{-2})$. Table 4 lists that the divergence errors in the electric and magnetic field still converge quadratically in the parameter $\epsilon = c_0/c_h$, as expected and as predicted by theorem 4.3, which indeed applies to general initial data, also not well-prepared ones, under the condition that $\phi = const.$ and $\psi = const.$ at $t = 0$.

Table 3. Test 1: well-prepared initial data, to verify property 4.2. L^2 error norms at $t = 0.1$ and corresponding convergence rates of the divergence errors of the magnetic and the electric field in terms of the dimensionless parameter $\epsilon = c_0/c_h$ on a fixed mesh of 40×40 elements obtained with the fully discrete semi-implicit scheme for increasing values of the cleaning speed c_h. In all simulations we have set $c_0 = 1$.

$c_0 = 1$		L^2 divergence errors		convergence order	
c_h	ϵ	$\nabla_p^c \times \mathrm{B}_c^{n+1}$	$\nabla_c^p \times \mathrm{E}_p^{n+1}$	$\nabla_p^c \times \mathrm{B}_c^{n+1}$	$\nabla_c^p \times \mathrm{E}_p^{n+1}$
10^2	10^{-2}	8.893666×10^{-5}	8.755024×10^{-5}	—	—
10^3	10^{-3}	9.513268×10^{-6}	9.512130×10^{-6}	2.0	2.0
10^4	10^{-4}	1.240058×10^{-8}	1.240058×10^{-8}	1.9	1.9
10^5	10^{-5}	1.243551×10^{-10}	1.243551×10^{-10}	2.0	2.0

Table 4. Test 2: not well-prepared initial data, to verify theorem 4.3. L^2 error norms at $t = 0.1$ and corresponding convergence rates of the divergence errors of the magnetic and the electric field in terms of the dimensionless parameter $\epsilon = c_0/c_h$ on a fixed mesh of 40×40 elements obtained with the fully discrete semi-implicit scheme for increasing values of the cleaning speed c_h. In all simulations we have set $c_0 = 1$.

$c_0 = 1$		L^2 divergence errors		convergence order	
c_h	ϵ	$\nabla_p^c \times \mathrm{B}_c^{n+\frac{1}{2}}$	$\nabla_c^p \times \mathrm{E}_p^{n+\frac{1}{2}}$	$\nabla_p^c \times \mathrm{B}_c^{n+\frac{1}{2}}$	$\nabla_c^p \times \mathrm{E}_p^{n+\frac{1}{2}}$
10^2	10^{-2}	3.831380×10^{-5}	3.831579×10^{-5}	—	—
10^3	10^{-3}	3.569500×10^{-6}	3.569623×10^{-6}	1.0	1.0
10^4	10^{-4}	4.351311×10^{-8}	4.351523×10^{-8}	1.9	1.9
10^5	10^{-5}	4.368280×10^{-10}	4.358525×10^{-10}	2.0	2.0

6. Conclusion

In this paper, we have provided an original variational derivation of the augmented hyperbolic GLM divergence cleaning system of Munz *et al.* [20], which is a very interesting SHTC system. We furthermore have provided a formal asymptotic analysis for the limit $\epsilon = c_0/c_h \to 0$ as well as its extension to general nonlinear energy potentials and to the case of special relativity. We also have established a direct connection to Hamiltonian mechanics and have found the associated Poisson bracket. To the best of our knowledge, the Maxwell–Munz system studied in this paper goes beyond the existing set of known SHTC systems. Concerning its numerical discretization we have discussed two different types of exactly structure-preserving schemes. A first set of semi-discrete cell-centred HTC finite volume schemes that employ collocated grids and that are exactly compatible with the total energy conservation law, but which do in general not respect the basic identities of vector calculus. We then have also proposed a second fully discrete method that uses vertex-based staggered grids, employs compatible mimetic finite difference operators for the discrete gradients so that all basic vector calculus identities are exactly satisfied also at the discrete level. Furthermore, the second scheme also conserves total energy exactly at the fully discrete level. It furthermore is asymptotically consistent with the limit when $\epsilon = c_0/c_h \to 0$, as shown by numerical evidence. We also have provided some numerical results concerning the extension of both schemes to the fully nonlinear case. In all cases, the numerical results confirm the theoretical expectations.

Future work will concern the extension of the staggered semi-implicit HTC schemes introduced in this work to other more complex nonlinear SHTC systems, including also numerical dissipation and entropy production, which were both not present in the methods developed in this paper. We also plan to apply the present schemes to the system of Buchman *et al.* in the context of new structure-preserving schemes for numerical general relativity.

Data accessibility. All computer codes are provided as ESM. The data are provided in electronic supplementary material [72].

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

Authors' contributions. M.D.: conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, resources, software, supervision, validation, visualization, writing—original draft and writing—review and editing; A.L.: conceptualization, data curation, formal analysis, investigation, methodology, resources, software, validation, visualization, writing—original draft and writing—review and editing; I.P.: conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, resources, software, validation, visualization, writing—original draft and writing—review and editing; and O.Z.: conceptualization, data curation, formal analysis, investigation, methodology, resources, software, validation, visualization, writing—original draft and writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

Conflict of interest declaration. We declare we have no competing interests.

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Appendix A

This appendix aims to verify the discrete vector calculus identities (equation (4.17)) and (equation (4.18)).

To simplify the computation, we restrict the discussion to two-dimensional case, i.e. we assume that $\partial/\partial z$ vanishes for all fields and thus, we assume a two-dimensional physical domain covered by a set of equidistant and non-overlapping Cartesian control volumes $\Omega_{i,j} = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right] \times \left[y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\right]$ with uniform mesh spacings of size $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and $\Delta y = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$ in x and y direction, respectively, where $x_{i\pm\frac{1}{2}} = x_i \pm \Delta x/2$ and $y_{j\pm\frac{1}{2}} = y_j \pm \Delta y/2$. For a schematic view of the grid, we refer again to figure 1 where c represents a pair of indices of the type (i,j), while p is of the form $(i \pm \frac{1}{2}, j \pm \frac{1}{2})$.

Let us introduce a scalar field in the barycentres of the primal control volume $\Omega_{i,j}$, i.e. $\phi_{i,j}$, then the discrete nabla operator (equation (4.12)) defines a discrete gradient of ϕ in all vertices of the primal mesh:

$$\nabla^c_{i+\frac{1}{2},j+\frac{1}{2}} \phi_c = \begin{pmatrix} \dfrac{1}{2\Delta x}(\phi_{i+1,j+1} + \phi_{i+1,j} - \phi_{i,j+1} - \phi_{i,j}) \\ \dfrac{1}{2\Delta y}(\phi_{i+1,j+1} + \phi_{i,j+1} - \phi_{i+1,j} - \phi_{i,j}) \\ 0 \end{pmatrix}. \tag{A1}$$

Applying the discrete curl operator to the last term, it is straightforward to verify that we obtain a null identity, that is,

$$\begin{aligned}
&(\nabla^p_{i,j} \times \nabla^c_p \phi_c) \cdot \mathbf{e_z}\\
&\quad = + \frac{\phi_{i+1,j+1} + \phi_{i,j+1} - \phi_{i+1,j} - \phi_{i,j} + \phi_{i+1,j} + \phi_{i,j} - \phi_{i+1,j+1} - \phi_{i,j-1}}{4\Delta x \Delta y}\\
&\qquad - \frac{\phi_{i,j-1} + \phi_{i-1,j+1} - \phi_{i,j} - \phi_{i-1,j} + \phi_{i,j} + \phi_{i-1,j} - \phi_{i,j-1} - \phi_{i-1,j-1}}{4\Delta x \Delta y}\\
&\qquad - \frac{\phi_{i+1,j+1} + \phi_{i+1,j} - \phi_{i,j+1} - \phi_{i,j} + \phi_{i,j+1} + \phi_{i,j} - \phi_{i-1,j+1} - \phi_{i-1,j}}{4\Delta x \Delta y}\\
&\qquad + \frac{\phi_{i+1,j} + \phi_{i+1,j-1} - \phi_{i,j} - \phi_{i,j-1} + \phi_{i,j} + \phi_{i,j-1} - \phi_{i-1,j} - \phi_{i-1,j-1}}{4\Delta x \Delta y}\\
&\quad = 0,
\end{aligned} \tag{A 2}$$

where $\mathbf{e_z} = (0, 0, 1)$ is the third unit vector in Cartesian coordinates. We have therefore shown that the continuous identity $\nabla \times \nabla\phi = 0$ has a discrete analogue represented by the relation

$$\nabla^p_c \times \nabla^a_p \phi_a = 0. \tag{A 3}$$

In the same way one can prove the dual identity $\nabla^c_p \times \nabla^q_c \phi_q = 0$. Similarly, we introduce a vector field $\mathbf{A}_{i,j}$ defined in the $\Omega_{i,j}$ barycentre and we compute $\nabla^c_p \times \mathbf{A}_c$, that is,

$$\begin{aligned}
&\nabla^c_{i+\frac{1}{2},j+\frac{1}{2}} \times \mathbf{A}_c\\
&\quad = \begin{pmatrix} \dfrac{A_{3,i+1,j+1} + A_{3,i,j+1} - A_{3,i+1,j} - A_{3,i,j}}{2\Delta y} \\ -\dfrac{A_{2,i+1,j+1} + A_{2,i+1,j} - A_{2,i,j+1} - A_{2,i,j}}{2\Delta x} \\ \dfrac{A_{2,i+1,j+1} + A_{2,i+1,j} - A_{2,i,j+1} - A_{2,i,j}}{2\Delta x} - \dfrac{A_{1,i+1,j+1} + A_{1,i,j+1} - A_{1,i+1,j} - A_{1,i,j}}{2\Delta y} \end{pmatrix}.
\end{aligned} \tag{A 4}$$

Applying the discrete divergence to the discrete curl of $\mathbf{A}$, it is possible to verify that the continuous identity $\nabla \cdot \nabla \times \mathbf{A} = 0$ holds also at the discrete level. Indeed, we have

$$\begin{aligned}
&\nabla^p_{i,j} \cdot \nabla^c_p \times \mathbf{A}_c\\
&\quad = + \frac{A_{3,i+1,j+1} + A_{3,i,j+1} - A_{3,i+1,j} - A_{3,i,j} + A_{3,i+1,j} + A_{3,i,j} - A_{3,i+1,j-1} - A_{3,i,j-1}}{4\Delta x \Delta y}\\
&\qquad - \frac{A_{3,i,j+1} + A_{3,i-1,j+1} - A_{3,i,j} - A_{3,i-1,j} + A_{3,i,j} + A_{3,i-1,j} - A_{3,i,j-1} - A_{3,i-1,j-1}}{4\Delta x \Delta y}\\
&\qquad - \frac{A_{3,i+1,j+1} + A_{3,i+1,j} - A_{3,i,j+1} - A_{3,i,j} + A_{3,i,j+1} + A_{3,i,j} - A_{3,i-1,j+1} - A_{3,i-1,j}}{4\Delta x \Delta y}\\
&\qquad + \frac{A_{3,i+1,j} + A_{3,i+1,j-1} - A_{3,i,j} - A_{3,i,j-1} + A_{3,i,j} + A_{3,i,j-1} - A_{3,i-1,j} - A_{3,i-1,j-1}}{4\Delta x \Delta y}\\
&\quad = 0.
\end{aligned} \tag{A 5}$$

In the same way one can prove the dual identity $\nabla^c_p \cdot \nabla^q_c \times \mathbf{A}_q = 0$.

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