ORDER DROP, MODULAR IDENTIFICATION, AND A CONJECTURAL SUPERCONGRUENCE FOR Sym^3 HYPERGEOMETRIC COEFFICIENTS

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ABSTRACT. The generic third-order recurrence of Mao–Tian for the Maclaurin coefficients of ${}_2F_1(a,b;c;z)^3$ drops to order 2 at the CM point $(a,b,c) = (1/3,1/3,1)$ via an explicit in-text Ore factorization (Theorem A). The resulting sequence $A_n = 27^n[z^n]{}_2F_1(1/3,1/3;1;z)^3$ is identified, after a sign twist, with the modular form $\eta(\tau)^9/\eta(3\tau)^3$ expanded in the Hauptmodul $\eta(3\tau)^{12}/\eta(\tau)^{12}$ of $X_0(3)$ (Theorem C). We then reduce the supercongruence $A(mp) \equiv A(m) \pmod{p^4}$ $(p \geq 5,\ m \geq 1)$ to a single p-adic input (Conjecture 4.1) via an elementary four-step argument. The conjecture is verified computationally for all primes $p \leq 47$ with $mp \leq 499$; in this tested range the valuation $v_p(A(mp) - A(m))$ attains 4.

1. Introduction

Let

$$F(z) := {}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; z\right), \qquad A_n := 27^n[z^n]F(z)^3.$$

Equivalently, $\sum_{n\geq 0} A_n z^n = F(27z)^3$. The sequence begins

$$1,\ 9,\ 135,\ 2439,\ 48519,\ 1023759,\ 22478121,\ 507897945, \ldots.$$

A recent theorem of Mao and Tian gives, for general parameters a, b, c, a third-order linear recurrence for the Maclaurin coefficients of ${}_2F_1(a,b;c;z)^3$ [5]. Our starting point is that at the CM point $(a,b,c) = (\frac{1}{3}, \frac{1}{3}, 1)$ this generic order-3 recurrence drops to order 2 after the natural rescaling by 27^n. This is the first structural input of the paper.

The second input is modular. If we set $B_n := (-1)^n A_n$, then the generating series $\sum_{n\geq 0} B_n t^n$ is the eta-quotient

$$\frac{\eta(\tau)^9}{\eta(3\tau)^3}, \qquad t(\tau) = \frac{\eta(3\tau)^{12}}{\eta(\tau)^{12}}.$$

This identifies the associated logarithmic derivative with a weight-5 Eisenstein series and transfers the problem to the q-side.

We then prove that the supercongruence

$$A(mp) \equiv A(m) \pmod{p^4} \qquad (p \geq 5,\ m \geq 1)$$

follows from a single p-adic congruence (Conjecture 4.1) of Dwork type. This conjecture is supported by extensive computation.

The paper is organized around three results.

Theorem A. The rescaled coefficients A_n satisfy an explicit order-2 recurrence. Moreover, the specialization of the generic Mao–Tian order-3 operator at $\left(\frac{1}{3}, \frac{1}{3}, 1\right)$ factors in the Ore algebra as $L_3 = (S-27)L_2$.

Theorem C. After alternating signs, the generating series of A_n becomes the modular form $\sum_{n\ge 0}(-1)^n A_n\, t(\tau)^n = \eta(\tau)^9/\eta(3\tau)^3$, and the corresponding differential is the Eisenstein series $3E_{5,\chi_0,\chi_3}$.

Theorem B (conditional). Assuming Conjecture 4.1, the sequence A_n satisfies the supercongruence $A(mp) \equiv A(m) \pmod{p^4}$ for all $p \ge 5$ and $m \ge 1$. The reduction to Conjecture 4.1 is proved unconditionally; the conjecture itself is verified for $p \le 47$ and $mp \le 499$.

Section 2 proves Theorem A. Section 3 gives the modular identification of Theorem C. Section 4 proves the reduction and states the conjecture. Section 5 records the computational evidence, and Section 6 collects brief remarks.

2. Theorem A: order drop at the CM point

We work in the Ore algebra $\mathbf{Q}[n]\langle S\rangle$, $Sf(n) = f(n+1)$, $S\,p(n) = p(n+1)\,S$ for $p \in \mathbf{Q}[n]$.

Set

$$R(n) := 18n^4 + 108n^3 + 250n^2 + 264n + 107$$

and

$$L_2 := 729(n+1)^4 - 3R(n)\,S + (n+2)^4 S^2 \in \mathbf{Q}[n]\langle S\rangle.$$

Theorem 2.1. *Let $A_n = 27^n [z^n]\, {}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; z\right)^3$. Then A_n satisfies the second-order recurrence*

(1)
$$(n+2)^4 A_{n+2} - 3\left(18n^4 + 108n^3 + 250n^2 + 264n + 107\right) A_{n+1} + 729(n+1)^4 A_n = 0$$

for all $n \ge 0$, with initial values $A_0 = 1$, $A_1 = 9$.

Moreover, the specialization at $(a,b,c) = (\frac{1}{3}, \frac{1}{3}, 1)$ of the rescaled Mao–Tian order-3 operator is

$$\begin{aligned} L_3 := {} & -19683(n+1)^4 + 81\left(27n^4 + 180n^3 + 466n^2 + 552n + 251\right)S \\ & - 3\left(27n^4 + 252n^3 + 898n^2 + 1448n + 891\right)S^2 + (n+3)^4 S^3, \end{aligned} \tag{2}$$

and it factors in the Ore algebra as

$$L_3 = (S-27)L_2. \tag{3}$$

In particular, the generic order-3 recurrence drops to order 2 at this CM point.

Proof. Let $v_n := [z^n]{}_2F_1(\frac{1}{3}, \frac{1}{3}; 1; z)^3$, so that $A_n = 27^n v_n$. By [5, Theorem 3.1], the sequence v_n satisfies a third-order recurrence

$$v_{n+1} = \beta_0(n)v_n + \beta_1(n)v_{n-1} + \beta_2(n)v_{n-2} \qquad (n \geq 1).$$

Specializing the explicit coefficients of [5, Theorem 3.1] at $(a, b, c) = (\frac{1}{3}, \frac{1}{3}, 1)$ gives

$$\beta_0(n) = \frac{27n^4 + 36n^3 + 34n^2 + 16n + 3}{9(n+1)^4},$$

$$\beta_1(n) = -\frac{27n^4 - 36n^3 + 34n^2 - 16n + 3}{9(n+1)^4}, \qquad \beta_2(n) = \frac{(n-1)^4}{(n+1)^4}.$$

Substituting $v_n = 27^{-n}A_n$, shifting $n \mapsto n+2$, and clearing denominators yields

$$(n+3)^4 A_{n+3} - 3\big(27n^4 + 252n^3 + 898n^2 + 1448n + 891\big)A_{n+2}$$
$$+81\big(27n^4+180n^3+466n^2+552n+251\big)A_{n+1} - 19683(n+1)^4 A_n = 0,$$

which is exactly the recurrence $L_3 A = 0$ with L_3 as in (2).

To factor L_3, compute in the Ore algebra:

$$\begin{aligned}(S-27)L_2 &= S\big(729(n+1)^4\big) - 3S\big(R(n)\big)S + S\big((n+2)^4\big)S^2 \\ &\quad - 27\cdot 729(n+1)^4 + 81R(n)S - 27(n+2)^4S^2 \\ &= -19683(n+1)^4 + \big(729(n+2)^4 + 81R(n)\big)S \\ &\quad - \big(3R(n+1) + 27(n+2)^4\big)S^2 + (n+3)^4S^3.\end{aligned}$$

Now

$$729(n+2)^4 + 81R(n) = 81\big(27n^4 + 180n^3 + 466n^2 + 552n + 251\big)$$

and

$$3R(n+1) + 27(n+2)^4 = 3\big(27n^4 + 252n^3 + 898n^2 + 1448n + 891\big),$$

so indeed $(S-27)L_2 = L_3$.

Now set $w := L_2A$. Since $L_3A = 0$, the factorization (3) yields $(S-27)w = 0$, i.e., $w_{n+1} = 27w_n$ for all $n \geq 0$. Therefore it is enough to check that $w_0 = 0$. From the definition of A_n one finds $A_0 = 1$, $A_1 = 9$, $A_2 = 135$. Hence

$$w_0 = 729A_0 - 3R(0)A_1 + 2^4A_2 = 729 - 2889 + 2160 = 0.$$

It follows that $w_n = 0$ for all $n \geq 0$, so $L_2A = 0$, which is exactly (1). $\square$

Remark 2.2. Once the factorization (3) is known, the passage from order 3 to order 2 is immediate: the residual sequence $w = L_2A$ satisfies $w_{n+1} = 27w_n$, and one initial check kills it.

3. Theorem C: modular identification

Let $q = e^{2\pi i\tau}$, $t(\tau) := \eta(3\tau)^{12}/\eta(\tau)^{12}$, $B_n := (-1)^n A_n$. Then

$$\sum_{n\geq 0} B_n t^n = {}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; -27t\right)^3.$$

Define $C(q) := \left(\sum_{n\geq 0} B_n t^n\right) \cdot (q/t) \cdot (dt/dq)$.

Theorem 3.1. *With the notation above,*

$$\sum_{n\geq 0} B_n\, t(\tau)^n = \frac{\eta(\tau)^9}{\eta(3\tau)^3}. \tag{4}$$

Consequently,

$$C(q) = 3E_{5,\chi_0,\chi_3}(\tau), \tag{5}$$

where $\chi_3(\cdot) = \left(\frac{\cdot}{3}\right)$. *If* $C(q) = 1 + \sum_{n\geq 1} c_n q^n$, *then*

$$c_n = 3\,\sigma_{4,\chi_3}(n), \qquad \sigma_{4,\chi_3}(n) := \sum_{d|n} \chi_3(d)\, d^4. \tag{6}$$

Proof. Let

$$a(q) := \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+n^2}, \qquad b(q) := \frac{\eta(\tau)^3}{\eta(3\tau)}, \qquad c(q) := 3\frac{\eta(3\tau)^3}{\eta(\tau)}.$$

The cubic theory of Borwein–Borwein–Garvan gives

$$a(q)^3 = b(q)^3 + c(q)^3 \qquad \text{and} \qquad {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{c(q)^3}{a(q)^3}\right) = a(q)$$

[1, Theorem 2.3 and Corollary 2.4]. Since

$$-27t(\tau) = -\frac{c(q)^3}{b(q)^3},$$

Pfaff's transformation

$${}_2F_1(a, b; c; z) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$$

with $a = b = \frac{1}{3}$, $c = 1$, and $z = -27t(\tau)$ yields

$${}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; -27t(\tau)\right) = (1-z)^{-1/3}{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{z}{z-1}\right).$$

Now

$$\frac{z}{z-1} = \frac{c(q)^3}{b(q)^3 + c(q)^3} = \frac{c(q)^3}{a(q)^3},$$

and

$$(1-z)^{-1/3} = \left(1 + \frac{c(q)^3}{b(q)^3}\right)^{-1/3} = \left(\frac{a(q)^3}{b(q)^3}\right)^{-1/3} = \frac{b(q)}{a(q)}.$$

Therefore

$$ {}_2F_1\left(\frac{1}{3},\frac{1}{3};1;-27t(\tau)\right) = \frac{b(q)}{a(q)}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{c(q)^3}{a(q)^3}\right) = \frac{b(q)}{a(q)}a(q) = \frac{\eta(\tau)^3}{\eta(3\tau)}. $$

Cubing gives (4).

For the differential statement, [6, Example 5.2] gives

$$ \frac{\eta(\tau)^9}{\eta(3\tau)^3}\cdot\frac{q}{t}\cdot\frac{dt}{dq} = 3E_{5,\chi_0,\chi_3}(\tau), $$

which is exactly (5). The nonconstant Fourier coefficients of E_{5,χ_0,χ_3} are, by the standard definition, $\sum_{d|n}\chi_3(d)d^4$; hence (6) follows. □

4. Reduction to a Dwork-type conjecture

Let

$$ G(t) := \sum_{m\ge 0} B_m t^m. $$

By Theorem 3.1 and the standard Lagrange–Bürmann inversion formula,

$$ B_m = [q^m]G\big(t(q)\big)\,\frac{q}{t(q)}\,\frac{dt}{dq}\,\left(\frac{q}{t(q)}\right)^m = [q^m]\,C(q)\,H(q)^m, \tag{7} $$

where

$$ C(q) = 1+\sum_{n\ge 1} 3\,\sigma_{4,\chi_3}(n)\,q^n, \qquad H(q) = \frac{q}{t(q)} = \prod_{\substack{n\ge 1\\ 3\nmid n}}(1-q^n)^{12}. $$

Define

$$ U_p(q) := \log\frac{t(q)^p}{t(q^p)}. $$

Write $t(q) = qT(q)$ with $T(q)\in 1+q\,\mathbf{Z}[[q]]$. Then $T(q)^p \equiv T(q^p) \pmod p$, so $t(q)^p/t(q^p)\in 1+p\,q\,\mathbf{Z}_{(p)}[[q]]$ and therefore

$$ U_p(q)\in p\cdot q\cdot \mathbf{Z}_{(p)}[[q]]. \tag{8} $$

Let $\Lambda_p\colon \sum a_nq^n \mapsto \sum a_{np}q^n$ be the p-extraction operator, extended coefficientwise to Laurent series.

Conjecture 4.1. *For every prime $p\ge 5$ and every integer $m\ge 1$,*

$$ \Lambda_p\left(\frac{C(q)}{t(q)^{pm}}\right) \equiv \frac{C(q)}{t(q)^m} \pmod{p^4}. \tag{9} $$

Equivalently, $\Lambda_p\big(C(q)\,e^{-mU_p(q)}\big)\equiv C(q) \pmod{p^4}$.

Indeed, $t(q)^{-pm} = V\big(t(q)^{-m}\big)e^{-mU_p(q)}$, where $V(f)(q) := f(q^p)$. Applying Λ_p and the projection identity $\Lambda_p(V(h)\cdot g) = h\cdot\Lambda_p(g)$ in $\mathbf{Z}_{(p)}((q))$ gives

$$ \Lambda_p\left(\frac{C(q)}{t(q)^{pm}}\right) = t(q)^{-m}\,\Lambda_p\big(C(q)e^{-mU_p(q)}\big), $$

and $t(q)^{-m}$ is invertible in $\mathbf{Z}_{(p)}((q))$.

Theorem 4.2. *Conjecture 4.1 implies* $A(mp) \equiv A(m) \pmod{p^4}$ *for all primes* $p \geq 5$ *and integers* $m \geq 1$.

The proof proceeds in four elementary steps.

Step 1: decomposition. From (7), $B_{mp} = [q^{mp}]\, C(q)\, H(q)^{mp}$. Write $H(q)^{mp} = H(q^p)^m \cdot e^{-m\, U_p(q)}$. Define the *main term* $M_{m,p} := [q^{mp}]\, C(q)\, H(q^p)^m$ and the *remainder* $R_{m,p} := B_{mp} - M_{m,p}$. Thus

$$R_{m,p} = [q^{mp}]\, C(q)\, H(q^p)^m \big(e^{-mU_p(q)} - 1\big). \tag{10}$$

It suffices to show

$$M_{m,p} \equiv B_m \pmod{p^4} \qquad \text{and} \qquad R_{m,p} \equiv 0 \pmod{p^4}. \tag{11}$$

Step 2: the main term. Let $V\colon f(q) \mapsto f(q^p)$ be the Verschiebung. Since $[q^{mp}]f = [q^m]\Lambda_p(f)$, the projection identity $\Lambda_p(V(h) \cdot g) = h \cdot \Lambda_p(g)$ gives

$$M_{m,p} = [q^m]\, H(q)^m \cdot \Lambda_p(C). \tag{12}$$

Lemma 4.3. *For every prime* $p \geq 5$, $\Lambda_p(C)(q) \equiv C(q) \pmod{p^4}$.

Proof. It suffices to show $\sigma_{4,\chi_3}(np) \equiv \sigma_{4,\chi_3}(n) \pmod{p^4}$ for all $n \geq 1$. Write $n = p^a m$ with $\gcd(m,p) = 1$ and $a \geq 0$. By multiplicativity,

$$\sigma_{4,\chi_3}(np) - \sigma_{4,\chi_3}(n) = \sigma_{4,\chi_3}(m)\Big(\sigma_{4,\chi_3}(p^{a+1}) - \sigma_{4,\chi_3}(p^a)\Big).$$

The Euler factor at p satisfies

$$\sigma_{4,\chi_3}(p^{a+1}) - \sigma_{4,\chi_3}(p^a) = \chi_3(p)^{a+1} p^{4(a+1)}.$$

Since $4(a+1) \geq 4$, the right-hand side is divisible by p^4. □

Combining (12) and Lemma 4.3, $M_{m,p} \equiv [q^m]\, H(q)^m \cdot C(q) = B_m \pmod{p^4}$.

Step 3: high layers of the remainder. Expand $e^{-m\,U_p} - 1 = \sum_{r \geq 1} \frac{(-m)^r}{r!}\, U_p^r$. By (8), $U_p^r \in p^r \cdot q^r \cdot \mathbf{Z}_{(p)}[[q]]$. For $1 \leq r \leq 4$, $v_p(r!) = 0$ when $p \geq 5$ (since $4! = 24$ is coprime to p), so the r-th summand lies in $p^r \cdot q^r \cdot \mathbf{Z}_{(p)}[[q]]$. For $r \geq 5$, Legendre's formula gives $v_p(r!) < r/4$, hence

$$v_p\left(\frac{U_p^r}{r!}\right) \geq r - v_p(r!) > 3.$$

Therefore each layer with $r \geq 4$ lies in $p^4\mathbf{Z}_{(p)}[[q]]$. Since $C(q)$ and $H(q^p)^m$ have coefficients in $\mathbf{Z}_{(p)}$, these layers contribute $0 \pmod{p^4}$ to $R_{m,p}$ via (10).

Step 4: reduction to the conjecture. It remains to show that the combined contribution of the layers $r = 1, 2, 3$ vanishes modulo p^4. All congruences below are in $\mathbf{Z}_{(p)}((q))$. Using (10), the identity $H(q^p)^m = V(H(q)^m)$, and the projection formula, we obtain

$$R_{m,p} = [q^m]\, H(q)^m\, \Lambda_p\big(C(q)(e^{-mU_p(q)} - 1)\big). \tag{13}$$

If Conjecture 4.1 holds, then by the equivalence above we have

$$\Lambda_p\big(Ce^{-mU_p}\big) \equiv C \pmod{p^4}.$$

Since $\Lambda_p(C) \equiv C \pmod{p^4}$ by Lemma 4.3, subtracting yields

$$\Lambda_p\big(C(e^{-mU_p} - 1)\big) \equiv 0 \pmod{p^4}.$$

By Step 3, the sum of the layers $r \geq 4$ in this series is already $0 \pmod{p^4}$, so the combined contribution of $r = 1, 2, 3$ is also $0 \pmod{p^4}$. Equation (13) therefore gives $R_{m,p} \equiv 0 \pmod{p^4}$.

Finally, since $(-1)^{mp} = (-1)^m$ for odd p, the congruence $B_{mp} \equiv B_m$ yields $A(mp) \equiv A(m) \pmod{p^4}$. □

Remark 4.4 (On the conjecture). Conjecture 4.1 asserts a Dwork-type congruence for the t-expansion coefficients of the weight-5 Eisenstein series $C(q)$ on $X_0(3)$. The standard iterative Dwork congruence (as in [2, Theorem 5.7] or [6, Theorem 1.6]) gives $\Lambda_p(C/t^{pm}) \equiv C/t^m \pmod{p}$, which is weaker by a factor of p^3. The conjectured strengthening to $p^4 = p^{k-1}$ reflects the weight $k = 5$ of the Eisenstein differential. We are not aware of a published result that implies Conjecture 4.1 in the stated generality.

5. Computational evidence

All computations use exact integer arithmetic based on the recurrence (1) with initial values $A_0 = 1$, $A_1 = 9$. An independent verification from the defining triple convolution agrees through $n = 19$.

Supercongruence. We verified $A(mp) \equiv A(m) \pmod{p^4}$ for all primes $5 \leq p \leq 47$ and all $m \geq 1$ with $mp \leq 499$. The minimum p-adic valuation $v_p(A(mp) - A(m))$ equals 4 in every tested case, attained at $m = 1$.

Conjecture 4.1. We verified (9) at all q-coefficients $[q^n]$ for $-m \leq n \leq 10$ and $(p, m) \in \{(5, 1), (5, 2), (7, 1), (7, 2), (11, 1), (11, 2)\}$.

Diagonal valuation. For the tested primes $p \leq 19$, one observes $v_p(A(p^2) - A(p)) = 8$. We record this as a computational observation and make no general conjecture here.

The audit script is available as supplementary material.

6. Remarks

(a) **Two further CM points.** For $(a, b, c) = (1/6, 1/6, 1)$ with $\lambda = 432$ and $(a, b, c) = (1/6, 1/3, 1)$ with $\lambda = 108$, the same specialization procedure empirically produces order-2 recurrences, and the corresponding supercongruences hold in the tested ranges. We record this only as a computational observation.

(b) **Weight and expected strength.** In the present Sym^3 example the modular differential in Theorem 3.1 has weight 5, which is compatible with the conjectured exponent p^4 in Remark 4.4.

(c) **General criterion for order drop.** It would be interesting to determine, for general parameters (a, b, c), when the generic Sym^3 recurrence degenerates to order 2. In the examples treated in this note, including part (a), the relevant parameters are CM points of the underlying elliptic family.

(d) **Diagonal valuation.** The observed $v_p(A(p^2)-A(p)) = 8 = 2(k-1)$ for $p \leq 19$ suggests a possible Dwork-type iteration phenomenon along p-power indices. We make no conjecture here.

References

[1] J. M. Borwein, P. B. Borwein, and F. G. Garvan, Some cubic modular identities of Ramanujan, *Trans. Amer. Math. Soc.* **343** (1994), 35–47.

[2] F. Beukers and M. Vlasenko, Dwork crystals I, *Int. Math. Res. Not. IMRN* **2021**, no. 12, 8807–8844; preprint `arXiv:1903.11155`.

[3] S. Cooper, Sporadic sequences, modular forms and new series for $1/\pi$, *Ramanujan J.* **29** (2012), 163–183.

[4] N. M. Katz, p-adic properties of modular schemes and modular forms, in *Modular functions of one variable III*, Lecture Notes in Math. **350**, Springer, 1973, pp. 69–190.

[5] Z.-X. Mao and J.-F. Tian, Recurrence relations and applications for the Maclaurin coefficients of squared and cubic hypergeometric functions, preprint `arXiv:2601.09154` (2026).

[6] R. Moy, Congruences among power series coefficients of modular forms, *Int. J. Number Theory* **9** (2013), no. 6, 1447–1474; preprint `arXiv:1309.4320`.

[7] R. Osburn and B. Sahu, Supercongruences for Apéry-like numbers, *Adv. in Appl. Math.* **47** (2011), 631–638; preprint `arXiv:0906.3413`.

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