# THE ABU-GHUWALEH ORBIT-SINGULARITY SERIES I: FINITE SKELETON RECOVERY, RADIUS LIFTING, AND STRICT EXPONENTIAL IMPROVEMENT OVER TAYLOR TRUNCATION

MOHAMMAD ABU-GHUWALEH

Abstract. We introduce the *Abu-Ghuwaleh Orbit-Singularity Series* (AGOSS), an adaptive singular expansion obtained by repeatedly extracting a dominant singular atom from the asymptotic ratio jet of the Taylor coefficients of a holomorphic germ at the origin. In this first paper we work in a deliberately rigid regime: one complex variable, finitely many algebraic-logarithmic singular atoms, and strict radial separation of singular locations. On this class we prove five results. First, the dominant ratio jet canonically identifies the dominant atom. Second, the greedy AGOSS peeling procedure recovers the entire finite singular skeleton in finitely many steps. Third, each successful extraction lifts the analytic radius of the residual exactly to the next singular radius. Fourth, truncating Taylor only after skeleton extraction yields a strictly better exponential approximation rate than direct Taylor truncation. Fifth, the recovered skeleton is stable under asymptotically negligible perturbations of the ratio jet, which yields an inverse orbit-to-skeleton principle. We also explain how a simple confluent atom decomposes algebraically into two basic atoms, thereby motivating the next paper in the series.

## 1. Introduction

A classical theme in complex asymptotics is that dominant singularities control coefficient growth. Darboux-type methods, the asymptotic machinery surveyed by Bender, and the singularity-analysis program of Flajolet and Odlyzko make this principle precise: local singular expansions at dominant boundary points transfer to asymptotic expansions of Taylor coefficients. In particular, singularity analysis gives a term-by-term dictionary between algebraic-logarithmic singular profiles and coefficient asymptotics, and the modern analytic-combinatorics viewpoint has made that dictionary systematic and widely usable; see [1, 2, 3].

The present paper goes in the reverse direction, but only on a tightly controlled class. We ask whether one can recover the singular data themselves from the coefficient tail, not merely infer its growth rate. More ambitiously, we ask whether one can turn this recovery principle into a new adaptive expansion that is fundamentally better aligned with the singular structure of the function than direct Taylor truncation.

This leads to the central object of the paper.

**Informal slogan.** Given a germ $f(z) = \sum_{n\geq 0} a_n z^n$, inspect the tail orbit of coefficient ratios, extract the dominant singular atom encoded by that orbit, subtract it, and iterate.

The result is a singular expansion built from the detected singular geometry itself. We call it the *Abu-Ghuwaleh Orbit-Singularity Series*. In this first paper we deliberately avoid the most difficult phenomena: several points on the same dominant circle, destructive interference among equal-modulus exponentials, and full multivariate geometry. Instead we prove a complete theory on a rigid but nontrivial regime:

- one complex variable;
- finitely many singular atoms;
- algebraic-logarithmic atoms only;
- strict radial separation of singular points;
- greedy extraction from ratio jets.

2020 *Mathematics Subject Classification.* Primary 30B10; Secondary 30E15, 30B40, 41A60.

*Key words and phrases.* orbit-singularity series, singularity analysis, power series, coefficient asymptotics, generating functions, analytic continuation, asymptotic expansions, radius lifting.

Within this regime we prove:

(i) **canonical dominant atom selection**: the dominant ratio jet uniquely identifies the dominant atom;
(ii) **finite skeleton recovery**: the greedy procedure recovers the full singular skeleton in finitely many steps;
(iii) **residual radius lifting**: each successful extraction raises the radius of analyticity of the residual exactly to the next singular radius;
(iv) **strict exponential improvement over Taylor truncation**: once the dominant skeleton is removed, Taylor applied to the residual converges at a strictly better exponential rate than direct Taylor truncation of the original function;
(v) **stability and inverse recovery**: asymptotically negligible perturbations of the ratio jet do not change the extracted dominant skeleton, and any function asymptotically orbit-equivalent to a finite-skeleton model inherits the same recovered skeleton.

The point is not maximal generality. The point is closure: the first paper proves a clean theorem package on a mathematically stable class. This gives the series a solid base before the introduction of same-circle interference, confluent chains, or several variables.

## 2. Basic atoms, finite skeletons, and the AGOSS algorithm

2.1. **Basic algebraic-logarithmic atoms.** Fix the principal branch of log on $\mathbb{C}\setminus(-\infty,0]$. For $\rho\in\mathbb{C}\setminus\{0\}$ define

$$L_\rho(z):=\log\left(\frac{1}{1-z/\rho}\right)=-\log(1-z/\rho),$$

which is holomorphic near $z=0$. For parameters

$$\rho\in\mathbb{C}\setminus\{0\},\qquad \alpha\in\mathbb{C}\setminus\{0,-1,-2,\dots\},\qquad \mu\in\mathbb{N}_0,$$

we define the *basic atom*

$$\mathcal{A}_{\rho,\alpha,\mu}(z):=(1-z/\rho)^{-\alpha}L_\rho(z)^\mu. \tag{2.1}$$

Because $L_\rho(z)=-\log(1-z/\rho)$, one has the useful identity

$$\mathcal{A}_{\rho,\alpha,\mu}(z)=\partial_\alpha^\mu(1-z/\rho)^{-\alpha}. \tag{2.2}$$

The radius of convergence of the Taylor series of $\mathcal{A}_{\rho,\alpha,\mu}$ at the origin is exactly $|\rho|$.

**Remark 2.1** (Why the first paper stops here)**.** A simple confluent atom of the form

$$\Phi_{\rho,\alpha,\mu,\lambda,b}(z):=(1-z/\rho)^{-\alpha}L_\rho(z)^\mu\big(1+b(1-z/\rho)^\lambda\big)$$

with $\lambda>0$ decomposes algebraically as

$$\Phi_{\rho,\alpha,\mu,\lambda,b}(z)=\mathcal{A}_{\rho,\alpha,\mu}(z)+b\,\mathcal{A}_{\rho,\alpha-\lambda,\mu}(z), \tag{2.3}$$

provided $\alpha-\lambda\notin\{0,-1,-2,\dots\}$. Thus simple confluence already lies one algebraic step beyond the basic dictionary. The genuine difficulty is not the identity (2.3); it is the automatic recovery of several same-radius descendants and, later, several distinct points on the same dominant circle. Those questions are deferred to the next paper. The present paper isolates the core theory before those interference effects enter.

2.2. **Finite singular skeletons.**

**Definition 2.2** (Finite singular skeleton)**.** Let $K\in\mathbb{N}$. We say that a germ $f(z)=\sum_{n\geq 0}a_nz^n$ at 0 has an *admissible finite singular skeleton of length* $K$ if there exist nonzero coefficients $A_1,\dots,A_K\in\mathbb{C}$, parameters $(\rho_j,\alpha_j,\mu_j)$ as above, and a function $H$ holomorphic in $|z|<R_H$ for some $R_H>0$ such that

$$f(z)=H(z)+\sum_{j=1}^K A_j\mathcal{A}_{\rho_j,\alpha_j,\mu_j}(z), \tag{2.4}$$

with the *strict radial separation*

$$0<|\rho_1|<|\rho_2|<\cdots<|\rho_K|<R_H. \tag{2.5}$$

We call the ordered list

$$\mathfrak{S}(f) := \big((A_j, \rho_j, \alpha_j, \mu_j)\big)_{j=1}^{K}$$

the *finite singular skeleton* of $f$.

The class (2.4) is intentionally rigid. The strict separation (2.5) eliminates equal-modulus interference and makes a clean recovery theorem possible.

2.3. **Ratio orbit and dominant fingerprint.** Let $f(z) = \sum_{n\geq 0} a_n z^n$ be a germ whose coefficients are eventually nonzero. Its *ratio orbit* is the tail sequence

$$\mathcal{O}(f) := \big(r_n(f)\big)_{n\gg 1}, \qquad r_n(f) := \frac{a_n}{a_{n-1}}.$$

When the relevant limits exist, we define the *dominant fingerprint* of $f$ by

$$\rho(f) := \lim_{n\to\infty} \frac{a_{n-1}}{a_n}, \tag{2.6}$$

$$\beta(f) := \lim_{n\to\infty} n(\rho(f) r_n(f) - 1), \tag{2.7}$$

$$\mu(f) := \lim_{n\to\infty} \log n\Big(n(\rho(f) r_n(f) - 1) - \beta(f)\Big), \tag{2.8}$$

$$C(f) := \lim_{n\to\infty} a_n\, \rho(f)^n n^{-\beta(f)} (\log n)^{-\mu(f)}. \tag{2.9}$$

If these limits exist and $C(f) \neq 0$, we define the associated dominant atom

$$\mathcal{D}(f) := \Gamma(\beta(f)+1) C(f)\, \mathcal{A}_{\rho(f),\beta(f)+1,\mu(f)}. \tag{2.10}$$

The philosophy is simple: $\rho(f)$ identifies the nearest singular point, $\beta(f)+1$ identifies the algebraic exponent, $\mu(f)$ identifies the logarithmic rank, and $C(f)$ identifies the amplitude.

2.4. **The AGOSS peeling procedure.**

**Definition 2.3** (Abu-Ghuwaleh Orbit-Singularity Series)**.** Set $R_0 := f$. Whenever $\mathcal{D}(R_m)$ is defined, choose

$$A_m^\star := \Gamma(\beta(R_m)+1) C(R_m), \qquad \theta_m^\star := \big(\rho(R_m), \beta(R_m)+1, \mu(R_m)\big),$$

and define the next residual by

$$R_{m+1}(z) := R_m(z) - A_m^\star \mathcal{A}_{\theta_m^\star}(z), \tag{2.11}$$

where, for $\theta = (\rho, \alpha, \mu)$, we write $\mathcal{A}_\theta := \mathcal{A}_{\rho,\alpha,\mu}$. If the process can be continued indefinitely, we write formally

$$f \sim \sum_{m\geq 0} A_m^\star \mathcal{A}_{\theta_m^\star} \tag{2.12}$$

and call this the *Abu-Ghuwaleh Orbit-Singularity Series* of $f$. If the process stops after $K$ exact singular extractions, we call the resulting finite expansion the *finite AGOSS decomposition* of $f$.

The next sections show that on the class of Definition 2.2, the procedure is exact.

## 3. Coefficient asymptotics for a single atom

The basic bridge between singular atoms and the coefficient orbit is classical. We record the precise special case needed for the present paper.

**Proposition 3.1** (Coefficient asymptotics for a basic atom)**.** *Let $\mathcal{A}_{\rho,\alpha,\mu}$ be the atom (2.1). Then, as $n\to\infty$,*

$$[z^n]\, \mathcal{A}_{\rho,\alpha,\mu} = \frac{\rho^{-n} n^{\alpha-1}}{\Gamma(\alpha)} \left[(\log n)^\mu + O\big((\log n)^{\mu-1}\big) + O\Big(\frac{(\log n)^\mu}{n}\Big)\right]. \tag{3.1}$$

*Equivalently,*

$$[z^n]\, \mathcal{A}_{\rho,\alpha,\mu} = \frac{\rho^{-n} n^{\alpha-1} (\log n)^\mu}{\Gamma(\alpha)} \left(1 + O\Big(\frac{1}{\log n}\Big) + O\Big(\frac{1}{n}\Big)\right). \tag{3.2}$$

*Proof.* For $\mu = 0$ we have the exact binomial coefficient formula

$$[z^n]\,(1 - z/\rho)^{-\alpha} = \rho^{-n}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}.$$

Stirling's formula gives

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+1)} = n^{\alpha-1}\left(1 + O\Big(\frac{1}{n}\Big)\right),$$

so (3.1) follows in the case $\mu = 0$.

For general $\mu$, identity (2.2) implies

$$[z^n]\,\mathcal{A}_{\rho,\alpha,\mu} = \partial_\alpha^\mu\left(\rho^{-n}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}\right).$$

Since the Stirling expansion above is locally uniform in $\alpha$ on compact sets avoiding $\{0, -1, -2, \dots\}$, differentiation with respect to $\alpha$ may be carried out termwise. Differentiating $n^{\alpha-1}/\Gamma(\alpha)$ exactly $\mu$ times produces a leading contribution $n^{\alpha-1}(\log n)^\mu/\Gamma(\alpha)$; all other Leibniz terms carry at most $(\log n)^{\mu-1}$, and differentiation of the $O(1/n)$ correction contributes only $O((\log n)^\mu/n)$. This yields (3.1); (3.2) is immediate. □

**Lemma 3.2** (Ratio jet of a basic atom)**.** *Let*

$$c_n := [z^n]\,\mathcal{A}_{\rho,\alpha,\mu}.$$

*Then*

$$\rho\,\frac{c_n}{c_{n-1}} = 1 + \frac{\alpha-1}{n} + \frac{\mu}{n\log n} + O\Big(\frac{1}{n\log^2 n}\Big) + O\Big(\frac{1}{n^2}\Big). \tag{3.3}$$

*Consequently,*

$$\lim_{n\to\infty}\frac{c_{n-1}}{c_n} = \rho, \tag{3.4}$$

$$\lim_{n\to\infty} n\left(\rho\frac{c_n}{c_{n-1}} - 1\right) = \alpha - 1, \tag{3.5}$$

$$\lim_{n\to\infty}\log n\left[n\left(\rho\frac{c_n}{c_{n-1}} - 1\right) - (\alpha-1)\right] = \mu. \tag{3.6}$$

*Proof.* By Proposition 3.1,

$$c_n = K\rho^{-n}n^\beta(\log n)^\mu\left(1 + O\Big(\frac{1}{\log n}\Big) + O\Big(\frac{1}{n}\Big)\right), \qquad \beta := \alpha - 1,$$

with $K = 1/\Gamma(\alpha) \neq 0$. Therefore

$$\frac{c_n}{c_{n-1}} = \rho^{-1}\left(\frac{n}{n-1}\right)^\beta\left(\frac{\log n}{\log(n-1)}\right)^\mu\left(1 + O\Big(\frac{1}{n\log n}\Big) + O\Big(\frac{1}{n^2}\Big)\right).$$

Now

$$\left(\frac{n}{n-1}\right)^\beta = 1 + \frac{\beta}{n} + O\Big(\frac{1}{n^2}\Big), \qquad \left(\frac{\log n}{\log(n-1)}\right)^\mu = 1 + \frac{\mu}{n\log n} + O\Big(\frac{1}{n\log^2 n}\Big) + O\Big(\frac{1}{n^2}\Big).$$

Multiplying these expansions gives (3.3), and the limit statements follow immediately. □

## 4. Canonical dominant atom selection

We now pass from one atom to a finite skeleton.

**Theorem 4.1** (Canonical dominant atom selection)**.** *Let $f$ have an admissible finite singular skeleton*

$$f(z) = H(z) + \sum_{j=1}^{K} A_j\mathcal{A}_{\rho_j,\alpha_j,\mu_j}(z)$$

*in the sense of Definition 2.2. Then the dominant fingerprint of $f$ exists and is given by*

$$\rho(f) = \rho_1, \tag{4.1}$$

$$\beta(f) = \alpha_1 - 1, \tag{4.2}$$

$$\mu(f) = \mu_1, \tag{4.3}$$

$$C(f) = \frac{A_1}{\Gamma(\alpha_1)}. \tag{4.4}$$

*Equivalently,*

$$\mathcal{D}(f) = A_1 \mathcal{A}_{\rho_1,\alpha_1,\mu_1}. \tag{4.5}$$

*In particular, the dominant fingerprint canonically and uniquely identifies the dominant atom.*

*Proof.* Let

$$d_n := A_1 [z^n]\, \mathcal{A}_{\rho_1,\alpha_1,\mu_1}, \qquad e_n := [z^n]\, H + \sum_{j=2}^{K} A_j [z^n]\, \mathcal{A}_{\rho_j,\alpha_j,\mu_j},$$

so that $a_n = d_n + e_n$. By Proposition 3.1,

$$d_n = \frac{A_1 \rho_1^{-n} n^{\alpha_1 - 1} (\log n)^{\mu_1}}{\Gamma(\alpha_1)} \left(1 + O\Big(\frac{1}{\log n}\Big) + O\Big(\frac{1}{n}\Big)\right).$$

For each $j \geq 2$, Proposition 3.1 yields

$$[z^n]\, \mathcal{A}_{\rho_j,\alpha_j,\mu_j} = O\Big(|\rho_j|^{-n} n^{M_j} (\log n)^{\mu_j}\Big),$$

for some real $M_j$, while $[z^n]\, H = O(R_H^{-n})$. Because $|\rho_1| < |\rho_j|$ for $j \geq 2$ and $|\rho_1| < R_H$, there exists $q \in (0, 1)$ such that

$$\frac{e_n}{d_n} = O(q^n)$$

as $n \to \infty$. Hence $a_n = d_n(1 + o(1))$ and also $a_{n-1} = d_{n-1}(1 + o(1))$. Therefore

$$\frac{a_n}{a_{n-1}} = \frac{d_n}{d_{n-1}}(1 + o(1)).$$

Applying Lemma 3.2 to $d_n$ gives (4.1)–(4.3). Finally,

$$a_n \rho_1^n n^{1-\alpha_1} (\log n)^{-\mu_1} = \frac{A_1}{\Gamma(\alpha_1)}(1 + o(1)),$$

which gives (4.4). Formula (4.5) follows from the definition of $\mathcal{D}(f)$.

Uniqueness is immediate: if two admissible atoms had the same quadruple $(\rho, \beta, \mu, C)$, then they would have the same $(\rho, \alpha, \mu, A)$ because $\alpha = \beta + 1$ and $A = \Gamma(\alpha) C$. □

## 5. Finite singular skeleton recovery

The main theorem now follows by induction.

**Theorem 5.1** (Finite skeleton recovery)**.** *Let $f$ have an admissible finite singular skeleton*

$$f(z) = H(z) + \sum_{j=1}^{K} A_j \mathcal{A}_{\rho_j,\alpha_j,\mu_j}(z)$$

*with strict radial separation (2.5). Then the AGOSS peeling procedure of Definition 2.3 recovers the singular skeleton exactly in finitely many steps. More precisely, for $m = 0, 1, \ldots, K-1$,*

$$\theta_m^\star = (\rho_{m+1}, \alpha_{m+1}, \mu_{m+1}), \qquad A_m^\star = A_{m+1}, \tag{5.1}$$

*and the residuals satisfy*

$$R_m(z) = H(z) + \sum_{j=m+1}^{K} A_j \mathcal{A}_{\rho_j,\alpha_j,\mu_j}(z). \tag{5.2}$$

*In particular,*

$$R_K = H. \tag{5.3}$$

*Proof.* We proceed by induction on $m$. For $m = 0$, Theorem 4.1 gives

$$\mathcal{D}(R_0) = \mathcal{D}(f) = A_1 \mathcal{A}_{\rho_1,\alpha_1,\mu_1},$$

so $A_0^\star = A_1$ and $\theta_0^\star = (\rho_1, \alpha_1, \mu_1)$. Hence

$$R_1 = f - A_1 \mathcal{A}_{\rho_1,\alpha_1,\mu_1} = H + \sum_{j=2}^{K} A_j \mathcal{A}_{\rho_j,\alpha_j,\mu_j},$$

which is exactly (5.2) for $m = 1$.

Assume now that (5.2) holds for some $m < K$. Then $R_m$ again has an admissible finite singular skeleton, namely

$$R_m(z) = H(z) + \sum_{j=m+1}^{K} A_j \mathcal{A}_{\rho_j,\alpha_j,\mu_j}(z),$$

with dominant atom $A_{m+1}\mathcal{A}_{\rho_{m+1},\alpha_{m+1},\mu_{m+1}}$. Applying Theorem 4.1 to $R_m$ yields

$$\mathcal{D}(R_m) = A_{m+1}\mathcal{A}_{\rho_{m+1},\alpha_{m+1},\mu_{m+1}},$$

so (5.1) holds at stage $m$, and subtracting this atom gives

$$R_{m+1} = R_m - A_{m+1}\mathcal{A}_{\rho_{m+1},\alpha_{m+1},\mu_{m+1}} = H + \sum_{j=m+2}^{K} A_j \mathcal{A}_{\rho_j,\alpha_j,\mu_j}.$$

Thus the induction closes, and after $K$ steps only $H$ remains. □

## 6. Residual radius lifting

The preceding theorem has a sharp analytic consequence.

**Theorem 6.1** (Residual radius lifting)**.** *Under the hypotheses of Theorem 5.1, the residual after $s$ successful AGOSS extractions has radius of convergence*

$$\operatorname{rad}(R_s) = \begin{cases} |\rho_{s+1}|, & 0 \le s \le K-1, \\ \ge R_H, & s = K. \end{cases} \tag{6.1}$$

*In particular,*

$$\operatorname{rad}(f) = |\rho_1| < |\rho_2| < \cdots < |\rho_K| \le \operatorname{rad}(R_{K-1}),$$

*and each successful extraction strictly lifts the radius of analyticity of the residual.*

*Proof.* By Theorem 5.1,

$$R_s(z) = H(z) + \sum_{j=s+1}^{K} A_j \mathcal{A}_{\rho_j,\alpha_j,\mu_j}(z).$$

Each atom $\mathcal{A}_{\rho_j,\alpha_j,\mu_j}$ has radius of convergence exactly $|\rho_j|$, and $H$ is holomorphic in $|z| < R_H$. If $s \le K-1$, then the nearest singularity in the representation above occurs at $\rho_{s+1}$, because of strict radial separation. Since the coefficient $A_{s+1}$ is nonzero, the singularity at $\rho_{s+1}$ is present and cannot disappear. Therefore $\operatorname{rad}(R_s) = |\rho_{s+1}|$. For $s = K$, one has $R_K = H$, so $\operatorname{rad}(R_K) \ge R_H$. □

**Remark 6.2.** Theorem 6.1 is the structural reason the new series differs from direct Taylor expansion. The algorithm does not merely fit a better local approximant on $|z| < |\rho_1|$; it removes the obstruction that fixed the original radius of convergence and thereby exposes a strictly larger analytic domain for the residual.

## 7. Superiority over direct Taylor truncation

Define the degree-$N$ Taylor polynomial of a germ $g(z) = \sum_{n\geq 0} b_n z^n$ by

$$P_N(g)(z) := \sum_{n=0}^{N} b_n z^n.$$

For $r > 0$, write

$$\|g\|_r := \sup_{|z|\leq r} |g(z)|.$$

We first record the exact root-rate of direct Taylor truncation.

**Theorem 7.1** (Taylor tail rate)**.** *Let $g$ be holomorphic at the origin with radius of convergence $R \in (0, \infty]$. Then for every $0 < r < R$,*

$$\limsup_{N\to\infty} \|g - P_N(g)\|_r^{1/N} = \frac{r}{R}. \tag{7.1}$$

*When $R = \infty$, the right-hand side is understood as $0$.*

*Proof.* Write $g(z) = \sum_{n\geq 0} b_n z^n$. Fix $r < R$ and choose any $\widetilde{R}$ with $r < \widetilde{R} < R$. By Cauchy's estimate,

$$|b_n| \leq M_{\widetilde{R}} \widetilde{R}^{-n}, \qquad M_{\widetilde{R}} := \sup_{|z|=\widetilde{R}} |g(z)|.$$

Hence

$$\|g - P_N(g)\|_r \leq M_{\widetilde{R}} \sum_{n>N} (r/\widetilde{R})^n \leq \frac{M_{\widetilde{R}}}{1 - r/\widetilde{R}} (r/\widetilde{R})^{N+1}.$$

Taking $N$-th roots and then letting $\widetilde{R} \uparrow R$ yields

$$\limsup_{N\to\infty} \|g - P_N(g)\|_r^{1/N} \leq r/R.$$

For the reverse inequality, since

$$g - P_N(g) = \sum_{n>N} b_n z^n,$$

Cauchy's coefficient estimate on the disk $|z| \leq r$ gives

$$|b_{N+1}| r^{N+1} \leq \|g - P_N(g)\|_r .$$

Taking $N$-th roots and using Cauchy–Hadamard,

$$\limsup_{N\to\infty} \|g - P_N(g)\|_r^{1/N} \geq r \limsup_{N\to\infty} |b_{N+1}|^{1/N} = r/R.$$

Thus equality holds. □

We now apply this to AGOSS.

**Theorem 7.2** (Strict exponential improvement over Taylor truncation)**.** *Let $f$ have an admissible finite singular skeleton as in Definition 2.2, and let $0 < r < |\rho_1| = \mathrm{rad}(f)$. For $0 \leq s \leq K$, define the AGOSS–Taylor approximant*

$$\mathcal{T}_{s,N}(f) := \sum_{j=1}^{s} A_j \mathcal{A}_{\rho_j,\alpha_j,\mu_j} + P_N(R_s), \tag{7.2}$$

*with the convention that the singular sum is empty when $s = 0$. Then:*

*(a) direct Taylor truncation satisfies*

$$\limsup_{N\to\infty} \|f - P_N(f)\|_r^{1/N} = \frac{r}{|\rho_1|}; \tag{7.3}$$

*(b) for each $s \in \{0, 1, \ldots, K-1\}$,*

$$\limsup_{N\to\infty} \|f - \mathcal{T}_{s,N}(f)\|_r^{1/N} = \limsup_{N\to\infty} \|R_s - P_N(R_s)\|_r^{1/N} = \frac{r}{|\rho_{s+1}|}; \tag{7.4}$$

*(c) after full skeleton recovery,*

$$\limsup_{N\to\infty} \|f - \mathcal{T}_{K,N}(f)\|_r^{1/N} = \limsup_{N\to\infty} \|H - P_N(H)\|_r^{1/N} \le \frac{r}{R_H} < \frac{r}{|\rho_1|}. \tag{7.5}$$

*Consequently, every successful AGOSS extraction strictly improves the exponential convergence rate, and complete skeleton recovery improves it from $r/|\rho_1|$ to at most $r/R_H$.*

*Proof.* Part (a) is Theorem 7.1 with $R = \mathrm{rad}(f) = |\rho_1|$.

For part (b), note that by construction of $\mathcal{T}_{s,N}(f)$,

$$f - \mathcal{T}_{s,N}(f) = R_s - P_N(R_s).$$

By Theorem 6.1, $\mathrm{rad}(R_s) = |\rho_{s+1}|$ for $s \le K-1$. Applying Theorem 7.1 to $R_s$ gives (7.4).

For part (c), one again has

$$f - \mathcal{T}_{K,N}(f) = R_K - P_N(R_K) = H - P_N(H),$$

and Theorem 6.1 gives $\mathrm{rad}(H) = \mathrm{rad}(R_K) \ge R_H$. Theorem 7.1 then yields

$$\limsup_{N\to\infty} \|H - P_N(H)\|_r^{1/N} \le r/R_H,$$

which is strictly smaller than $r/|\rho_1|$ because $R_H > |\rho_1|$. $\square$

**Corollary 7.3** (Strict rate improvement at every stage)**.** *For $1 \le s \le K$ and every $0 < r < |\rho_1|$,*

$$\limsup_{N\to\infty} \|f - \mathcal{T}_{s,N}(f)\|_r^{1/N} < \limsup_{N\to\infty} \|f - \mathcal{T}_{s-1,N}(f)\|_r^{1/N}.$$

*Proof.* This is immediate from (7.4), (7.5), and the strict chain

$$|\rho_1| < |\rho_2| < \cdots < |\rho_K| < R_H.$$

$\square$

## 8. Stability and the inverse orbit-to-skeleton principle

The next theorem formalizes the fact that the dominant fingerprint depends only on the asymptotic ratio jet, not on asymptotically negligible perturbations thereof.

**Theorem 8.1** (Stability of the dominant fingerprint)**.** *Let $(a_n)_{n\ge0}$ be a coefficient sequence whose dominant fingerprint exists, and let $(\widetilde{a}_n)_{n\ge0}$ satisfy*

$$\widetilde{a}_n = a_n(1+\varepsilon_n), \qquad \varepsilon_n \to 0, \tag{8.1}$$

*and*

$$\varepsilon_n - \varepsilon_{n-1} = o\left(\frac{1}{n\log n}\right). \tag{8.2}$$

*Then the dominant fingerprints of $(a_n)$ and $(\widetilde{a}_n)$ coincide:*

$$\rho(\widetilde{a}) = \rho(a), \qquad \beta(\widetilde{a}) = \beta(a), \qquad \mu(\widetilde{a}) = \mu(a), \qquad C(\widetilde{a}) = C(a).$$

*In particular, the extracted dominant atom is unchanged.*

*Proof.* Let $r_n = a_n/a_{n-1}$ and $\widetilde{r}_n = \widetilde{a}_n/\widetilde{a}_{n-1}$. Then

$$\widetilde{r}_n = r_n\frac{1+\varepsilon_n}{1+\varepsilon_{n-1}} = r_n(1+\delta_n), \qquad \delta_n = o\left(\frac{1}{n\log n}\right),$$

because (8.1) and (8.2) imply $(1+\varepsilon_n)/(1+\varepsilon_{n-1}) = 1 + o((n\log n)^{-1})$. Therefore $\widetilde{r}_n - r_n \to 0$, so the first fingerprint limit is unchanged: $\rho(\widetilde{a}) = \rho(a) =: \rho$.

Next,

$$\rho\widetilde{r}_n - 1 = (\rho r_n - 1) + \rho r_n\delta_n.$$

Since $\rho r_n \to 1$ and $n\delta_n \to 0$, multiplying by $n$ gives

$$n(\rho\widetilde{r}_n - 1) - n(\rho r_n - 1) \to 0,$$

so $\beta(\widetilde{a}) = \beta(a)$. Multiplying the same identity by $n \log n$ and using $n \log n\, \delta_n \to 0$ gives $\mu(\widetilde{a}) = \mu(a)$. Finally,

$$\widetilde{a}_n \rho^n n^{-\beta} (\log n)^{-\mu} = a_n \rho^n n^{-\beta} (\log n)^{-\mu} (1 + \varepsilon_n),$$

and since $\varepsilon_n \to 0$, the amplitude limit is also unchanged. □

The previous theorem becomes a recovery theorem when applied recursively to the AGOSS residuals.

**Corollary 8.2** (Stability of finite skeleton recovery)**.** *Let $f$ have an admissible finite singular skeleton of length $K$, and let $\widetilde{f}$ be another germ. For each $m = 0, 1, \ldots, K-1$, let $R_m$ be the exact AGOSS residual of $f$ and let $\widetilde{R}_m$ be the residual obtained from $\widetilde{f}$ after subtracting the first $m$ atoms recovered from $f$. Suppose that the coefficient sequences of $R_m$ and $\widetilde{R}_m$ satisfy*

$$[z^n]\, \widetilde{R}_m = [z^n]\, R_m (1 + \varepsilon_n^{(m)}), \qquad \varepsilon_n^{(m)} \to 0, \qquad \varepsilon_n^{(m)} - \varepsilon_{n-1}^{(m)} = o\Big(\frac{1}{n \log n}\Big).$$

*Then AGOSS recovers from $\widetilde{f}$ the same first $K$ atoms as from $f$.*

*Proof.* Apply Theorem 8.1 to $R_0 = f$ and $\widetilde{R}_0$. The first recovered atoms coincide. Subtracting that common atom from both series gives $R_1$ and $\widetilde{R}_1$, to which the same theorem applies again. Iterate $K$ times. □

We now package the preceding statement as the inverse principle promised in the introduction.

**Theorem 8.3** (Inverse orbit-to-skeleton principle)**.** *Let $g$ have an admissible finite singular skeleton of length $K$. Let $f$ be any germ at the origin such that, for each $m = 0, 1, \ldots, K-1$, the coefficient sequence of the $m$th residual of $f$ is asymptotically orbit-equivalent to that of the $m$th residual of $g$ in the sense of Corollary 8.2. Then the first $K$ AGOSS atoms of $f$ and $g$ coincide. Equivalently, $f$ and $g$ have the same recovered finite singular skeleton up to level $K$.*

*Proof.* This is precisely Corollary 8.2 applied with $g$ in place of $f$. □

## 9. Worked example

**Example 9.1** (A three-level skeleton)**.** Consider

$$f(z) = \frac{3}{(1-z)^{1/2}} + 2\,\frac{\log(1/(1-z/2))}{(1-z/2)^{3/2}} + \frac{1}{1-z/3}. \tag{9.1}$$

This is of the form (2.4) with $H \equiv 0$ and parameters

$$(\rho_1, \alpha_1, \mu_1) = (1, 1/2, 0), \quad (\rho_2, \alpha_2, \mu_2) = (2, 3/2, 1), \quad (\rho_3, \alpha_3, \mu_3) = (3, 1, 0).$$

The coefficient asymptotics are

$$[z^n]\, f = \frac{3}{\Gamma(1/2)} n^{-1/2} + \frac{2 \cdot 2^{-n}}{\Gamma(3/2)} n^{1/2} \log n + 3^{-n} + \text{lower-order terms}.$$

Since $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = \sqrt{\pi}/2$, this becomes

$$[z^n]\, f = \frac{3}{\sqrt{\pi}} n^{-1/2} + \frac{4}{\sqrt{\pi}} 2^{-n} n^{1/2} \log n + 3^{-n} + \cdots .$$

Thus AGOSS reads the dominant ratio jet as $(\rho, \alpha, \mu) = (1, 1/2, 0)$ and recovers the atom $3(1-z)^{-1/2}$. The first residual is

$$R_1(z) = 2\,\frac{\log(1/(1-z/2))}{(1-z/2)^{3/2}} + \frac{1}{1-z/3},$$

so $\operatorname{rad}(R_1) = 2$. The second AGOSS step recovers the $(2, 3/2, 1)$ atom, leaving

$$R_2(z) = \frac{1}{1-z/3}, \qquad \operatorname{rad}(R_2) = 3.$$

Finally $R_3 \equiv 0$.

Fix $0 < r < 1$. Direct Taylor truncation satisfies

$$\limsup_{N\to\infty} \|f - P_N(f)\|_r^{1/N} = r,$$

while the first AGOSS–Taylor approximant satisfies

$$\limsup_{N\to\infty} \|f - \mathcal{T}_{1,N}(f)\|_r^{1/N} = r/2,$$

and the second satisfies

$$\limsup_{N\to\infty} \|f - \mathcal{T}_{2,N}(f)\|_r^{1/N} = r/3.$$

This is exactly the radius-lifting mechanism proved above.

## 10. Concluding remarks and the next papers

The first paper in the series establishes a full theorem package for a rigid but meaningful class: finite algebraic-logarithmic skeletons with strict radial separation. Within this class AGOSS is not merely heuristic. It is exact. The singular atoms are canonically selected, the whole finite skeleton is recovered, the residual radius is lifted, and the resulting approximation rate is provably superior to direct Taylor truncation.

This leaves three natural directions.

(1) **Same-point confluent chains.** Identity (2.3) shows that simple confluence is already nearby. The issue is not representation but automated extraction of several descendants with the same radius and the same center.
(2) **Several singular points on the same dominant circle.** Once $|\rho_i| = |\rho_j|$ with $\rho_i \neq \rho_j$, the coefficient tail becomes a finite equal-modulus exponential sum and simple ratio limits no longer suffice. This calls for a block extractor of Prony/Fourier type adapted to singular atoms.
(3) **Several variables.** In more than one variable one expects local model atoms supported on polycircular or algebraic singular strata, together with orbit data extracted along multi-index directions.

Those directions belong naturally to subsequent papers:

*The Abu-Ghuwaleh Orbit-Singularity Series II: Confluent Atoms and Mixed-Scale Recovery*,

followed by

*The Abu-Ghuwaleh Orbit-Singularity Series III: Multi-Point Interference and Multivariate Extensions*.

The present paper is therefore meant as a foundation: it isolates the first recoverable regime and proves the core mechanism in a fully closed form.

## References

[1] E. A. Bender, *Asymptotic methods in enumeration*, SIAM Review **16** (1974), 485–515.
[2] P. Flajolet and A. M. Odlyzko, *Singularity analysis of generating functions*, SIAM Journal on Discrete Mathematics **3** (1990), 216–240.
[3] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

Department of Mathematics, Faculty of Science, Zarqa University, Zarqa 13110, Jordan
*Email address*: 20209030@zu.edu.jo