Reduction of the Multivariate Quadratic (MQ) System to a Trivial Equation

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The MQ System

Let $\mathbb{F}$ be a field. Consider the following system of m multivariate quadratic equations in n variables $x_1, \ldots, x_n \in \mathbb{F}^n$:

$$\begin{aligned}
f_1(x_1, \ldots, x_n) &= \sum_{1 \le i \le j \le n} a_{ij}^{(1)}\, x_i x_j + \sum_{1 \le i \le n} b_i^{(1)}\, x_i + c^{(1)} = 0, \\
f_2(x_1, \ldots, x_n) &= \sum_{1 \le i \le j \le n} a_{ij}^{(2)}\, x_i x_j + \sum_{1 \le i \le n} b_i^{(2)}\, x_i + c^{(2)} = 0, \\
&\;\vdots \\
f_m(x_1, \ldots, x_n) &= \sum_{1 \le i \le j \le n} a_{ij}^{(m)}\, x_i x_j + \sum_{1 \le i \le n} b_i^{(m)}\, x_i + c^{(m)} = 0.
\end{aligned}$$

The goal is to find a solution $v = [v_1, \ldots, v_n] \in \mathbb{F}^n$ of the above system.

Reduction: $n = 2$, $m = 1$

For $n = 2$ variables x_1, x_2 and a single quadratic equation, the MQ system reduces to:

$$f(x_1, x_2) = a_{11}\, x_1^2 + a_{12}\, x_1 x_2 + a_{22}\, x_2^2 + b_1\, x_1 + b_2\, x_2 + c_1 = 0.$$

Treating this as a quadratic in x_1, `FullSimplify` yields:

$$a_{11}\, x_1^2 + x_1\,(a_{12}\, x_2 + b_1) + x_2\,(a_{22}\, x_2 + b_2) + c_1 = 0,$$

and completing the square in x_1 gives:

$$\frac{(2\, a_{11}\, x_1 + a_{12}\, x_2 + b_1)^2}{4\, a_{11}} - \frac{-4\, a_{11}\, a_{22}\, x_2^2 - 4\, a_{11}\, b_2\, x_2 - 4\, a_{11}\, c_1 + a_{12}^2\, x_2^2 + 2\, a_{12}\, b_1\, x_2 + b_1^2}{4\, a_{11}} = 0.$$

Resultant for Two Quadratics in Two Variables

Given two quadratic polynomials in x, y:

$$
\begin{aligned}
p_1(x,y) &= a_1\,x^2 + a_2\,xy + a_3\,y^2 + b_1\,x + b_2\,y + c_1,\\
p_2(x,y) &= d_1\,x^2 + d_2\,xy + d_3\,y^2 + e_1\,x + e_2\,y + c_2,
\end{aligned}
$$

the resultant $\mathrm{Res}_y(p_1,p_2)$ eliminates y and yields a univariate polynomial in x. The full expanded form is:

$$
\begin{aligned}
\mathrm{Res}_y(p_1,p_2) = {} & a_1^2d_3^2\,x^4 - a_1a_2d_2d_3\,x^4 - a_1a_2d_3e_2\,x^3 - 2\,a_1a_3c_2d_3\,x^2\\
& - 2\,a_1a_3d_1d_3\,x^4 + a_1a_3d_2^2\,x^4 + 2\,a_1a_3d_2e_2\,x^3 - 2\,a_1a_3d_3e_1\,x^3\\
& + a_1a_3e_2^2\,x^2 + 2\,a_1c_1d_3^2\,x^2 + a_2^2c_2d_3\,x^2 + a_2^2d_1d_3\,x^4\\
& + a_2^2d_3e_1\,x^3 - a_2a_3c_2d_2\,x^2 - a_2a_3c_2e_2\,x - a_2a_3d_1d_2\,x^4\\
& - a_2a_3d_1e_2\,x^3 - a_2a_3d_2e_1\,x^3 - a_2a_3e_1e_2\,x^2 - a_2b_1d_3e_2\,x^2\\
& + a_2b_1d_2d_3\,x^3 + a_2b_2d_1d_3\,x^3 - a_2b_2d_2^2\,x^3 + 2\,a_2c_1d_3e_2\,x\\
& + a_3^2d_1^2\,x^4 + 2\,a_3^2d_1e_1\,x^3 + a_3^2e_1^2\,x^2 - 2\,a_3b_1c_2d_3\,x\\
& - 2\,a_3b_1d_1d_3\,x^3 + 2\,a_3b_1d_1e_2\,x^2 - 2\,a_3b_1d_2e_1\,x^2 + a_3b_1d_2^2\,x^3\\
& + a_3b_1e_2^2\,x - a_3b_2c_2d_2 - a_3b_2c_2e_2\,x + 2\,a_3b_2d_1d_2\,x^2 - 2\,a_3c_1c_2d_3\\
& - 2\,a_3c_1d_1d_3\,x^2 + a_3c_1d_2^2\,x^2 + 2\,a_3c_1d_2e_2\,x - 2\,a_3c_1d_3e_1\,x\\
& + a_3c_1e_2^2 + b_1^2d_3^2\,x^2 - b_1b_2d_2d_3\,x^2 + 2\,b_1c_1d_3^2\,x\\
& + b_2^2d_1d_3\,x^2 + b_2^2d_3e_1\,x - b_2c_1d_2d_3\,x - b_2c_1d_3e_2 + c_1^2d_3^2.
\end{aligned}
$$

The alternative factored form is:

$$
\begin{aligned}
\mathrm{Res}_y(p_1,p_2) = d_3\Big(& x^2\big(-a_2(a_1x+b_1)(d_2x+e_2) + d_3(a_1x+b_1)^2 + a_2^2\big(c_2 + x(d_1x+e_1)\big)\big)\Big)\\
& - b_2\Big(x(a_1x+b_1)(d_2x+e_2) - 2\,a_2\,x\big(c_2 + x(d_1x+e_1)\big)\Big)\\
& + c_1(d_2x+e_2) + c_1x\big(2\,a_1d_3x - a_2(d_2x+e_2) + 2\,b_1d_3\big)\\
& + b_2^2\big(c_2 + x(d_1x+e_1)\big) + c_1^2\,d_3\\
& - a_3\Big(x\Big(a_1x\big(c_2d_3 + x(2\,d_3(d_1x+e_1) - d_2^2x) - 2\,d_2e_2x - e_2^2\big)\\
& \qquad + a_2(d_2x+e_2)\big(c_2 + x(d_1x+e_1)\big) - d_3\big(d_2^2x - e_2^2\big)\Big)\\
& \quad + b_2(d_2x+e_2)\big(c_2 + x(d_1x+e_1)\big)\\
& \quad + c_1\big(2\,c_2d_3 + x\big(2\,d_3(d_1x+e_1) - d_2^2x\big) - 2\,d_2e_2x - e_2^2\big)\Big)\\
& + a_3^2\big(c_2 + x(d_1x+e_1)\big)^2.
\end{aligned}
$$

Setting $\mathrm{Res}_y(p_1,p_2) = 0$ yields a univariate polynomial in x alone, solvable by standard root-finding methods. Each root x^* is back-substituted into p_1 or p_2 to recover y, completing the reduction of the MQ system to a trivial univariate equation.