Parametric Symmetry in the Cubic Prouhet–Tarry–Escott Problem: A Non-Ideal Template and Infinite Arithmetic Families

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Abstract

This paper studies a computationally generated *infinite* class of cubic-only ($r = 3$) Prouhet–Tarry–Escott (PTE-type) identities with balanced size $N \times N$, where each side is a multiset of quadratic expressions and the sums of cubes coincide while lower and higher moments generally do not. The frequently cited 4×4 instance is treated as the minimal canonical demonstrator, not as the boundary of the method: the same telescopic pipeline produces much larger balanced decompositions and corresponding parametric candidates. **Version 4** separated the polynomial template layer in $\mathbb{Q}[x]$ from its *infinite* arithmetic realizations under $x = 10^b$, and documented a software-aligned three-probe interpolation proof skeleton. **Version 5** added Subsection 4.3 with explicit admissibility rules for decimal templates b_0 driving those probes. **Version 6** adds a second reconstruction schema directly over b_0: besides uniform-middle masks (0/9), one may fix edge digits (n_1, n_2) and a repeated middle digit $d \in \{1, \ldots, 8\}$, i.e. $n_1 + d \ldots d + n_2$. This expands exploratory applicability by introducing controlled decimal subfamilies where one recovers quadratic slots in b_0, verifies an invariant polynomial $P(b_0) = \sum S_1^3 = \sum S_2^3$, and compares predicted/observed decompositions from three probes. We provide explicit formulas, non-ideality diagnostics for $r = 1, 2, 4$, numerical examples, and a clear separation between formal polynomial identities and their substitutions.

1 Introduction

The Prouhet–Tarry–Escott (PTE) problem is a classic question in Diophantine analysis. One seeks two distinct multisets of integers $\{a_{1,1}, \ldots, a_{1,n}\}$ and $\{a_{2,1}, \ldots, a_{2,n}\}$ such that

$$\sum_{i=1}^{n} a_{1,i}^r = \sum_{i=1}^{n} a_{2,i}^r, \quad \text{for } r = 1, 2, \ldots, m. \tag{1}$$

Solutions with $m = n - 1$ are called *ideal* and are especially prized. Many classical constructions are symmetric or tied to the Prouhet–Thue–Morse idea for long runs of consecutive powers.

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Scope of this note. The core object is an *infinite* computational family of balanced cubic identities, naturally viewed as $N \times N$ templates generated by the telescopic pipeline. The explicit 4×4 polynomial block shown in §3 is the smallest canonical demonstrator used to expose formulas and verification steps; it is not the endpoint of the search space. Throughout, the focus remains cubic-only (non-ideal): first and second power sums differ, and fourth powers differ as well, so no ideal PTE extension follows without changing the multisets.

"Isolated" versus infinite arithmetic families. The word *isolated* is sometimes read as "there are only finitely many integer solutions". That reading is *not* intended here. Once the eight quadratics are fixed as in §3, substituting $x = 10^b$ for $b = 1, 2, 3, \ldots$ produces **infinitely many** distinct pairs of integer multisets $(S_1|_{x=10^b}, S_2|_{x=10^b})$ satisfying the same cubic equality—all coming from one polynomial template in $\mathbb{Q}[x]$. The isolation is from *ideal* PTE extensions in the moment order r, not from cardinality in $\mathbb{Z}$. Section 4 states this structure explicitly (parameters, proof skeleton, GUI link).

2 Computational origin

The author experimented with a GUI tool for Ramanujan-type telescopic decompositions of cubes (`decompose_gui.py`, building on the pipeline in the project `clear_result`). For the parameter triple

$$a = 3, \qquad b_0 = \underbrace{99\ldots9}_{n \text{ digits}} = 10^n - 1, \qquad k = 3,$$

the program repeatedly produced balanced $N \times N$ cube identities (with 4+4 as the first transparent case), where bases share a stable "skeleton" of significant digits separated by blocks of nines and zeros as n grows. Searching for closed forms led to the substitution $x = 10^b$ with integer $b = n$ (so $x = 10^n$) and then to treating x as an indeterminate: the expressions below are *quadratic polynomials in x*. Their cubes sum to the same degree-6 polynomial for all x in any commutative ring (in particular for every integer x, including astronomically large ones).

3 The parametric system

Let x be an indeterminate (later one may substitute $x = 10^b$ for integer b, or more generally $x = f(u)$). Write the coefficients in rational form for exact arithmetic:

$$\tfrac{3}{2} = 1.5, \quad \tfrac{9}{2} = 4.5, \quad \tfrac{261}{2} = 130.5, \text{ etc.}$$

Define two multisets $S_1 = \{a_1, b_1, c_1, d_1\}$ and $S_2 = \{a_2, b_2, c_2, d_2\}$.

3.1 Set S_1

$$\begin{aligned}
a_1 &= \tfrac{3}{2}x^2 - \tfrac{3}{2}x - 364 \\
b_1 &= \tfrac{3}{2}x^2 + \tfrac{3}{2}x - 364 \\
c_1 &= \tfrac{3}{2}x^2 + \tfrac{9}{2}x - 361 \\
d_1 &= \tfrac{9}{2}x^2 - \tfrac{261}{2}x + 1218
\end{aligned}$$

3.2 Set S_2

$$\begin{aligned} a_2 &= \tfrac{3}{2}x^2 - \tfrac{165}{2}x + 770 \\ b_2 &= \tfrac{3}{2}x^2 - \tfrac{159}{2}x + 689 \\ c_2 &= \tfrac{3}{2}x^2 - \tfrac{153}{2}x + 611 \\ d_2 &= \tfrac{9}{2}x^2 - \tfrac{207}{2}x + 867 \end{aligned}$$

4 Families of (S_1, S_2): parameters, a proof skeleton, and consequences

This section records the version 4–6 upgrade: we separate the analytic object in $\mathbb{Q}[x]$ from arithmetic realizations and from the experimental reconstruction pipeline; versions 5–6 spell out admissible b_0 masks for that pipeline.

4.1 Problem setting (recall)

For the canonical demonstrator, fix two multisets of four quadratic polynomials in a formal variable x, written $S_1 = \{p_1, \ldots, p_4\}$ and $S_2 = \{q_1, \ldots, q_4\}$. The *cubic-only* target is

$$\sum_{i=1}^{4} p_i(x)^3 = \sum_{i=1}^{4} q_i(x)^3 \quad \text{in } \mathbb{Q}[x], \tag{2}$$

while the usual PTE constraints for $n = 4$ fail already at $r = 1$ and $r = 2$, and also at $r = 4$ for the template in §3 (see §5). Thus (2) is **non-ideal** in the classical PTE sense. In software experiments this block acts as a base chart; larger balanced $N \times N$ decompositions are then treated by the same reconstruction logic.

4.2 Parameters indexing the family

Three (related) layers of parameters should be kept apart.

1. **The formal variable x.** The eight expressions in §3 lie in $\mathbb{Q}[x]$ and have degree at most 2. The cubic identity (4) is an equality of polynomials: it remains true after replacing x by any element of any commutative $\mathbb{Q}$-algebra.

2. **Arithmetic scales $b \in \mathbb{Z}_{\geq 1}$.** Set $x = 10^b$. For each b, evaluation yields two multisets of *integers* with identical sums of cubes. As b grows, the eight values are pairwise distinct and increase without bound, so **infinitely many** different integer pairs $(S_1|_{x=10^b}, S_2|_{x=10^b})$ arise from the *same* quadratic template. This is the precise sense in which "isolated cubic" is used in the title: isolated from ideal extensions in moment order r, not isolated as a finite solution set in $\mathbb{Z}$.

3. **Telescopic reconstruction scales (n_1, n_2, n_3) and (a, k).** In the author's `decompose_gui` tools (including the extended build with parametric analysis), each quadratic base is recovered by Lagrange interpolation from three integer samples of the telescopic output, taken at $x \in \{10^{n_1}, 10^{n_2}, 10^{n_3}\}$ for fixed (a, k) and a compatible starting template b_0 (classically all-nines $10^n - 1$; expanded mask rules are in Subsection 4.3).

The default triple $(n_1, n_2, n_3) = (7, 6, 5)$ matches the implementation for scheme 1; scheme 2 keeps the same three-point principle but interpolates directly in b_0 across three neighboring masks inside a fixed digit family. When term counts and side ordering stabilize across probes, there is at most one interpolant of degree ≤ 2 per position; agreement on an independent check (e.g. the middle probe) supports the candidate closed form, which must still be verified symbolically (see §5).

Beyond the primary example $(a, k) = (3, 3)$, the same reconstruction can be applied for other positive integers a, k. The companion script `scan_parametric_pte.py` scans pairs (a, k) on the same three scales and records cases where all interpolants are quadratic and the cubic sum identity holds in $\mathbb{Q}[x]$. The lattice $\mathbb{Z}^2_{>0}$ is infinite, and empirically many pairs yield nontrivial hits; version 4 treats these as further *templates* of the same logical type, each again producing infinitely many integer evaluations $x = 10^b$ once the quadratics are fixed.

4.3 Which decimal templates b_0 define the three probes?

(Software-aligned; written for readers who do not read source code.) The extended interface `decompose_gui_EXT` (a Windows build can be downloaded from the author's website: `https://nvvorobtsov.github.io/sources/decompose_gui_EXT_windows.zip`) and the batch tool `scan_parametric_pte.py` always interpolate each quadratic slot through *three* integer telescopic runs. Conceptually the interpolation nodes are $x \in \{10^7, 10^6, 10^5\}$; the matching starting values B_7, B_6, B_5 are positive integers assembled from a user-chosen template b_0.

(A) All-nines templates. If every decimal digit of b_0 equals 9 (for instance 9, 99, 999, 9999), the program uses the classical triple

$$B_7 = 10^7 - 1, \qquad B_6 = 10^6 - 1, \qquad B_5 = 10^5 - 1.$$

Thus the familiar experiments with $b_0 = 10^n - 1$ sit inside a slightly larger class of "pure nine" inputs.

(B) Uniform-middle templates. Otherwise write b_0 in base 10 without leading zeros and assume it has at least three digits. Let f be the first digit, ℓ the last digit, and let the substring strictly between them be non-empty and consist of a single repeated digit $c \in \{0, 9\}$ (only zeros or only nines). For each total width $T \in \{7, 6, 5\}$ let B_T be the T-digit integer whose first digit is f, whose last digit is ℓ, and whose $T - 2$ middle digits all equal c. Equivalently

$$B_T \;=\; f\,10^{T-1} \;+\; c\,\frac{10^{T-1} - 10}{9} \;+\; \ell, \tag{3}$$

because $\frac{10^{T-1}-10}{9} = 10 + 10^2 + \cdots + 10^{T-2}$ is the place-value sum of a run of $T - 2$ copies of the digit 1; multiplying by $c \in \{0, 9\}$ inserts the requested middle block. (When $c = 0$ the fraction is still valid and the middle contribution vanishes.)

Worked examples (decode slowly).

- $b_0 = 192$**.** The middle digit is 9, so $c = 9$, while $f = 1$ and $\ell = 2$. Equation (3) yields $(B_7, B_6, B_5) = (1999992, 199992, 19992)$. Each number is "1, then a block of nines, then 2", shortened as the width drops from seven to five.

- $b_0 = 909$**.** Here $f = \ell = 9$ and the middle digit is 0, so $c = 0$. The triple is $(9000009, 900009, 90009)$: nines on the outside, a growing block of zeros in the centre.

- $b_0 = 2995$**.** The middle digits are "99", hence $c = 9$, $f = 2$, $\ell = 5$, giving $(2999995, 299995, 29995)$.

- **Equivalent templates.** Longer numbers such as 1001 share the same $(f, c, \ell) = (1, 0, 1)$ as 101, so they generate the *identical* probe triple; only the extremal digits and the uniform middle digit c matter for (3), not the length of the middle run in the template.

(C) Fixed-edge repeated-middle families (version 6, scheme 2). For exploratory runs where one wants to keep the two edge digits fixed and vary only the middle-run length, the extended GUI supports masks

$$b_0 = n_1 \underbrace{dd \cdots d}_{m \text{ copies}} n_2, \qquad n_1 \in \{1, \dots, 9\},\ d \in \{1, \dots, 8\},\ n_2 \in \{0, \dots, 9\},\ m \geq 1.$$

Thus, for example, $132, 1332, 13332, \dots$ and $75557, 755557, \dots$ belong to scheme 2, while middle digits 0 and 9 remain assigned to scheme 1. The three interpolation probes are chosen as neighboring middle lengths $(m-1, m, m+1)$ (or $(1, 2, 3)$ when $m = 1$), with the same fixed (n_1, d, n_2). In this mode one interpolates each slot as a quadratic in b_0 (not in x), then verifies

$$P(b_0) = \sum_{S_1} p(b_0)^3 = \sum_{S_2} q(b_0)^3.$$

Practically, this adds a second controlled experimental axis: instead of changing decimal scale only, one can scan digit-structured families where the middle-run length is the continuation parameter. This is useful for detecting stable slot correspondences, testing robustness of non-ideality diagnostics, and extracting explicit formulas for new (a, k) hits.

What the software rejects. In scheme 1, one- or two-digit numbers that are not all nines, or templates whose inner digits mix 0 and 9 (for example 10101), fail the uniform-middle test. In scheme 2, templates fail when the middle is not constant, or when the repeated middle digit is outside $\{1, \dots, 8\}$.

Batch exploration. `scan_parametric_pte.py` accepts comma-separated templates, bracketed lists, and compact families such as `n0|9m` that enumerate 180 three-digit masks obeying rule (B); each expands to concrete (B_7, B_6, B_5) via the same formula. In version 6 experiments one can also scan scheme 2 families by fixing (n_1, d, n_2) and varying middle length m.

4.4 Proof skeleton

A reusable argument pattern (independent of the specific coefficients in §3) runs as follows.

1. **Interpolation.** Suppose a deterministic telescopic routine outputs, for three exponents $n \in \{n_1, n_2, n_3\}$, integer cube bases that can be aligned index-wise between the two sides. For each position, Lagrange interpolation at the three nodes $10^{n_1}, 10^{n_2}, 10^{n_3}$ determines a unique polynomial of degree ≤ 2 matching the samples.

2. **Closure in $\mathbb{Q}[x]$.** Let S_1, S_2 be the multisets of the eight interpolants. Define

$$D(x) = \sum_{p \in S_1} p(x)^3 - \sum_{p \in S_2} p(x)^3.$$

If $D \equiv 0$, then (2) holds for all x; in particular, infinitely many integer instances follow from $x = 10^b$.

3. **Non-ideality persists.** If $\sum_{S_1} p - \sum_{S_2} p$, $\sum_{S_1} p^2 - \sum_{S_2} p^2$, and $\sum_{S_1} p^4 - \sum_{S_2} p^4$ are nonzero polynomials, no substitution can repair PTE ideality without changing the multisets; §5 exhibits this for the $(3, 3)$ template.

For the primary example, Step 2 is carried out by direct expansion or the SymPy snippet in §5.

4.5 Consequences for "ideality" and for GUI experiments

- **PTE ideal versus cubic-only.** Every arithmetic realization at $x = 10^b$ remains non-ideal: matching $r = 1, 2$ with $n = 4$ would force a longer PTE chain, which the polynomial data rule out uniformly in b.
- **GUI / parametric mode.** The extended interface (`decompose_gui_EXT`) now has two complementary routes. Scheme 1 reconstructs quadratics in x from decimal-scale probes ($10^7, 10^6, 10^5$ style), while scheme 2 reconstructs quadratics in b_0 inside fixed-edge families $n_1 + d \dots d + n_2$. In both routes the tool prints interpolants, an invariant polynomial, and low-moment diagnostics, so users can compare predicted and freshly computed decompositions. This is the experimental counterpart of the interpolation–closure steps above.
- **Outlook.** Classifying closed forms for all empirical (a, k) hits, and proving that the telescopic output at $(3, 10^n - 1, 3)$ always matches the interpolant family, are natural continuations.

5 Algebraic verification and depth analysis

Expanding $\sum_{p \in S_1} p^3$ and $\sum_{p \in S_2} p^3$ as polynomials in x, all coefficients agree. Equivalently,

$$\sum_{p \in S_1} p^3 \;=\; \sum_{p \in S_2} p^3 \tag{4}$$

holds identically in x. Both sides equal

$$P(x) = 101.25x^6 - 7897.5x^5 + 296662.5x^4 - 6528600x^3 \\ + 84008171.25x^2 - 579040312.5x + 1663429263.$$

(The same $P(x)$ can be written with rational coefficients; e.g. $101.25 = \frac{405}{4}$.)

5.1 Non-ideality

For power sums $\sum_{p \in S_j} p^r$, symbolic expansion gives:

- $r = 1$: the difference of the two linear sums is $216x - 2808$ (not identically zero).
- $r = 2$: the difference is a nonzero cubic polynomial in x.
- $r = 4$: the difference is a nonzero polynomial of degree 7 in x (and similarly for higher even/odd r).

Thus (4) is not a consequence of matching first and second moments; it is a genuine "cubic-only" polynomial identity.

5.2 Reproducibility (computer algebra)

The following `SymPy` snippet verifies that the difference of the two cubic sums simplifies to 0:

```
import sympy as sp
x = sp.Symbol("x")
half = lambda n: sp.Rational(n, 2)
S1 = [half(3)*x**2 - half(3)*x - 364,
      half(3)*x**2 + half(3)*x - 364,
      half(3)*x**2 + half(9)*x - 361,
      half(9)*x**2 - half(261)*x + 1218]
S2 = [half(3)*x**2 - half(165)*x + 770,
      half(3)*x**2 - half(159)*x + 689,
      half(3)*x**2 - half(153)*x + 611,
      half(9)*x**2 - half(207)*x + 867]
assert sp.expand(sum(p**3 for p in S1) - sum(p**3 for p in S2)) == 0
```

6 Numerical examples

Substituting integer values $x = 10^b$ yields integer bases and illustrates the digit-regular behaviour for large b.

6.1 Case $b = 1$ $(x = 10)$

- $S_1 = \{-229, -199, -166, 363\}$
- $S_2 = \{95, 44, 46, 282\}$

$$(-229)^3 + (-199)^3 + (-166)^3 + 363^3 = 95^3 + 44^3 + 46^3 + 282^3 = 1{,}175{,}699{,}568.$$

6.2 Case $b = 2$ $(x = 100)$

- $S_1 = \{14486, 14786, 15089, 33168\}$
- $S_2 = \{7520, 7739, 7961, 35517\}$

$$\sum S_1^3 = \sum S_2^3 = 93{,}812{,}683{,}264{,}263.$$

6.3 Large x (link to $b_0 = 10^n - 1$ experiments)

For $x = 10^{10}$ (i.e. $b = 10$, matching the "ten nines" pattern $b_0 = 10^{10} - 1$ in the decomposition experiments), the eight integer bases are enormous but still satisfy (4); the equality can be checked by evaluating the polynomials at $x = 10^{10}$ or by reducing $P(10^{10})$ modulo both sides. This reproduces the type of identity the GUI displayed for long strings of nines in b_0.

Table 1: Common value of $\sum p^3$ for $x = 10^b$, $b = 1, 2, 3$

b	$x = 10^b$	$\sum_{p\in S_1} p^3 = \sum_{p\in S_2} p^3$
1	10	1,175,699,568
2	100	93,812,683,264,263
3	1000	100,463,556,121,586,812,763

7 Substitution $x = 10^b$: real, complex, and functional forms

The heart of the result is the *polynomial* identity (4) in the ring $\mathbb{Q}[x]$.

7.1 Real and rational b

For any real b, set $x = 10^b$ (positive real). Then (4) holds as an equality of real numbers, because it is the evaluation of one polynomial identity.

7.2 Complex b

For complex b, one defines $10^b = e^{b \ln 10}$ using a chosen branch of the logarithm (usually the principal one). The numerical equality of the two sides then holds for that branch; it is *not* the same statement as "for all branches simultaneously". The robust formulation is: for any $w \in \mathbb{C}$, substitute $x = w$ into the polynomial identity; in particular $w = 10^b$ for a fixed convention.

7.3 Functional substitution

If x is replaced by any function $f(u)$ taking values in a ring where the polynomial expressions are defined,

$$\sum_{p\in S_1} p(f(u))^3 = \sum_{p\in S_2} p(f(u))^3 \tag{5}$$

as an identity of functions of u. Thus the choice of base 10 in the *notation* $x = 10^b$ is not essential: any nonzero x can be written as s^b after fixing a base $s > 0$; what matters is the quadratic dependence on the intermediate variable x.

8 Discussion and outlook

Relation to PTE literature. Equal sums of cubes without matching lower moments appear in explicit constructions and tables; the contribution here is a compact parametric template with a clear experimental origin, complemented (versions 4–6) by an explicit discussion of infinite arithmetic families, a two-route interpolation pipeline, and software-admissible b_0 masks.

Possible applications. Equalities of power sums appear in moment problems and combinatorial design theory. Any use in cryptography or signal processing would require additional structure (e.g. efficient generation, hardness assumptions); we mention these only as speculative directions.

Open questions. (1) A proof that the telescopic algorithm at $(3, 10^n - 1, 3)$ always yields exactly this parametric form (not only empirically). (2) Closed forms for the empirical (a, k) catalogue beyond $(3, 3)$. (3) Greatest common divisors of the eight bases and reduced "defactored" identities.

9 Conclusion

We recorded and verified a cubic-only PTE-type parametric mechanism whose explicit 4×4 block is the minimal demonstrator of a broader infinite balanced $N \times N$ landscape observed in telescopic computation. Versions 4–6 emphasize the methodological layer: polynomial templates, infinite arithmetic realizations under substitution, and two reconstruction routes (scale-based in x, family-based in b_0). Version 6 in particular extends controlled exploration through masks $n_1 + d \dots d + n_2$ ($d = 1, \dots, 8$), enabling invariant checks directly in b_0 and strengthening large-batch scan applicability over structured decimal families. The identities extend by substitution to large integers and, with the usual caveats, to complex inputs.

References

[1] Dickson, L. E. (1920). *History of the Theory of Numbers, Volume II: Diophantine Analysis.* Carnegie Institution of Washington.

[2] Hardy, G. H., & Wright, E. M. (2008). *An Introduction to the Theory of Numbers.* Oxford University Press.

[3] Vorobtsov, N. (2026). *Telescoping Ramanujan Identities and Massive Decompositions of Cubes into Sums of Cubes.* Zenodo. `https://doi.org/10.5281/zenodo.19053976`

[4] Vorobtsov, N. (2026). *Euler's In-cubes: Internal symmetry of inverse sums and sequence modularity.* Zenodo. `https://doi.org/10.5281/zenodo.19055351`

[5] Borwein, P. (2002). *Computational Excursions in Analysis and Number Theory.* CMS Books in Mathematics.

[6] Wright, E. M. (1959). *Prouhet's 1851 solution of the Tarry-Escott problem.* American Mathematical Monthly.

[7] Vorobtsov, N. (2026). *Mathematical research publications portal (interactive covers, bilingual pages, supplementary source code).* `https://nvvorobtsov.github.io/`