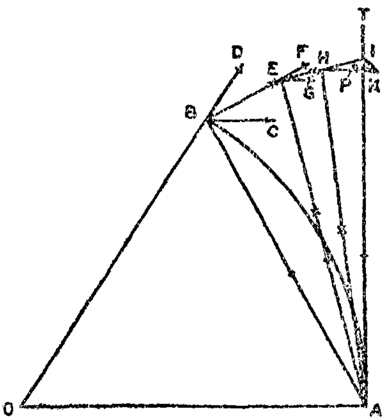
LII. *Notes on a Geometrical Construction for rectifying any Arc of a Circle.* *By* F. A. LINDEMANN *.

NOTES have been published recently by M. de Pulligny and by R. E. Baynes giving geometrical constructions for the ratio π or some simple function of π. All of these are based upon some numerical coincidence which enables π or the function in question to be represented very closely by a ratio of fairly small whole numbers such as 355/113. The following construction may perhaps be of interest as it allows any arc of a circle to be rectified, and as it is based upon no such numerical coincidence but represents an extremely rapidly converging series. In principle, an extraordinary degree of accuracy is obtainable in a very short time; in practice, it need scarcely be said, it is of no more value for this purpose than any of the constructions whose accuracy can only be verified *a posteriori*.



Let AB be the arc whose length is to be determined.

Draw AT the tangent to AB at the point A.

Continue OB to D and draw BC parallel to OA.

Bisect ∢ DBC by line BF and ∢ BAT by line AE which cuts BF at E.

Draw EG parallel to OA.

Bisect ∢ FEG by line EH and ∢ EAT by line AH which cuts EH at H.

This process may be repeated as often as desired. In the present instance, for the sake of clearness in the diagram, no

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further bisection will be undertaken, and the point H will be used to determine the final result.

Draw HK parallel to OA meeting AT at K and continue EH until it cuts AT at I.

Divide KI in the ratio 1 : 2 at point P.

Then the straight line AP will be very nearly equal in length to the arc AB.

It is easy to demonstrate that this result is true. If $\measuredangle$ AOB is called α and OA = 1, then AI $= 2^n \tan \frac{\alpha}{2^n}$, where n represents the number of times (in the present instance 2) that the process of bisecting the angles took place. Similarly

$$AK = 2 \ \sin \frac{\alpha}{2^n}.$$

Therefore

$$AP = 2^n \left\{ \sin \frac{\alpha}{2^n} + 1/3 \left(\tan \frac{\alpha}{2^n} - \sin \frac{\alpha}{2^n} \right) \right\}.$$

Expanding this one finds

$$\begin{aligned} AP = 2^n\{ & (\alpha/2^n) - 1/6(\alpha/2^n)^3 + 1/120(\alpha/2^n)^5 - \ldots \\ & + 1/3((\alpha/2^n) + 1/3(\alpha/2^n)^3 + 2/15(\alpha/2^n)^5 + \ldots \\ & - (\alpha/2^n) + 1/6(\alpha/2^n)^3 - 1/120(\alpha/2^n)^5 + \ldots)\} \\ = 2^n\{ & (\alpha/2^n) + 1/20(x/2^n)^5 + \ldots\} = \alpha(1 + 1/20(\alpha/2^n)^4). \end{aligned}$$

The residual error $1/20(\alpha/2^n)^4$ is obviously reduced to 1/16 by each repetition of the bisecting process, and may therefore in theory be made very small indeed in a very short time. Even with but two bisections as in the above diagram, the error is only of the order of 1 part in 5000. A similar construction with a 90° arc would give π to 6 places of decimals if the bisecting process were repeated 5 times.

Farnborough.
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[NOTE.

The method is interesting though hardly practical.

The details seem to be these :—

(1) The angles ABE, AEH are right angles.

For $A\hat{B}C = B\hat{A}O = 90° - \frac{\alpha}{2}$, as is seen by dropping a perpendicular from O on AB.

Whence also, incidentally, $B\hat{A}T = \frac{1}{2}\alpha,$

$$E\hat{A}T = \tfrac{1}{4}\alpha,$$

$$H\hat{A}T = \tfrac{1}{8}\alpha,$$

$$\left.\begin{array}{ll}\text{Now,} & A\hat{B}C = 90^\circ - \frac{\alpha}{2} \\ \text{and} & C\hat{B}E = \frac{\alpha}{2}\end{array}\right\} \text{therefore } A\hat{B}E = 90^\circ.$$

Similarly

$$A\hat{E}H = A\hat{E}G + G\hat{E}H = E\hat{A}O + G\hat{E}H$$

$$= 90^\circ - \frac{\alpha}{4} + \frac{\alpha}{4} = 90^\circ.$$

(2) $AB = 2 \sin \frac{\alpha}{2}$, seen by dropping same perpendicular from O on AB. Therefore

$$AE = 2 \sin \frac{\alpha}{2} \div \cos \frac{\alpha}{4} = 2^2 \sin \frac{\alpha}{2^2} = AK$$

$\left(AH = 2^2 \sin \frac{\alpha}{2^2} \div \cos \frac{\alpha}{2^3} = 2^3 \sin \frac{\alpha}{2^3}\right.$, not needed here though)

$$AI = AE \div \cos \frac{\alpha}{2^2} = 2^2 \tan \frac{\alpha}{2^2}.$$

It is interesting to note that with A as origin and AT the initial line, the points B, E, H, ... all lie on the curve whose polar equation is $r = \frac{\alpha \sin\theta}{\theta}$. For, taking any one of the radii vectores, say AH, when $\theta = \frac{\alpha}{2^3}$ then $2^3 = \frac{\alpha}{\theta}$, whence $AH = \frac{\alpha}{\theta} \sin\theta$.]