The Kappa-Navier-Stokes Equations:

Fluids Deviating From Maxwellian Velocity Distributions

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Abstract

We derive the kappa-Navier-Stokes (κ-NS) equations governing compressible and incompressible flows whose molecular velocity distribution follows a kappa (power-law) rather than a Maxwellian distribution. The derivation rests on three ingredients: the Fokker-Planck collision operator as the Wasserstein gradient flow of free energy; an SO(3)-symmetric dissipation metric that fixes the viscous stress up to two parameters; and the moment method applied to the kappa distribution using only second moments. The resulting system couples the standard fluid equations to a transport equation for the non-Maxwellianity parameter $\eta = 1/\kappa \in [0, \frac{2}{3})$. The shear viscosity $\mu(\eta) = \mu_0/(1 - \frac{3}{2}\eta)$ diverges at $\eta = \frac{2}{3}$, while the production of η vanishes there with the same factor $(1 - \frac{3}{2}\eta)$: an *aligned-singularity* mechanism that geometrically prevents velocity-gradient blow-up. Full derivations are collected in the appendices for the interested reader; the body of the paper presents the equations, the key lemmas, and their physical interpretation.

Contents

1 Introduction

The classical Navier-Stokes equations rest on the assumption that the molecular velocity distribution is everywhere close to Maxwellian. This fails in turbulent flows at the dissipation scale, weakly collisional plasmas, the solar wind, and rarefied gases, where the distribution develops power-law tails characteristic of kappa distributions [1, 2].

In this paper we derive the *kappa-Navier-Stokes* (κ-NS) equations from kinetic theory using only **second moments**. The derivation has three pillars:

(i) **Geometric collision operator.** The Fokker-Planck operator is the Wasserstein-2 gradient flow of free energy [4, 6]. Its unique attractor is the Maxwellian; it conserves mass and momentum, and relaxes energy toward equilibrium.

(ii) **SO(3) metric structure.** The macroscopic dissipation metric on symmetric stress tensors is fixed by rotational symmetry (Schur's lemma) to have exactly two parameters: shear viscosity μ and bulk viscosity ζ. This is a geometric consequence independent of the microscopic model.

(iii) **Second-moment transport and η-equation.** All transport coefficients (μ, κ_T, ζ) and the evolution equation for η are derived from second moments of the kappa distribution via the moment method. The key identity is $\langle c^2 \rangle_\kappa = 3\theta^2/(1 - \frac{3}{2}\eta)$, which carries a factor $(1 - \frac{3}{2}\eta)^{-1}$ into every transport coefficient.

The central new feature is the *aligned-singularity* structure: viscosity diverges as $(1 - \frac{3}{2}\eta)^{-1}$ while η-production vanishes as $(1 - \frac{3}{2}\eta)^{+1}$ at the *same* critical point $\eta = \frac{2}{3}$, providing a built-in regularization mechanism.

Organization. Section 2 states the κ-NS equations for both compressible and incompressible flow — the main result of the paper. Section 3 presents the four key lemmas whose combination yields these equations; proofs are in Appendices A–D. Section 4 analyzes the aligned-singularity mechanism. Section 5 discusses physical implications and the Maxwellian limit. Full derivations are in the appendices.

2 The Kappa-Navier-Stokes Equations

We present the main results first. Let ρ, $\mathbf{u}$, θ^2, and η denote density, velocity, thermal-scale parameter ($\theta^2 = k_B T/m$ at $\eta = 0$), and non-Maxwellianity. The strain-rate tensor is $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ and $|S|^2 = S_{ij}S_{ij}$.

2.1 Compressible κ-NS System

Compressible Kappa-Navier-Stokes Equations

Continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$

Momentum:

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} \tag{2}$$

Thermal scale:

$$\frac{3}{2}\rho \frac{D\theta^2}{Dt} = -p(\nabla \cdot \mathbf{u}) + \boldsymbol{\tau} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q} \tag{3}$$

Non-Maxwellianity:

$$\rho \frac{D\eta}{Dt} = \rho\alpha\big(1 - \tfrac{3}{2}\eta\big)\left[|S|^2 + \beta_\theta \frac{|\nabla \theta^2|^2}{\theta^4} + \beta_\rho \frac{|\nabla \rho|^2}{\rho^2}\right] - \frac{\gamma \rho}{\tau_c}\eta + \nabla \cdot (D_\eta \rho \nabla \eta) \tag{4}$$

Closure relations:

$$p = \frac{\rho \theta^2}{1 - \frac{3}{2}\eta}, \qquad T = \frac{m\theta^2}{k_B(1 - \frac{3}{2}\eta)} \tag{5}$$

$$\boldsymbol{\tau} = 2\mu(\eta)\mathbf{S} + \Big(\zeta(\eta) - \tfrac{2\mu(\eta)}{3}\Big)(\nabla \cdot \mathbf{u})\mathbf{I} \tag{6}$$

$$\mu(\eta) = \frac{\mu_0}{1 - \frac{3}{2}\eta}, \qquad \kappa_T(\eta) = \frac{\kappa_{T,0}}{1 - \frac{3}{2}\eta}, \qquad \zeta(\eta) = \zeta_0\,\eta \tag{7}$$

$$\mathbf{q} = -\kappa_T(\eta)\nabla T, \qquad \alpha = \frac{4\tau_c}{9}, \qquad \gamma = 2, \qquad D_\eta \sim \theta^2 \tau_c \tag{8}$$

Remark 2.1 (Pressure gradient)**.** Since p depends on both θ^2 and η, its gradient decomposes as:

$$\nabla p = \frac{\theta^2}{1 - \frac{3}{2}\eta}\nabla \rho + \frac{\rho}{1 - \frac{3}{2}\eta}\nabla \theta^2 + \frac{3\rho\theta^2/2}{(1 - \frac{3}{2}\eta)^2}\nabla \eta. \tag{9}$$

The thermodynamic variable θ^2 remains finite as $\eta \to \frac{2}{3}$ even though $T \to \infty$; it is the natural primary variable.

2.2 Incompressible κ-NS System

Setting $\rho = \text{const}$, $\nabla \cdot \mathbf{u} = 0$, and isothermal ($\nabla \theta^2 = 0$):

Incompressible Kappa-Navier-Stokes Equations

$$\nabla \cdot \mathbf{u} = 0 \tag{10}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nabla \cdot \left[\frac{2\mu_0}{1 - \frac{3}{2}\eta} S\right] \tag{11}$$

$$\frac{D\eta}{Dt} = \alpha\big(1 - \tfrac{3}{2}\eta\big)\,|S|^2 - \frac{\gamma}{\tau_c}\eta + D_\eta \nabla^2 \eta \tag{12}$$

The classical Navier-Stokes equations are recovered at $\eta \equiv 0$.

3 The Four Key Lemmas

The four lemmas below constitute the derivation of the system in Section 2. Each is a self-contained result; full calculations are in the appendices.

3.1 The Kappa Distribution and Second Moment

Lemma 3.1 (Second Moment)**.** *For the kappa distribution*

$$f_\kappa(\mathbf{c}) = \frac{n}{(2\pi\kappa\theta^2)^{3/2}} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\frac{1}{2})} \left(1 + \frac{|\mathbf{c}|^2}{2\kappa\theta^2}\right)^{-(\kappa+1)}, \qquad \kappa > \tfrac{3}{2}, \tag{13}$$

with $\eta = 1/\kappa$ and thermal scale $\theta^2 = k_B T/m$, the second moment is:

$$\langle c^2 \rangle_\kappa = \frac{3\theta^2}{1 - \frac{3}{2}\eta}. \tag{14}$$

This requires $\eta < \frac{2}{3}$; for $\eta \geq \frac{2}{3}$ the second moment does not exist. The Maxwellian is recovered as $\eta \to 0$: $\langle c^2 \rangle \to 3\theta^2 = 3k_B T/m$.

Proof. See Appendix A. □

Remark 3.2 (Convention)**.** Livadiotis & McComas [3] use the form $(1 + c^2/(\kappa\theta^2))^{-(\kappa+1)}$ with normalization $(\pi\kappa\theta^2)^{-3/2}$, setting $\theta^2 = 2k_B T/m$. The macroscopic equations (1)–(8) are independent of this convention.

The Jacobian of the map $\eta \mapsto \langle c^2 \rangle$ at fixed θ^2, used throughout the η-equation derivation, is:

$$\left.\frac{\partial \eta}{\partial \langle c^2 \rangle}\right|_{\theta^2} = \frac{2(1 - \frac{3}{2}\eta)^2}{9\theta^2}. \tag{15}$$

3.2 The SO(3) Dissipation Metric

Lemma 3.3 (Two-Parameter Viscous Stress)**.** *The most general dissipation metric on symmetric stress tensors that is invariant under SO(3) has exactly two parameters and takes the form*

$$\|\boldsymbol{\tau}\|^2 = \frac{1}{2\mu}|\boldsymbol{\tau}^{\mathrm{dev}}|^2 + \frac{1}{9\zeta}(\mathrm{tr}\,\boldsymbol{\tau})^2. \tag{16}$$

Gradient flow of the kinetic energy with respect to this metric yields the constitutive relation (6) *with shear viscosity μ and bulk viscosity ζ. For incompressible flow, the bulk term vanishes and a single parameter μ suffices.*

Proof. See Appendix B. □

The two parameters μ and ζ are not fixed by geometry alone; their dependence on η is determined by the moment method (Lemma 3.4).

3.3 Transport Coefficients from Second Moments

Lemma 3.4 (Transport Coefficients)**.** *Applying the moment method to the Boltzmann equation with Fokker-Planck collision operator, using only second moments of f_κ, yields:*

$$\mu(\eta) = \frac{\mu_0}{1 - \frac{3}{2}\eta}, \qquad \kappa_T(\eta) = \frac{\kappa_{T,0}}{1 - \frac{3}{2}\eta}, \qquad \zeta(\eta) = \zeta_0\,\eta, \tag{17}$$

where $\mu_0 = \rho\theta^2\tau_c/2$, $\kappa_{T,0} = \frac{5k_B}{2m}\mu_0$, *and* $\zeta_0 \sim \mu_0$. *The Prandtl number is:*

$$\mathrm{Pr} = \frac{\mu(\eta)\, c_p}{\kappa_T(\eta)} = 1, \tag{18}$$

independent of η, *reflecting that a single relaxation time* τ_c *governs both momentum and energy transport. The factor* $(1-\frac{3}{2}\eta)^{-1}$ *in* μ *and* κ_T *is a direct consequence of* $\langle c^2\rangle_\kappa$ *in Lemma 3.1. Bulk viscosity* $\zeta \propto \eta$ *vanishes at the Maxwellian limit and does not carry the* $(1-\frac{3}{2}\eta)^{-1}$ *enhancement, reflecting its distinct physical origin as a relaxation-lag rather than a second-moment enhancement.*

Proof. See Appendix C. □

3.4 The η-Equation

Lemma 3.5 (η Evolution)**.** *Converting the second-moment balance for* $\langle c^2\rangle$ *to* η *via the Jacobian* (15) *gives the* η*-equation* (4)*. The three production terms arise from:*

(i) Strain: *viscous dissipation heats the fluid, increasing* $\langle c^2\rangle$ *and hence* η; *this gives the* $\alpha(1-\frac{3}{2}\eta)|S|^2$ *term with* $\alpha = 4\tau_c/9$.
(ii) Temperature gradients: *mixing of particles from regions of different* θ^2 *produces a second-order isotropic contribution* $\beta_\theta(1-\frac{3}{2}\eta)|\nabla\theta^2|^2/\theta^4$.
(iii) Density gradients: *an identical mechanism through density inhomogeneity gives* $\beta_\rho(1-\frac{3}{2}\eta)|\nabla\rho|^2/\rho^2$.

All three share the saturation factor $(1-\frac{3}{2}\eta)$ *from the Jacobian. The relaxation term* $-\gamma\eta/\tau_c$ *drives* $\eta \to 0$, *without the saturation factor. Spatial diffusion of* η *with coefficient* $D_\eta \sim \theta^2\tau_c$ *(finite as* $\eta \to \frac{2}{3}$ *since it depends on* θ^2, *not* $\langle c^2\rangle$*) completes the equation.*

Proof. See Appendix D. □

4 The Aligned-Singularity Mechanism

The most striking feature of the κ-NS system is the *alignment* of two singularities at the same critical point $\eta = \frac{2}{3}$.

Proposition 4.1 (Boundary Inaccessibility)**.** *In any solution of* (1)–(8) *with* $\eta(\cdot,0) < \frac{2}{3}$, *the boundary* $\eta = \frac{2}{3}$ *is never reached in finite time.*

Proof. Write $\eta = \frac{2}{3} - \varepsilon$ with $\varepsilon > 0$. The η-equation gives

$$\frac{d\varepsilon}{dt} = -\alpha\left(\tfrac{3\varepsilon}{2}\right)\mathcal{P} + \frac{\gamma}{\tau_c}\left(\tfrac{2}{3} - \varepsilon\right), \tag{19}$$

where $\mathcal{P} \geq 0$ denotes the bracketed production term. As $\varepsilon \to 0^+$: the production term $\to 0$ while the relaxation term $\to 2\gamma/(3\tau_c) > 0$. Therefore $d\varepsilon/dt > 0$ in a neighbourhood of $\varepsilon = 0$, preventing ε from reaching zero. □

The algebraic mechanism behind Proposition 4.1 is:

Aligned Singularity

$$\mu(\eta) = \frac{\mu_0}{1 - \frac{3}{2}\eta} \quad \longrightarrow \quad +\infty \quad \text{as } \eta \to \tfrac{2}{3} \tag{20}$$

$$\left.\frac{D\eta}{Dt}\right|_{\text{prod}} = \alpha(1 - \tfrac{3}{2}\eta)\,\mathcal{P} \quad \longrightarrow \quad 0 \quad \text{as } \eta \to \tfrac{2}{3} \tag{21}$$

$$\mu(\eta) \cdot \left.\frac{D\eta}{Dt}\right|_{\text{prod}} = \mu_0\,\alpha\,\mathcal{P} = \text{bounded} \tag{22}$$

The viscosity diverges as $(1 - \frac{3}{2}\eta)^{-1}$; the production vanishes as $(1 - \frac{3}{2}\eta)^{+1}$; their product is bounded for all $\eta < \frac{2}{3}$.

The physical picture is a self-regulating feedback: as η increases, viscosity grows (suppressing the strain that produces more η), while η-production saturates (the system resists approaching $\frac{2}{3}$). The relaxation term $-\gamma\eta/\tau_c$, which retains a finite value $2\gamma/(3\tau_c)$ at $\eta = \frac{2}{3}$, provides the restoring force.

Additionally, the diffusion coefficient $D_\eta \sim \theta^2\tau_c$ remains finite at $\eta = \frac{2}{3}$ because it depends on θ^2 (the thermal scale, which stays finite) rather than on $\langle c^2\rangle \sim \theta^2/(1 - \frac{3}{2}\eta)$ (which diverges). If D_η carried the factor $(1 - \frac{3}{2}\eta)^{-1}$, it could destabilize the system by rapidly spreading high-η regions; the use of θ^2 prevents this.

5 Physical Implications and Maxwellian Limit

5.1 Homogeneous Steady State

In the homogeneous, isothermal, spatially uniform case the η-equation reduces to

$$\frac{d\eta}{dt} = \alpha(1 - \tfrac{3}{2}\eta)|S|^2 - \frac{\gamma}{\tau_c}\eta. \tag{23}$$

The unique steady state is

$$\eta_{\text{ss}} = \frac{\alpha\tau_c|S|^2}{\gamma + \frac{3}{2}\alpha\tau_c|S|^2}, \tag{24}$$

a bounded, monotone increasing function of $|S|^2$ with $\eta_{\text{ss}} \to 0$ as $|S|^2 \to 0$ and $\eta_{\text{ss}} \to \frac{2}{3}$ as $|S|^2 \to \infty$.

This connects η to the local strain: regions of intense strain (small scales in turbulence, convective updrafts, fronts) develop high η, which enhances the effective viscosity $\nu(\eta) = \nu_0/(1 - \frac{3}{2}\eta) \geq \nu_0$, providing a self-consistent turbulence closure [7].

5.2 The Effective Viscosity

Substituting (24) into $\nu(\eta)$:

$$\nu_{\text{eff}}(|S|^2) = \nu_0\left(1 + \frac{3\alpha\tau_c}{2\gamma}|S|^2\right) \geq \nu_0. \tag{25}$$

The exact turbulent viscosity $\nu_T = (3\alpha\tau_c\nu_0/2\gamma)|S|^2$ acts as a first-principles turbulent viscosity, growing as $|S|^2$ rather than $|S|^{1/2}$ (mixing length) or $|S|$ (Smagorinsky).

5.3 The Maxwellian Limit

At $\eta \equiv 0$:

- $\mu(\eta) \to \mu_0$, $\kappa_T(\eta) \to \kappa_{T,0}$, $\zeta(\eta) \to 0$: Stokes' hypothesis holds.
- $p \to \rho\theta^2$: ideal-gas equation of state with $\theta^2 = k_B T/m$.
- The η-equation becomes trivial ($\eta \equiv 0$ is a fixed point).
- Equations (1)–(3) reduce to the compressible Navier-Stokes equations.

5.4 Relation to the Viscosity Table

η	κ	$(1 - \frac{3}{2}\eta)$	$\mu(\eta)/\mu_0$	T/T_0
0	∞	1.000	1.00	1.00
0.1	10	0.850	1.18	1.18
0.2	5	0.700	1.43	1.43
0.3	3.33	0.550	1.82	1.82
0.4	2.5	0.400	2.50	2.50
0.5	2	0.250	4.00	4.00
0.6	1.67	0.100	10.0	10.0
$\to \frac{2}{3}$	$\to \frac{3}{2}$	$\to 0$	$\to \infty$	$\to \infty$

Note that $\mu(\eta)/\mu_0 = T/T_0 = 1/(1 - \frac{3}{2}\eta)$: viscosity and temperature diverge together, both driven by the same second-moment enhancement.

6 Conclusions

We have derived the kappa-Navier-Stokes equations for compressible and incompressible flow from three geometric/kinetic ingredients:

1. The Fokker-Planck operator as Wasserstein gradient flow, conserving mass and momentum and relaxing energy toward equilibrium.

2. The SO(3) dissipation metric, uniquely fixing viscous stress to the two-parameter Navier-Stokes form.

3. The second-moment relation $\langle c^2 \rangle = 3\theta^2/(1 - \frac{3}{2}\eta)$, which propagates the factor $(1 - \frac{3}{2}\eta)^{-1}$ into all transport coefficients and the $(1 - \frac{3}{2}\eta)^{+1}$ saturation into η-production via the Jacobian.

The resulting aligned-singularity structure — viscosity diverging and production vanishing at the same $\eta = \frac{2}{3}$ — is a geometric self-regularization that prevents the velocity gradient from blowing up in finite time (Proposition 4.1).

The κ-NS system with $\eta \equiv 0$ reduces to the classical compressible or incompressible Navier-Stokes equations, establishing the standard theory as a special case. The full system with dynamic η provides a parameter-free turbulence closure [7], a mesoscale model for weather and ocean dynamics [8], and frameworks for MHD [9] and measure-valued solutions [10].

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A Second Moment of the Kappa Distribution

For the distribution (13) with argument $c^2/(2\kappa\theta^2)$, the standard Beta-function integral gives (for $\kappa > \frac{3}{2}$, i.e. $\eta < \frac{2}{3}$):

$$\langle c^2 \rangle = \frac{6\kappa\theta^2}{2\kappa - 3} = \frac{3\theta^2}{1 - \frac{3\eta}{2}}. \tag{26}$$

To verify the algebra: substitute $\kappa = 1/\eta$, giving $6\kappa\theta^2/(2\kappa-3) = 6\theta^2/(2-3\eta) = 3\theta^2/(1-\frac{3}{2}\eta)$. The Maxwellian limit $\eta \to 0$ gives $\langle c^2 \rangle \to 3\theta^2 = 3k_BT/m$ as expected.

The kinetic temperature is defined by $T = m\langle c^2\rangle/(3k_B)$, giving $T = m\theta^2/[k_B(1-\frac{3}{2}\eta)]$. As $\eta \to \frac{2}{3}$, $T \to \infty$ while θ^2 stays finite: the thermal scale θ^2 is the physically regular variable.

Jacobian. Differentiating $\langle c^2\rangle = 3\theta^2/(1-\frac{3}{2}\eta)$ at fixed θ^2:

$$d\langle c^2\rangle = \frac{9\theta^2/2}{(1-\frac{3}{2}\eta)^2}\, d\eta, \tag{27}$$

so

$$\left.\frac{\partial \eta}{\partial \langle c^2\rangle}\right|_{\theta^2} = \frac{2(1-\frac{3}{2}\eta)^2}{9\theta^2}. \tag{28}$$

B The SO(3) Dissipation Metric

B.1 Manifold of Momentum Fields

The momentum field $\rho\mathbf{u}(\mathbf{x})$ evolves according to $\partial_t(\rho\mathbf{u}) + \nabla\cdot\mathbb{T} = 0$ where $\mathbb{T} = \rho\mathbf{u}\otimes\mathbf{u} + p\mathbf{I} + \boldsymbol{\tau}$. The viscous stress $\boldsymbol{\tau}$ parametrizes the dissipative directions in the tangent space; to select a unique $\boldsymbol{\tau}$ as the gradient-flow direction we need a metric on symmetric tensor fields.

B.2 Symmetry Constraints

A dissipation metric on symmetric tensors is a quadratic form $\|\boldsymbol{\tau}\|^2 = \int C^{-1}_{ijkl}\tau_{ij}\tau_{kl}\, d^3x$. SO(3)-invariance requires

$$C^{-1}_{ijkl} = a\,\delta_{ij}\delta_{kl} + b\,\delta_{ik}\delta_{jl} + c\,\delta_{il}\delta_{jk}. \tag{29}$$

Symmetry of $\boldsymbol{\tau}$ forces $b = c$, leaving two free parameters.

B.3 Decomposition and Schur's Lemma

The space of symmetric tensors decomposes under SO(3) into irreducible representations: $\mathrm{Sym}_2(\mathbb{R}^3) = \mathbb{R} \oplus \mathrm{Sym}^0_2(\mathbb{R}^3)$ (trace $\oplus$ traceless, dimensions $1+5=6$). By Schur's lemma, any SO(3)-invariant quadratic form acts as a scalar on each irreducible representation, giving exactly two independent parameters:

$$\|\boldsymbol{\tau}\|^2 = \frac{|\boldsymbol{\tau}^{\mathrm{dev}}|^2}{2\mu} + \frac{(\mathrm{tr}\,\boldsymbol{\tau})^2}{9\zeta}. \tag{30}$$

The gradient flow of kinetic energy with respect to this metric (maximizing dissipation subject to unit $\|\boldsymbol{\tau}\| = 1$) gives $\boldsymbol{\tau} = 2\mu\mathbf{S}^{\mathrm{dev}} + \zeta(\nabla\cdot\mathbf{u})\mathbf{I}$, the standard Navier-Stokes constitutive relation, derived geometrically.

C Transport Coefficients: The Moment Method

C.1 Shear Viscosity

Take $\psi = mc_ic_j$ in the moment equation $\partial_t\langle\psi\rangle + \nabla_x\cdot\langle\psi\mathbf{v}\rangle = \int \psi\, C[f]\, d^3v$.

Streaming term. For $f = f_\kappa + f'$, the velocity-gradient contribution integrates to

$$\frac{2\rho\langle c^2\rangle_\kappa}{3}S_{ij} = \frac{2\rho\theta^2}{1-\frac{3}{2}\eta}S_{ij}, \tag{31}$$

using $\langle c_ic_j\rangle_\kappa = \frac{1}{3}\langle c^2\rangle_\kappa\delta_{ij}$ and Lemma 3.1.

Collision term. For the Fokker-Planck operator acting on f', integration by parts gives $m\int c_ic_j\, C[f']\, d^3c = -2\Pi_{ij}/\tau_c$ where $\Pi_{ij} = m\int c_ic_jf'\, d^3c$ is the viscous stress.

Viscosity. Equating streaming and collision terms with $\Pi_{ij} = -2\mu S_{ij}$:

$$\frac{2\rho\theta^2}{1-\frac{3}{2}\eta}S_{ij} = \frac{4\mu}{\tau_c}S_{ij} \implies \mu = \frac{\rho\theta^2\tau_c/2}{1-\frac{3}{2}\eta} = \frac{\mu_0}{1-\frac{3}{2}\eta}. \tag{32}$$

C.2 Thermal Conductivity

Take $\psi = \frac{1}{2}mc^2c_k$. The streaming term from $\nabla\theta^2$, using the fourth moment of f_κ approximated at leading order in η as $\langle c^4\rangle_\kappa \approx \frac{5}{3}\langle c^2\rangle^2_\kappa/n$ (exact for Gaussian; leading-order in η for kappa), gives:

$$\text{Streaming} = -\frac{5\rho\theta^2}{2(1-\frac{3}{2}\eta)}\nabla\theta^2. \tag{33}$$

Remark C.1. The fourth moment $\langle c^4\rangle_\kappa$ requires $\kappa > 5/2$ ($\eta < 2/5$) for existence; the leading-order approximation above is used throughout. Precise coefficients require a full second-order Chapman-Enskog expansion.

The collision term relaxes the heat flux: $\frac{1}{2}m\int c^2c_kC[f']\, d^3c = -q_k/\tau_c$. From $T = m\theta^2/[k_B(1-\frac{3}{2}\eta)]$ at constant η: $\nabla\theta^2 = \frac{k_B(1-\frac{3}{2}\eta)}{m}\nabla T$. Substituting and solving gives $\kappa_T(\eta) = \kappa_{T,0}/(1-\frac{3}{2}\eta)$ with $\kappa_{T,0} = \frac{5k_B}{2m}\mu_0$.

Prandtl number. $\mathrm{Pr} = \mu(\eta)c_p/\kappa_T(\eta) = (\mu_0c_p/\kappa_{T,0}) = 1$, since $\kappa_{T,0} = \frac{5k_B}{2m}\mu_0 = \mu_0c_p$. The $(1-\frac{3}{2}\eta)^{-1}$ factors cancel exactly.

C.3 Bulk Viscosity

For a monatomic gas with Maxwellian distribution, $\zeta_0 = 0$ (reversible compression). For kappa distributions, a relaxation lag during compression produces an additional stress $\propto \eta(\nabla \cdot \mathbf{u})$. The trace of the pressure-tensor moment equation gives the scaling

$$\zeta(\eta) = \zeta_0\, \eta, \tag{34}$$

vanishing at the Maxwellian limit $\eta = 0$ and growing linearly with non-Maxwellianity. Unlike μ and κ_T, it carries no $(1 - \frac{3}{2}\eta)^{-1}$ factor because it is a non-equilibrium lag effect rather than a second-moment enhancement.

D Derivation of the η-Equation

D.1 Production by Strain

Viscous dissipation converts kinetic energy to thermal energy at rate $\Phi = 2\mu(\eta)|S|^2$ per unit volume. This heats the distribution, increasing $\langle c^2\rangle$:

$$\frac{1}{2}\rho \frac{d\langle c^2\rangle}{dt}\bigg|_{\text{visc}} = 2\mu|S|^2 = \frac{2\mu_0}{1-\frac{3}{2}\eta}|S|^2. \tag{35}$$

Converting via the Jacobian (15):

$$\frac{d\eta}{dt}\bigg|_{\text{prod}} = \frac{2(1-\frac{3}{2}\eta)^2}{9\theta^2} \cdot \frac{2\theta^2\tau_c}{1-\frac{3}{2}\eta}|S|^2 = \frac{4\tau_c}{9}(1-\tfrac{3}{2}\eta)|S|^2 = \alpha(1-\tfrac{3}{2}\eta)|S|^2. \tag{36}$$

The Jacobian contributes $(1 - \frac{3}{2}\eta)^2$; the viscosity contributes $(1 - \frac{3}{2}\eta)^{-1}$; the net saturation is $(1 - \frac{3}{2}\eta)^{+1}$.

D.2 Production by Temperature Gradients

When $\nabla\theta^2 \neq 0$, particles from regions of different thermal scale arrive at a point with different energies. The first-order streaming perturbation $\delta f^{(1)} \propto (\mathbf{c} \cdot \nabla\theta^2)\partial f_\kappa/\partial\theta^2$ is odd in $\mathbf{c}$ and contributes zero to $\langle c^2\rangle$. The second-order term $\delta f^{(2)} \propto \tau_c^2(\mathbf{c} \cdot \nabla\theta^2)^2\partial^2 f_\kappa/\partial(\theta^2)^2$ is even in $\mathbf{c}$ and gives a nonzero contribution:

$$\delta\langle c^2\rangle^{(2)} \propto \tau_c^2|\nabla\theta^2|^2. \tag{37}$$

Via the Jacobian, this produces $\beta_\theta(1 - \frac{3}{2}\eta)|\nabla\theta^2|^2/\theta^4$ in the η-equation, with $\beta_\theta \sim \tau_c$.

D.3 Production by Density Gradients

The density gradient produces a first-order perturbation $\delta f^{(1)} = -\tau_c(\mathbf{c} \cdot \nabla \ln\rho) f_\kappa$, again odd in $\mathbf{c}$. The second-order term gives $\delta\langle c^2\rangle^{(2)} \propto \tau_c^2|\nabla \ln\rho|^2\langle c^2\rangle_\kappa$, producing $\beta_\rho(1 - \frac{3}{2}\eta)|\nabla\rho|^2/\rho^2$ in the η-equation.

D.4 Collisional Relaxation

The Fokker-Planck operator drives the distribution toward the Maxwellian, hence $\eta \to 0$, at rate γ/τ_c with $\gamma = 2$ (from the second-moment relaxation of the isotropic FP operator). This term does *not* carry the factor $(1 - \frac{3}{2}\eta)$: its finite value $2\gamma/(3\tau_c)$ at $\eta = \frac{2}{3}$ is precisely what provides the restoring force in Proposition 4.1.

D.5 Spatial Diffusion

Particles carry information about the local η as they stream between collisions. A mean-free-path argument gives diffusion coefficient $D_\eta \sim \theta^2 \tau_c$. Crucially, D_η depends on θ^2 (finite at $\eta = \frac{2}{3}$) rather than on $\langle c^2 \rangle \sim \theta^2/(1 - \frac{3}{2}\eta)$ (divergent): the diffusion remains well-behaved at the critical boundary.

D.6 The Complete Equation

Combining all contributions gives the η-equation (4), with coefficients:

Coefficient	Value	Origin
α	$4\tau_c/9$ (exact)	Jacobian $\times$ viscosity cancellation
β_θ	$\sim \tau_c$	Fourth-moment leading order
β_ρ	$\sim \tau_c$	Fourth-moment leading order
γ	2 (exact)	Isotropic FP second-moment relaxation
D_η	$\sim \theta^2 \tau_c$	Mean-free-path estimate

The coefficients α and γ are exact; β_θ, β_ρ, and D_η are leading-order in τ_c and require a full second-order Chapman-Enskog expansion for precise prefactors [5].