

# Formula for Motion Threshold per Grain Size for Graded Sediments in Steady Flows

Daniel Buscombe, Daniel Conley

School of Marine Science and Engineering, University of Plymouth, Plymouth, United Kingdom



## Introduction

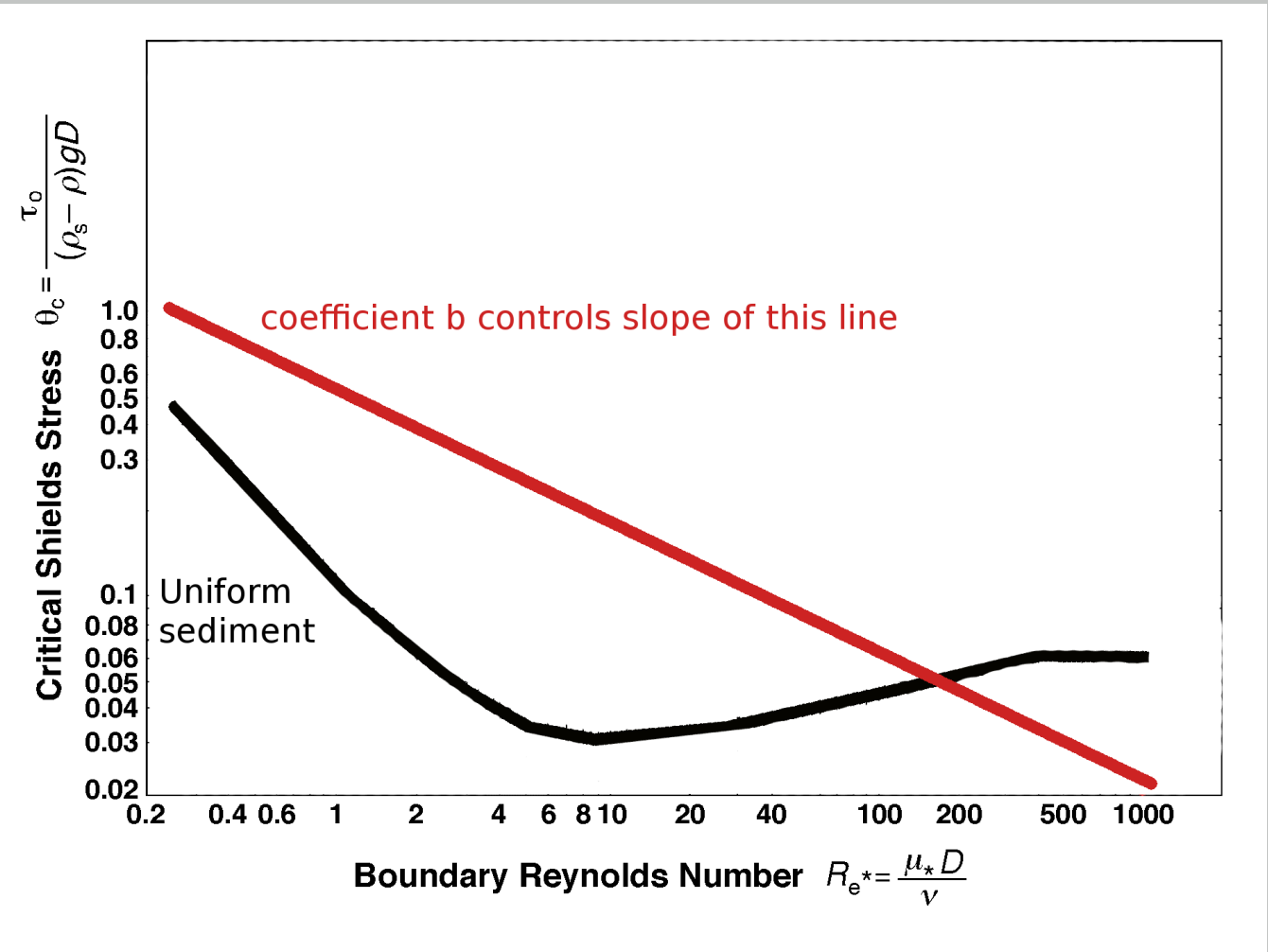
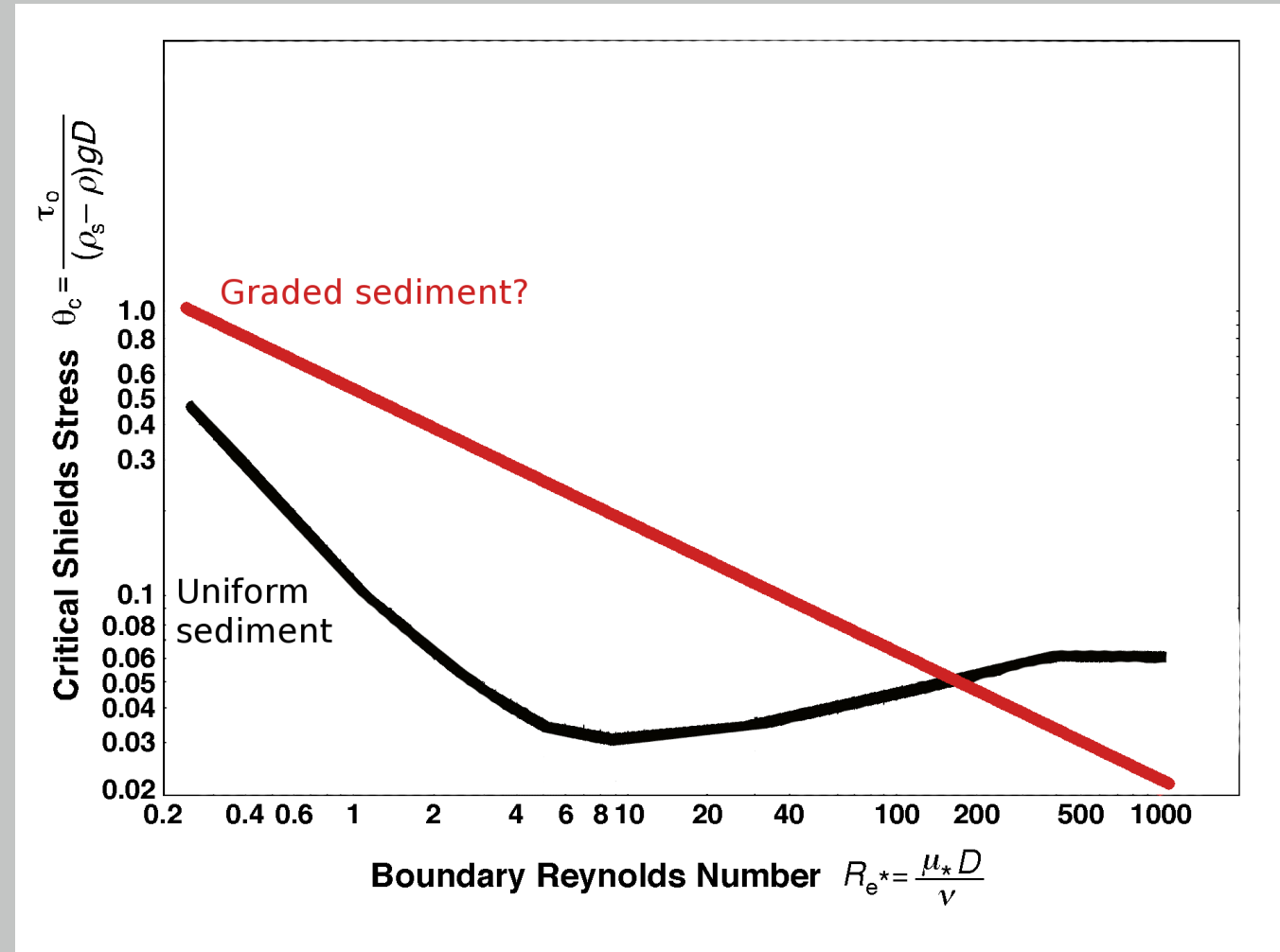
- ▶ The response of grain size at a given shear stress different from what it would be if the bed were uniform
- ▶ Most sediment mobilisation models assume a single ‘representative’ grain size
- ▶ Those parameterisations based on a distribution of sizes are unsatisfactory

## Different Approaches

- ▶ Physical: balance of all forces
- ▶ Complicated to formulate/measure, especially mixtures
- ▶ Engineering:  $\theta > \theta_c$  for sediment to move
- ▶  $Q = F[\theta - \theta_c]$  = nonlinear function of excess shear stress
- ▶  $\theta = F\left(\frac{u_*^2}{D}\right)$ ,  $\theta_c = F\left(\frac{u_{*c}^2}{\nu}\right)$

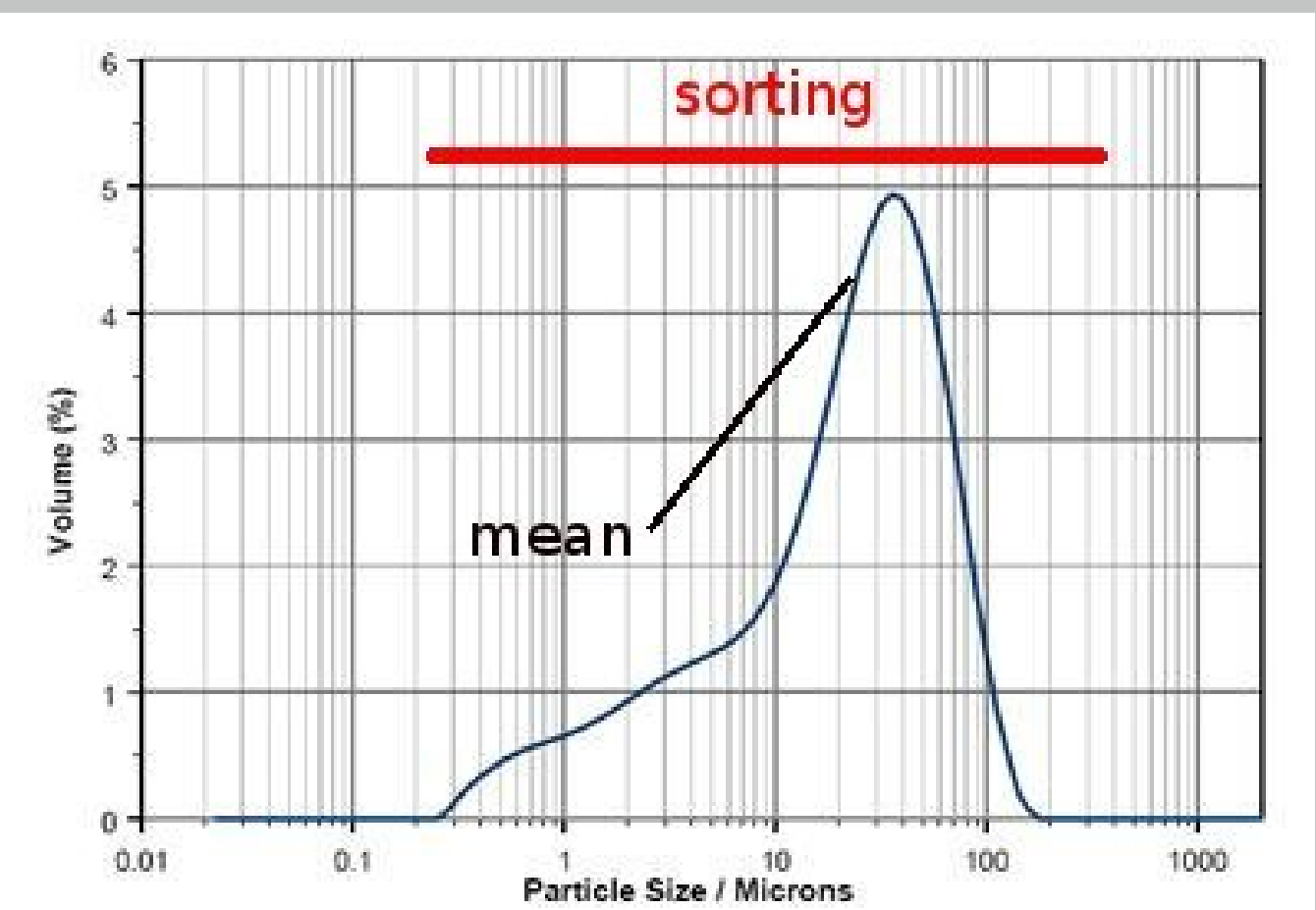
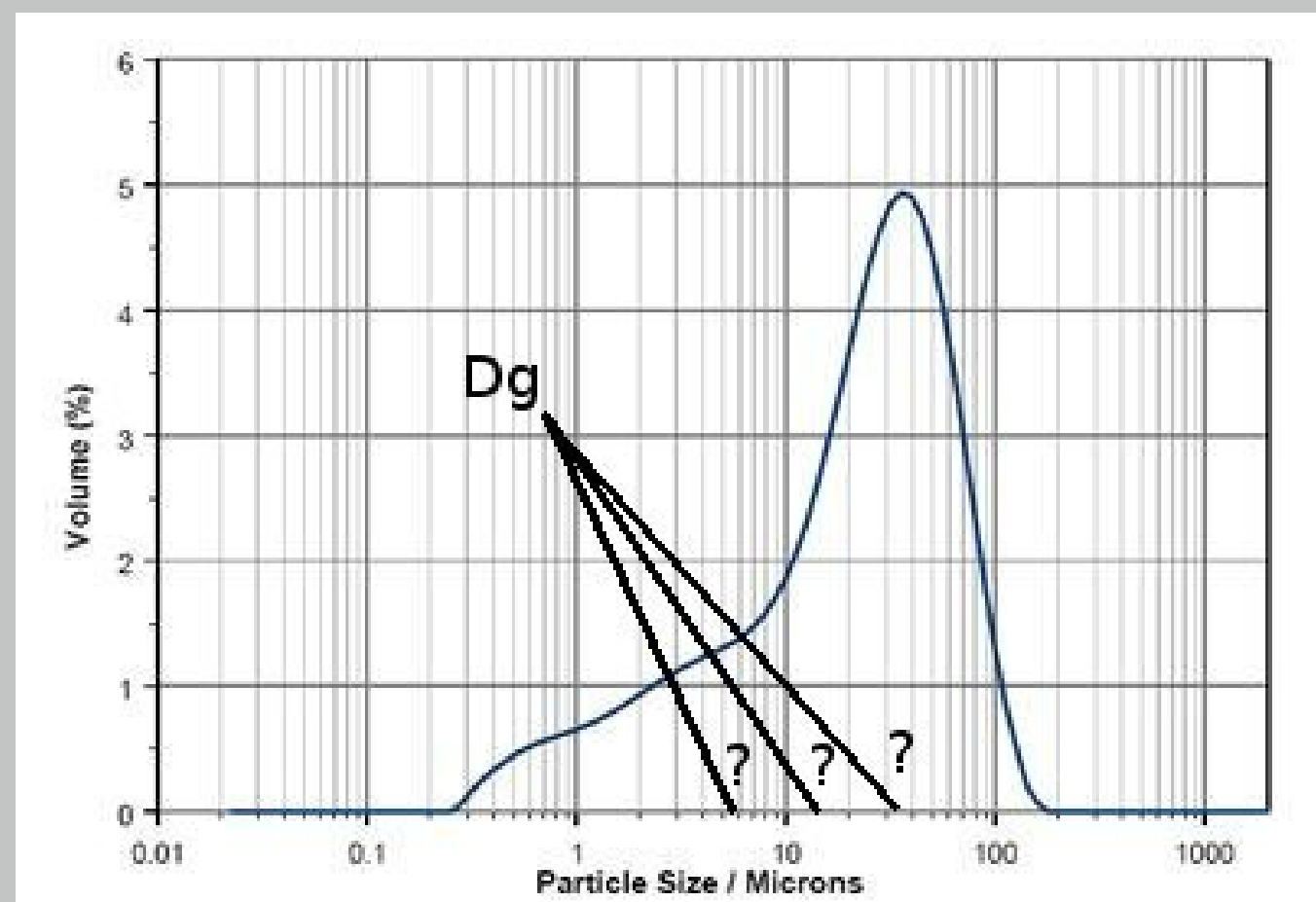
## Apparent Threshold for i Grain Sizes

- ▶  $Q_i = F[\theta_g - \theta_{ci}] P_i$  typically over-predicts fines
- ▶ Instead, threshold for fines raised and for coarse lowered
- ▶ Einstein (1950):  $\frac{\theta_{c1}}{\theta_{c2}} \approx \frac{D_2}{D_1}$
- ▶ Define apparent threshold:
- ▶  $\theta'_{ci} = \theta_{cg} \left(\frac{D_i}{D_g}\right)^{b-1}$



## Problems with ‘b’

- ▶ **b** determined by matching modelled and measured  $Q_i$ 
  - ▶ This introduces an unnecessary level of abstraction
- ▶ all include arbitrary breakpoints at some **D**
- ▶ 3 known parameterizations, none universally applied. All are based on matching predicted and observed sediment transport rates
- ▶ Large degree of empiricism
- ▶ 2 do not include sorting,  $\sigma$



## A New Approach

- ▶ We propose an approach based on the inferred measured particle size-distribution of the mobilized sediment,  $P_{mi}$ , rather than transport rate per fraction  $Q_i$ 
  - ▶ This avoid uncertainties in direct modelling of  $Q_i$  (specification of **F**)
  - ▶ Depth-integrated transport rate  $Q = U_s L$
  - ▶ Depth-integrated transport load  $L = \int_0^h C(z) dz$
  - ▶ Assume **L** is linearly proportional to  $(\theta - \theta_c)$
  - ▶ Rate per fraction  $Q_i = U_s (\theta_g - \theta'_{ci}) P_i$
- ▶ Infer  $P_{mi} = \frac{\int U_s (\theta_g - \theta'_{ci}) P_i dt}{\int \sum U_s (\theta_g - \theta'_{ci}) P_i dt}$
- ▶ Given  $\theta_g$ ,  $L_i$ ,  $P_i$  and  $D_g$ , we fit **b** using:
$$\theta_g - \theta_{cg} \left(\frac{D_i}{D_g}\right)^{b-1} P_i$$
- ▶ 
$$= \frac{\theta_g - \theta_{cg} \left(\frac{D_i}{D_g}\right)^{b-1} P_i}{\sum \left(\theta_g - \theta_{cg} \left(\frac{D_i}{D_g}\right)^{b-1} P_i\right)}$$

## Summary of experimental conditions for the data used in this study

- ▶ Non-linear optimization of this simple equation is used to find the optimal form of a weighting (so-called ‘hiding’) function used to modify critical entrainment criteria to provide the best possible fit with data.
- ▶ 12 sets of published data from flume experiments. Total of 81 different mixed sand and gravel beds and flow conditions (grain Reynolds numbers between 1 and 10,000)

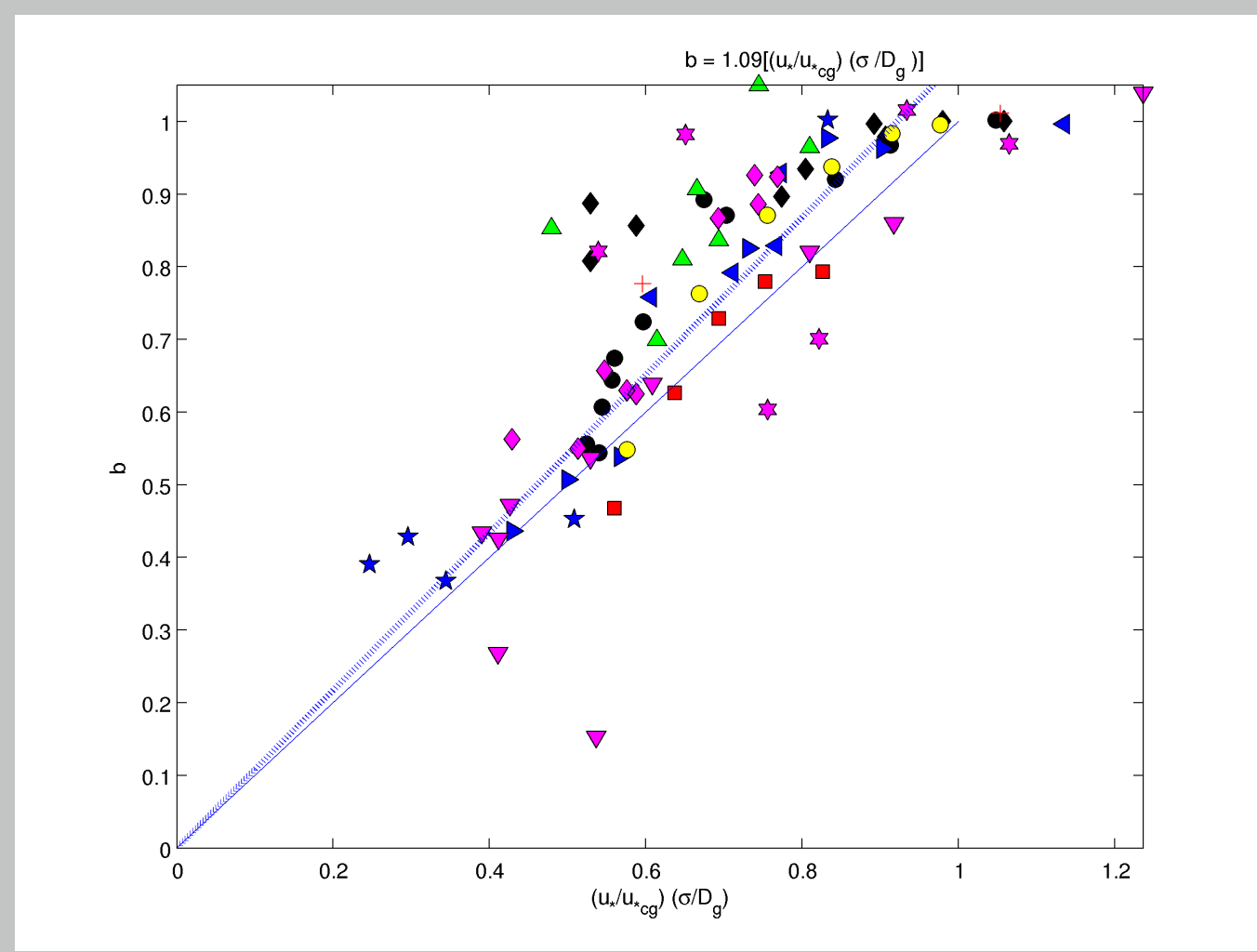
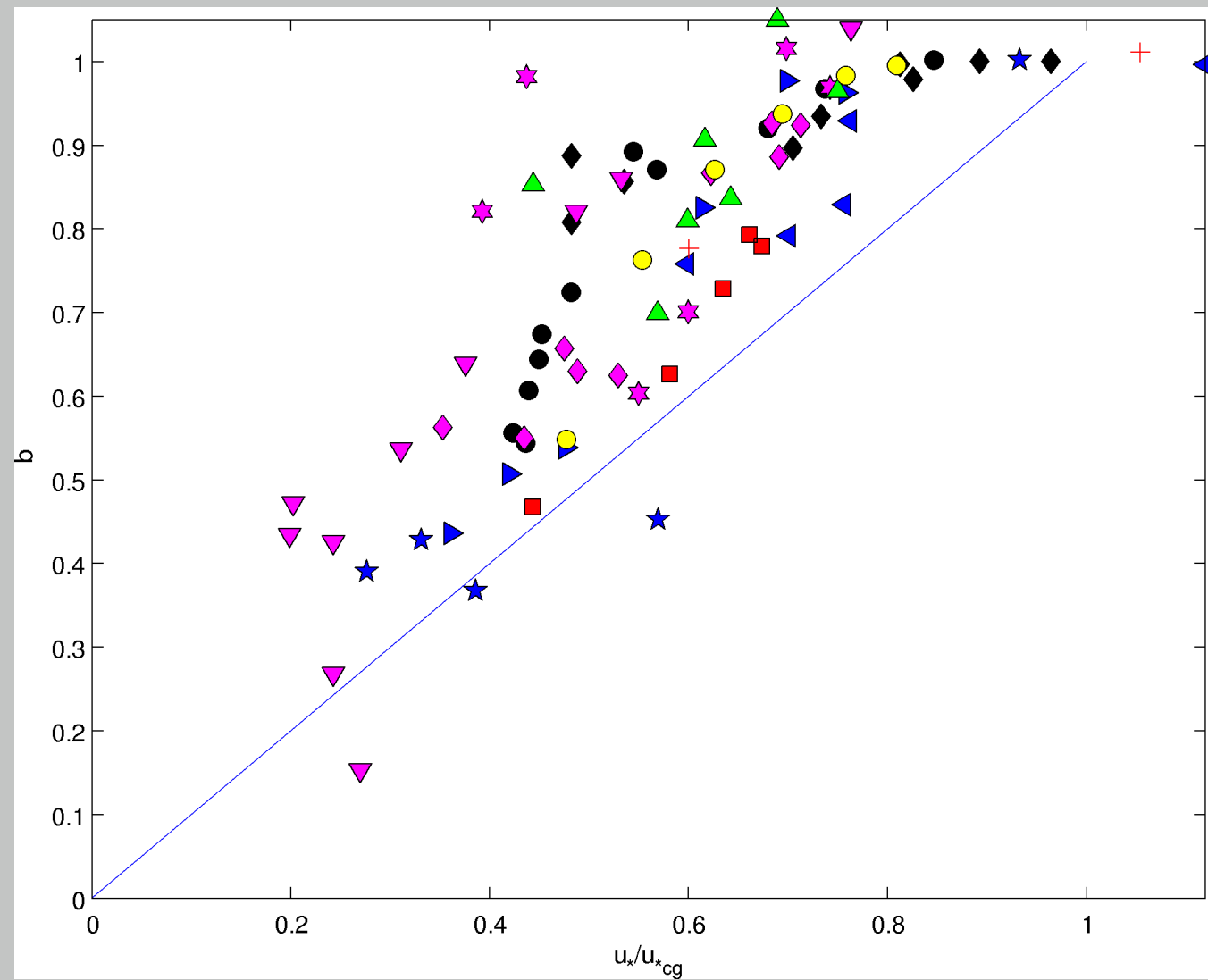
Experiment	$\theta_g \times 10^{-3}$	$D_i/D_g$	$D_g$ (mm)	$\sigma$ (mm)	mean B
Published					
Day (1980)	5.2→20.8	0.02→3.49	2.5	2.48	3.36
Day (1980)	7.07→28.3	0.03→3.03	1.84	1.51	9.62
Blom & Kleinans (1999)	8.36→20.9	0.055→4.97	2.8→3.5	3.55→3.95	5.49
Kuhnle (1993)	4.7→53.47	0.22→8.35	0.97	1.52	0.52
Kuhnle (1993)	17.06→59.67	0.12→4.79	1.7	2.14	1.24
Kuhnle (1993)	3.92→17.34	0.081→3.21	2.6	2.45	3.65
Wilcock & McArdeU (1993)	0.53→7.86	0.017→9.12	5.1→6	7.97→13.42	6.34
Wilcock et al (2001)	4.77→20.91	0.0058→3.11	14.5→17.1	9.95→12	0
Wilcock et al (2001)	4.6→18.81	0.0062→3.19	14.2→16.2	11.2→11.9	0
Sun & Donahue (2000)	6.07→17.32	0.18→7.9	0.7→1.08	0.55→1.16	0
Tait et al. (1992)	11.22→32.46	0.03→2.05	3.1→3.4	1.48→1.57	0
Kuhnle (1992)	12.34→35.51	0.0079→4.26	15.72	18.96	7.91

$D_g$  = arithmetic mean;  $\sigma$  = arithmetic sorting; **B** = bimodality index (0=unimodal)

## Analysis reveals functional form of b

▶  $b = \left(\frac{u_*}{u_{*cg}}\right)$

▶  $b = 1.09 \left(\frac{u_*}{u_{*cg}}\right) \left(\frac{\sigma}{D_g}\right)$

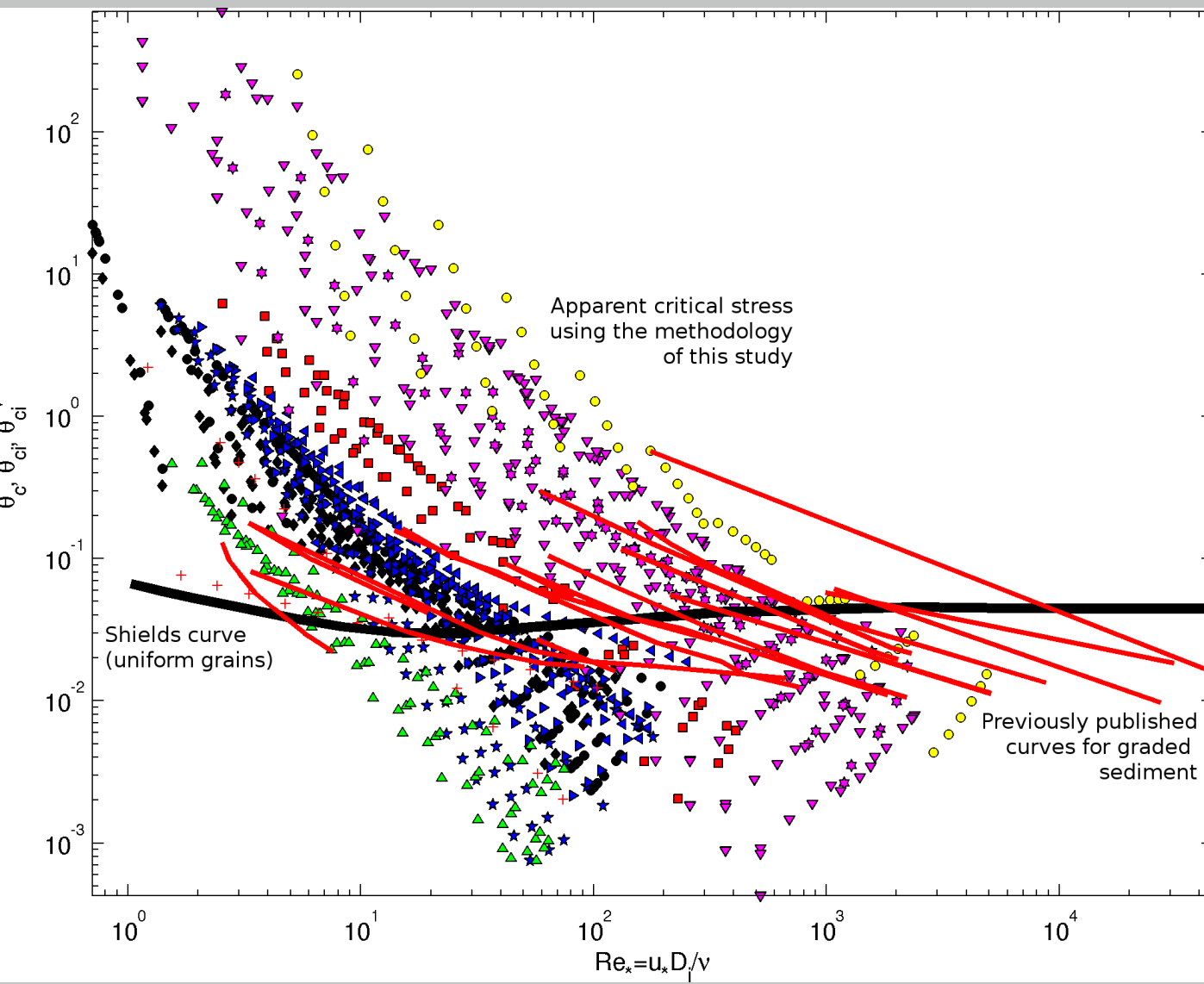


## Results

- ▶ Analysis reveals that motion threshold is dependent on:
  1. excess shear stress
  2. ratio of particle (arithmetic) sorting and mean grain size
- ▶ Our new formula for the mobilization of graded sediment:
$$\theta'_{ci} = \theta_{cg} \left(\frac{D_i}{D_g}\right)^{\left(1.09 \left(\frac{u_*}{u_{*cg}}\right) \left(\frac{\sigma}{D_g}\right)\right) - 1}$$
- ▶ Based on routinely measured quantities, and easily calculable based on mean grain size
- ▶ This new relation outperforms existing formulae in 11 out of 12 data sets, and with an expected error of  $\pm 20\%$

## Synthesis

- ▶ A simple deterministic equation, in non-dimensional form, is proposed for fraction-specific apparent critical shear stress and mobilized particle size distribution
- ▶ It predicts that  $\theta'_{ci}$  varies over more than 5 orders of magnitude for graded sediment, compared to a 1 order variation in  $\theta_{cg}$  for non-graded sediment (dark solid line in Figure adjacent) and 2 orders in  $\theta_{ci}$  found by previous studies (the light solid lines in Figure).
- ▶ The slope of the relation between  $R_e$  and threshold condition is apparently steeper than previously thought





# Formula for Motion Threshold per Grain Size for Graded Sediments in Steady Flows

Daniel Buscombe, Daniel Conley

School of Marine Science and Engineering, University of Plymouth, Plymouth, United Kingdom



## Introduction

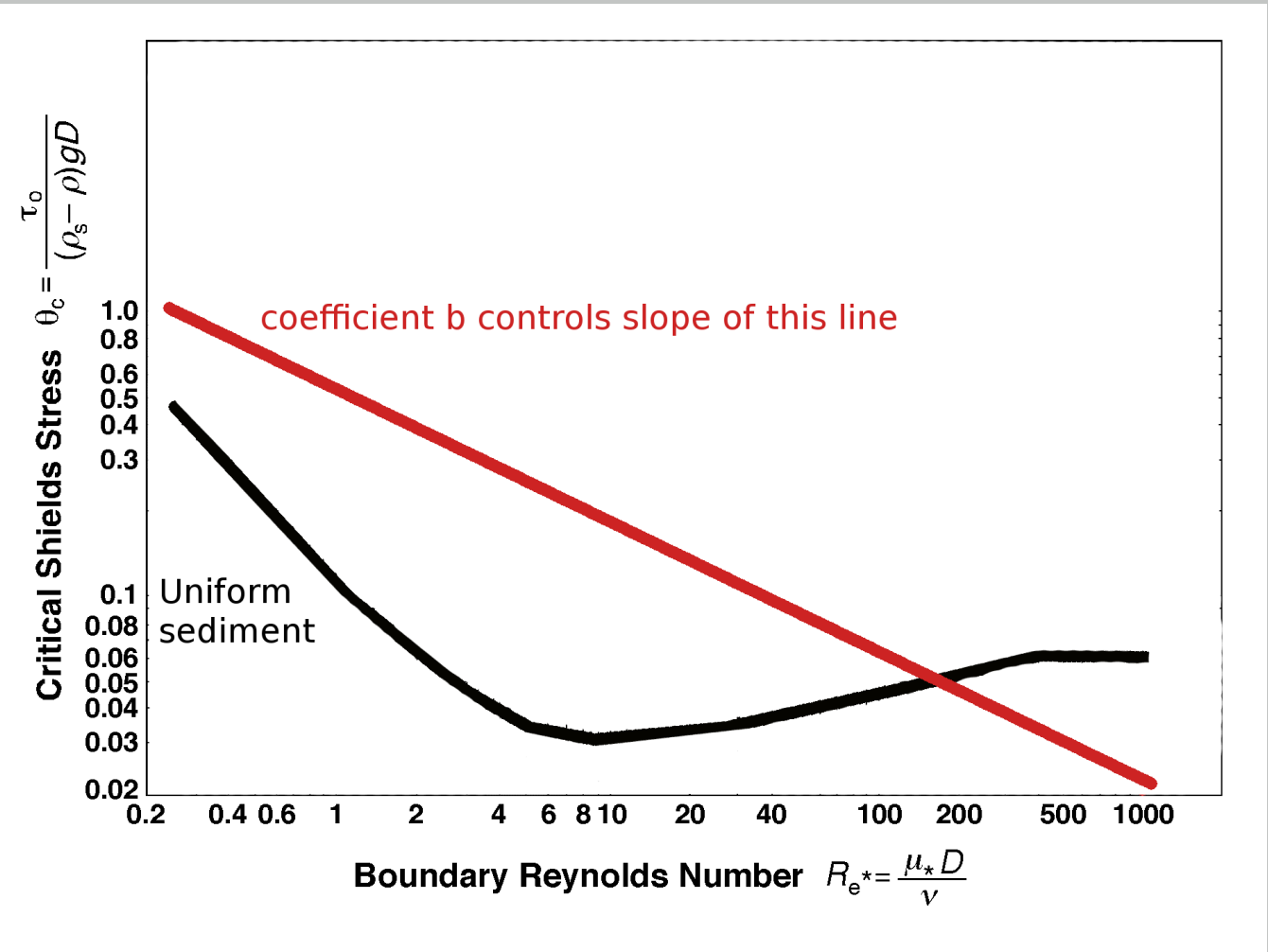
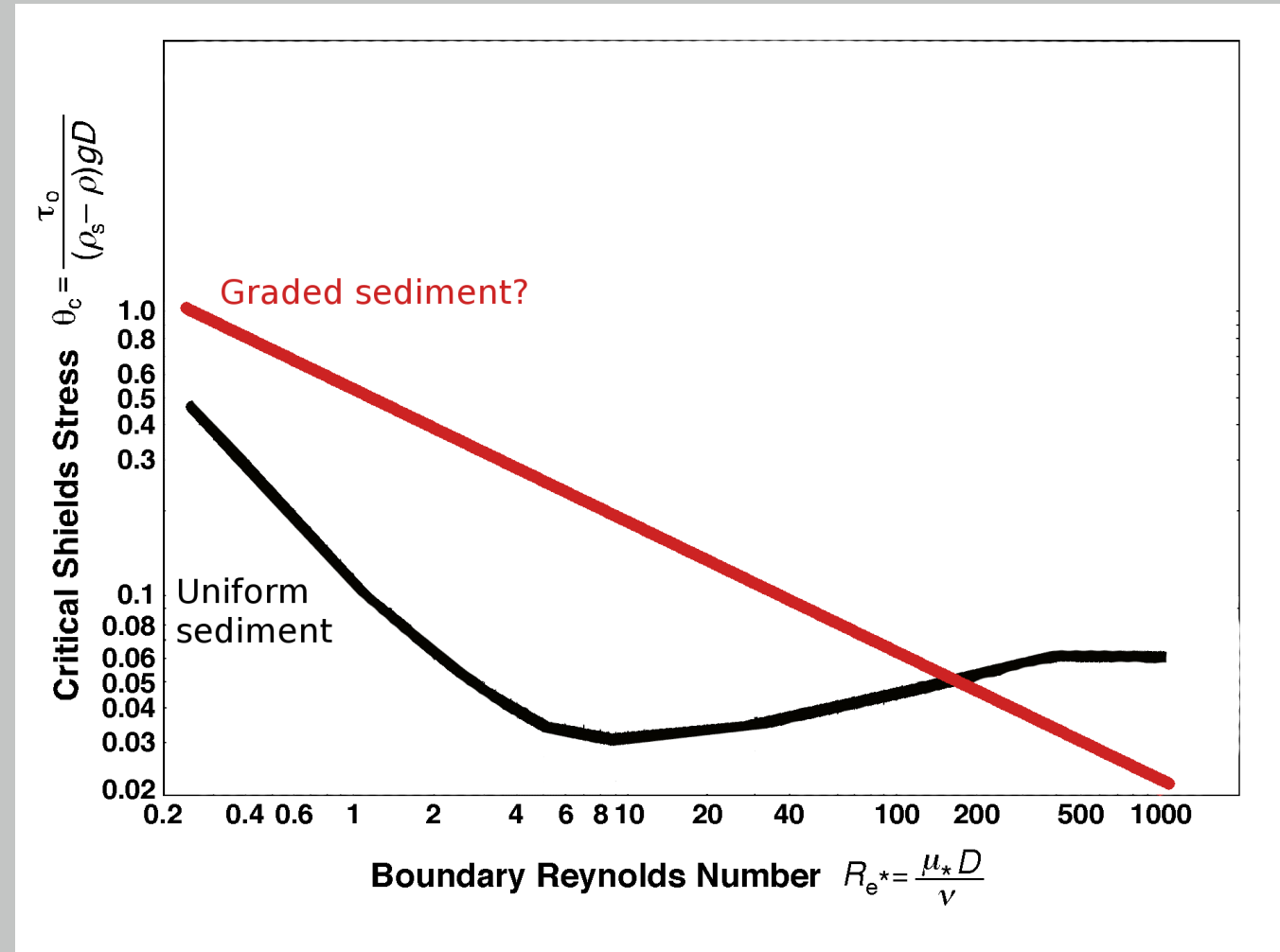
- ▶ The response of grain size at a given shear stress different from what it would be if the bed were uniform
- ▶ Most sediment mobilisation models assume a single ‘representative’ grain size
- ▶ Those parameterisations based on a distribution of sizes are unsatisfactory

## Different Approaches

- ▶ Physical: balance of all forces
- ▶ Complicated to formulate/measure, especially mixtures
- ▶ Engineering:  $\theta > \theta_c$  for sediment to move
- ▶  $Q = F[\theta - \theta_c]$  = nonlinear function of excess shear stress
- ▶  $\theta = F\left(\frac{u_*^2}{D}\right)$ ,  $\theta_c = F\left(\frac{u_{*D}}{\nu}\right)$

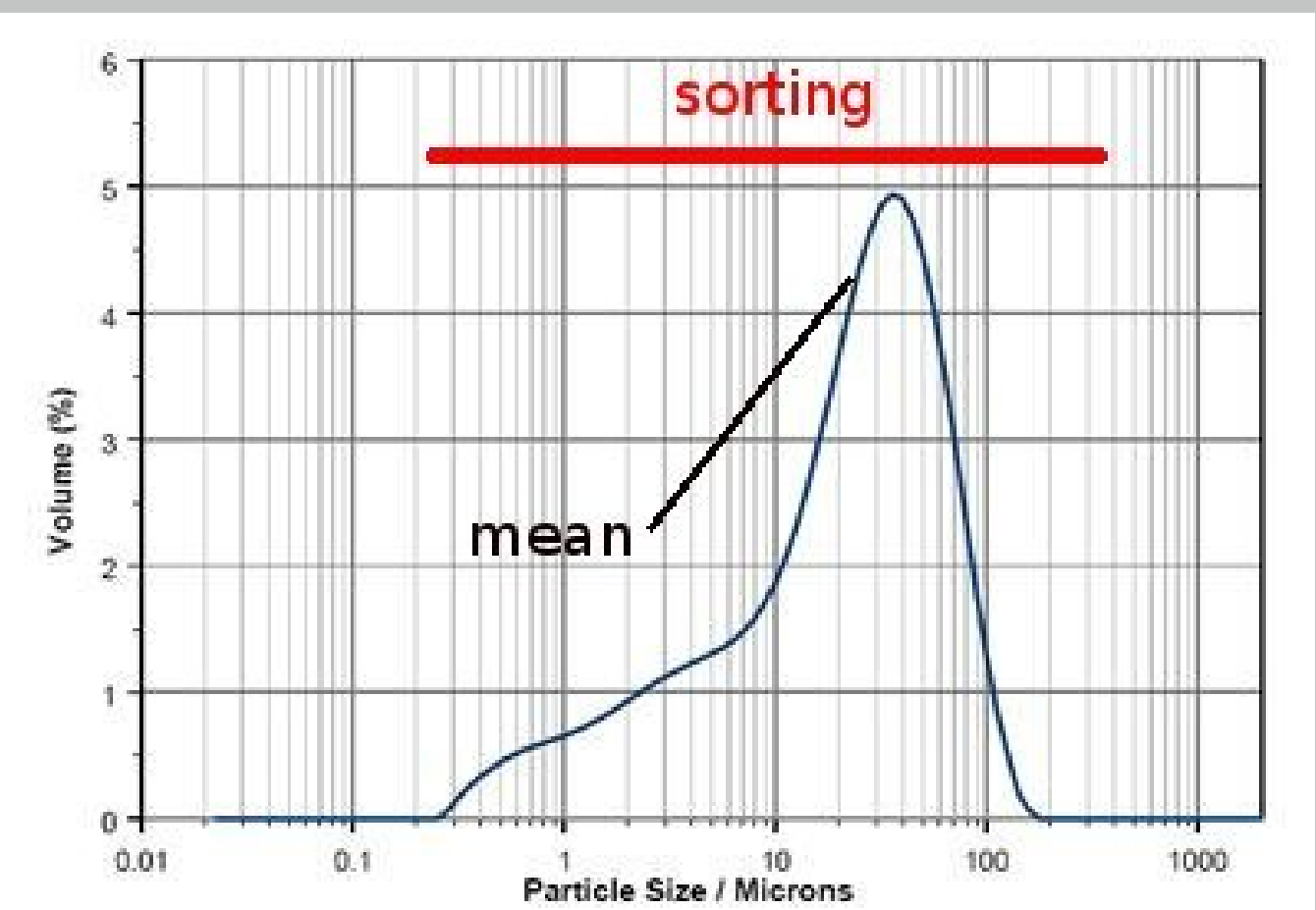
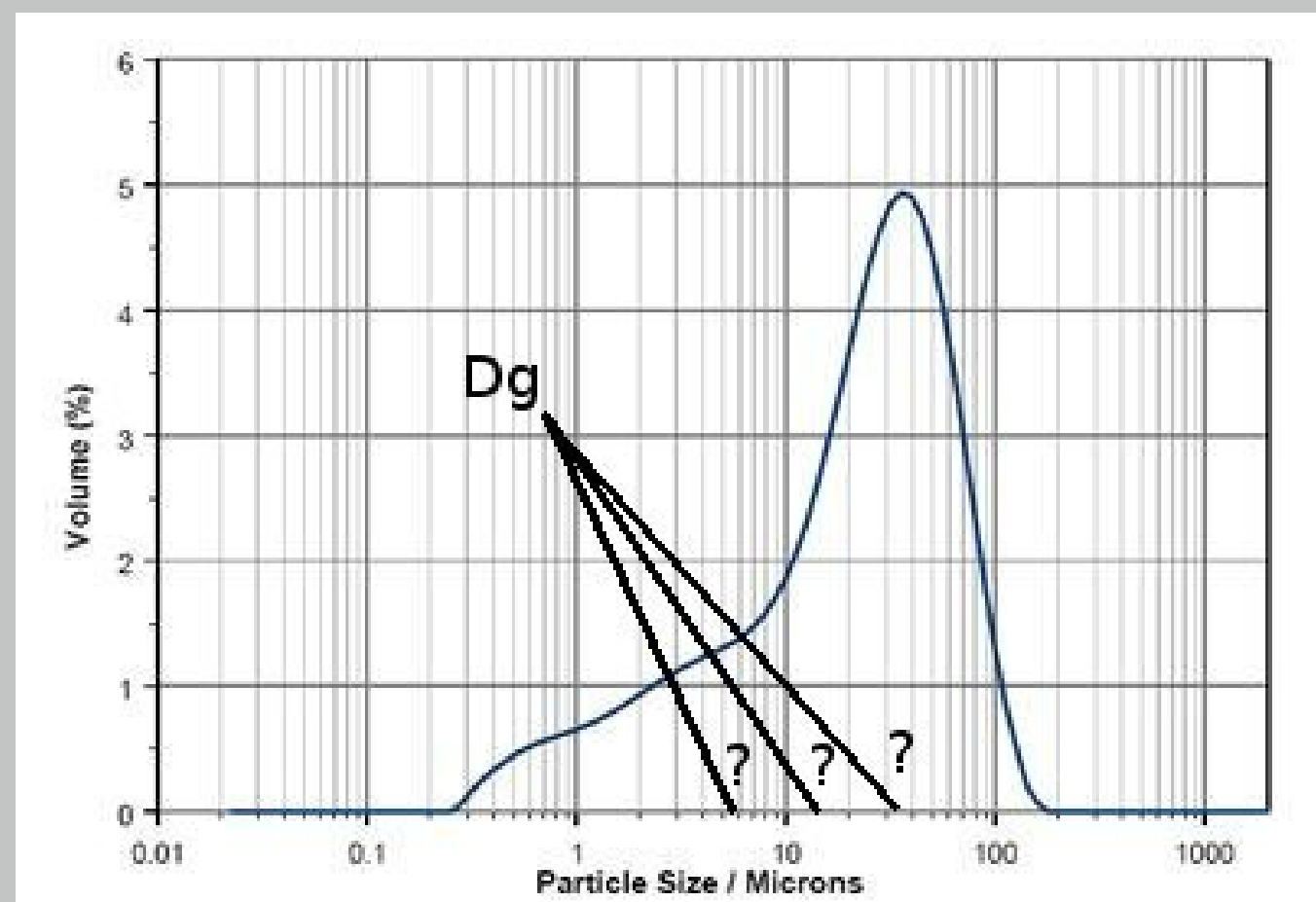
## Apparent Threshold for i Grain Sizes

- ▶  $Q_i = F[\theta_g - \theta_{ci}] P_i$  typically over-predicts fines
- ▶ Instead, threshold for fines raised and for coarse lowered
- ▶ Einstein (1950):  $\frac{\theta_{c1}}{\theta_{c2}} \approx \frac{D_2}{D_1}$
- ▶ Define apparent threshold:  $\theta'_{ci} = \theta_{cg} \left(\frac{D_i}{D_g}\right)^{b-1}$



## Problems with ‘b’

- ▶ **b** determined by matching modelled and measured  $Q_i$ 
  - ▶ This introduces an unnecessary level of abstraction
- ▶ all include arbitrary breakpoints at some **D**
- ▶ 3 known parameterizations, none universally applied. All are based on matching predicted and observed sediment transport rates
- ▶ Large degree of empiricism
- ▶ 2 do not include sorting,  $\sigma$



## A New Approach

- ▶ We propose an approach based on the inferred measured particle size-distribution of the mobilized sediment,  $P_{mi}$ , rather than transport rate per fraction  $Q_i$ 
  - ▶ This avoid uncertainties in direct modelling of  $Q_i$  (specification of **F**)
  - ▶ Depth-integrated transport rate  $Q = U_s L$
  - ▶ Depth-integrated transport load  $L = \int_0^h C(z) dz$
  - ▶ Assume **L** is linearly proportional to  $(\theta - \theta_c)$
  - ▶ Rate per fraction  $Q_i = U_s (\theta_g - \theta'_{ci}) P_i$
- ▶ Infer  $P_{mi} = \frac{\int U_s (\theta_g - \theta'_{ci}) P_i dt}{\int \sum U_s (\theta_g - \theta'_{ci}) P_i dt}$
- ▶ Given  $\theta_g$ ,  $L_i$ ,  $P_i$  and  $D_g$ , we fit **b** using:
$$\theta_g - \theta_{cg} \left(\frac{D_i}{D_g}\right)^{b-1} P_i$$
- ▶ 
$$= \frac{\theta_g - \theta_{cg} \left(\frac{D_i}{D_g}\right)^{b-1} P_i}{\sum \left(\theta_g - \theta_{cg} \left(\frac{D_i}{D_g}\right)^{b-1} P_i\right)}$$

## Summary of experimental conditions for the data used in this study

- ▶ Non-linear optimization of this simple equation is used to find the optimal form of a weighting (so-called ‘hiding’) function used to modify critical entrainment criteria to provide the best possible fit with data.
- ▶ 12 sets of published data from flume experiments. Total of 81 different mixed sand and gravel beds and flow conditions (grain Reynolds numbers between 1 and 10,000)

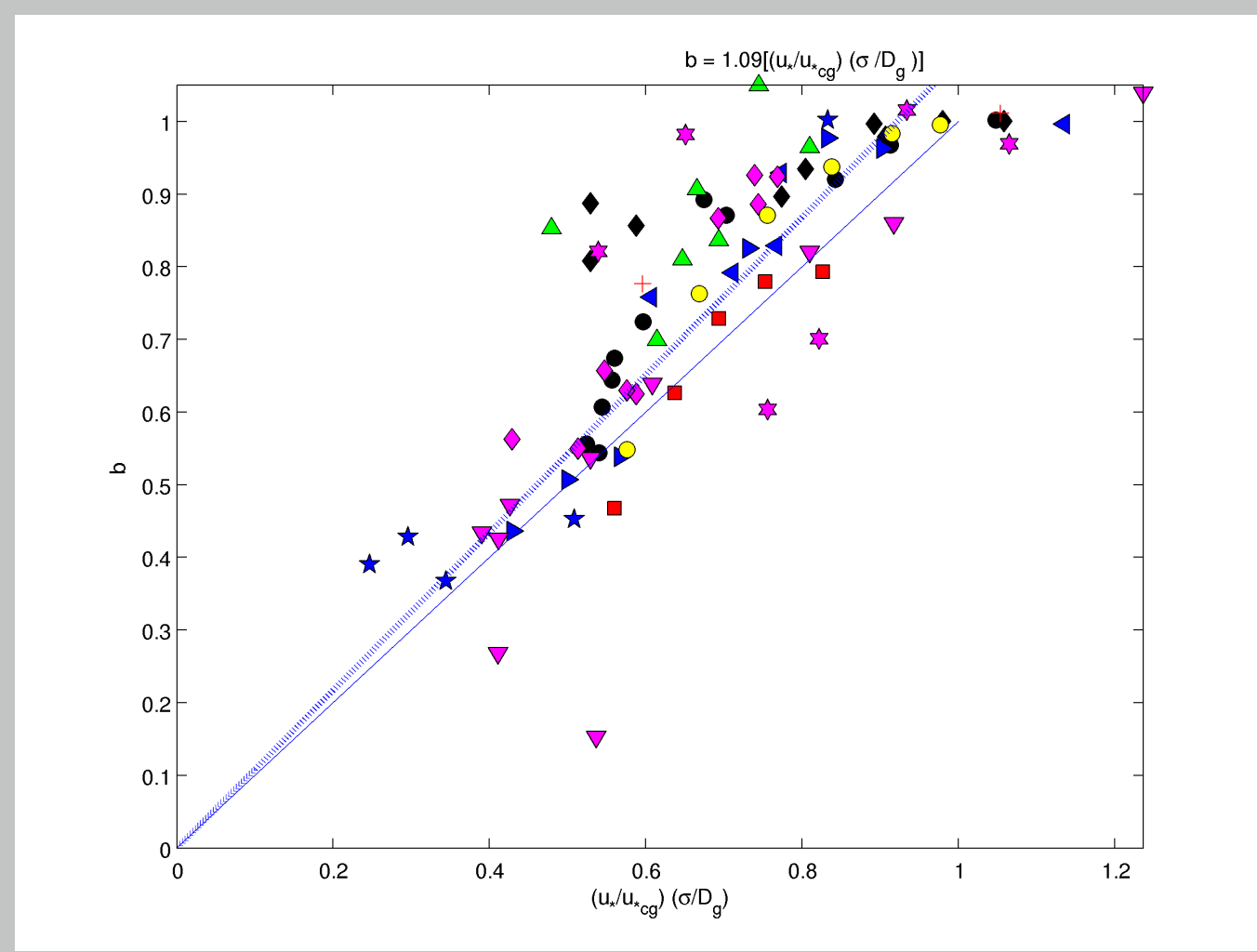
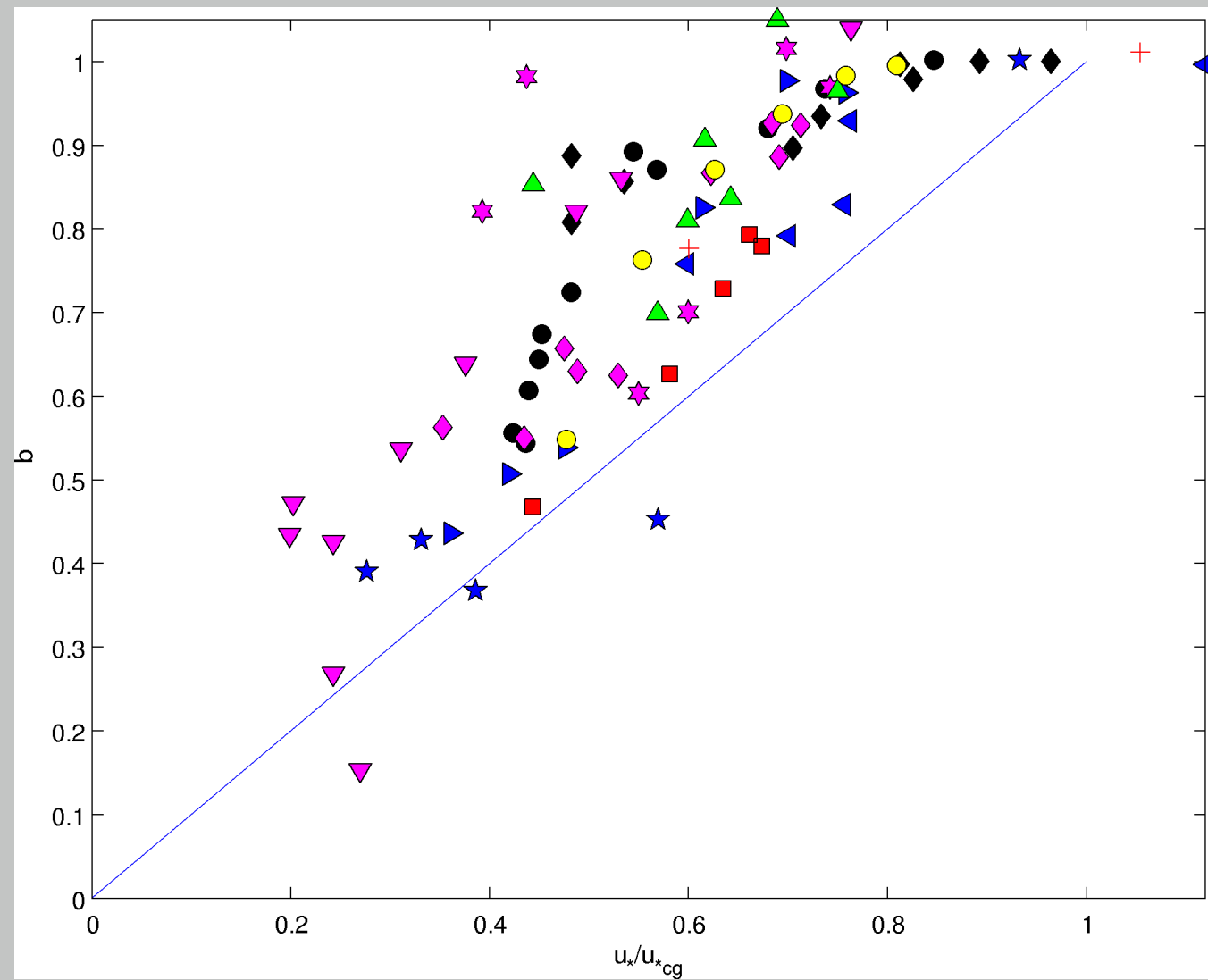
Experiment	$\theta_g \times 10^{-3}$	$D_i/D_g$	$D_g$ (mm)	$\sigma$ (mm)	mean B
Published					
Day (1980)	5.2→20.8	0.02→3.49	2.5	2.48	3.36
Day (1980)	7.07→28.3	0.03→3.03	1.84	1.51	9.62
Blom & Kleinans (1999)	8.36→20.9	0.055→4.97	2.8→3.5	3.55→3.95	5.49
Kuhnle (1993)	4.7→53.47	0.22→8.35	0.97	1.52	0.52
Kuhnle (1993)	17.06→59.67	0.12→4.79	1.7	2.14	1.24
Kuhnle (1993)	3.92→17.34	0.081→3.21	2.6	2.45	3.65
Wilcock & McArdeU (1993)	0.53→7.86	0.017→9.12	5.1→6	7.97→13.42	6.34
Wilcock et al (2001)	4.77→20.91	0.0058→3.11	14.5→17.1	9.95→12	0
Wilcock et al (2001)	4.6→18.81	0.0062→3.19	14.2→16.2	11.2→11.9	0
Sun & Donahue (2000)	6.07→17.32	0.18→7.9	0.7→1.08	0.55→1.16	0
Tait et al. (1992)	11.22→32.46	0.03→2.05	3.1→3.4	1.48→1.57	0
Kuhnle (1992)	12.34→35.51	0.0079→4.26	15.72	18.96	7.91

$D_g$  = arithmetic mean;  $\sigma$  = arithmetic sorting; **B** = bimodality index (0=unimodal)

## Analysis reveals functional form of b

▶  $b = \left(\frac{u_*}{u_{*cg}}\right)$

▶  $b = 1.09 \left(\frac{u_*}{u_{*cg}}\right) \left(\frac{\sigma}{D_g}\right)$



## Results

- ▶ Analysis reveals that motion threshold is dependent on:
  1. excess shear stress
  2. ratio of particle (arithmetic) sorting and mean grain size
- ▶ Our new formula for the mobilization of graded sediment:
$$\theta'_{ci} = \theta_{cg} \left(\frac{D_i}{D_g}\right)^{1.09 \left(\frac{u_*}{u_{*cg}}\right) \left(\frac{\sigma}{D_g}\right) - 1}$$
- ▶ Based on routinely measured quantities, and easily calculable based on mean grain size
- ▶ This new relation outperforms existing formulae in 11 out of 12 data sets, and with an expected error of  $\pm 20\%$

## Synthesis

- ▶ A simple deterministic equation, in non-dimensional form, is proposed for fraction-specific apparent critical shear stress and mobilized particle size distribution
- ▶ It predicts that  $\theta'_{ci}$  varies over more than 5 orders of magnitude for graded sediment, compared to a 1 order variation in  $\theta_{cg}$  for non-graded sediment (dark solid line in Figure adjacent) and 2 orders in  $\theta_{ci}$  found by previous studies (the light solid lines in Figure).
- ▶ The slope of the relation between  $R_e$  and threshold condition is apparently steeper than previously thought

