A Rational Coordinate Space for the Exponential Period

Formal construction, hidden-coordinate extension, compactification, algebraic adjunction, and Abel linearization

Abstract

A finite-dimensional rational coordinate space is constructed for the principal exponential period. The construction begins with the principal logarithmic value

$$g := \mathrm{Log}(-e) = 1 + i\pi,$$

and studies the $\mathbf{Q}$-vector space

$$V_{\exp} := \mathrm{span}_{\mathbf{Q}}\{1, i, g\} \subset \mathbf{C}.$$

It is proved that $\{1, i, g\}$ is a $\mathbf{Q}$-basis, that

$$V_{\exp} = \{x + i(y + k\pi) : x, y, k \in \mathbf{Q}\},$$

that the coordinates are explicit and unique, and that every principal logarithm of a root of unity belongs to $V_{\exp}$. The same data are then recast as an intrinsic coordinate model

$$E_{\mathbf{Q}} := \mathbf{Q}^3, \qquad \varepsilon_{\mathbf{Q}}(x, y, k) = x + i(y + \pi k),$$

whose realification fits into the exact sequence

$$0 \longrightarrow K \longrightarrow E_{\mathbf{R}} \xrightarrow{\varepsilon_{\mathbf{R}}} \mathbf{C} \longrightarrow 0, \qquad K = \mathbf{R}(0, -\pi, 1).$$

It is proved that $E_{\mathbf{Q}} \cap K = \{0\}$, so the model is genuinely larger than $\mathbf{C}$ while remaining arithmetically rigid on its rational lattice. An explicit splitting

$$E_{\mathbf{R}} \cong \mathbf{C} \times \mathbf{R}$$

is constructed, and this yields a fiberwise compactification

$$\widehat{\mathbf{C}} \times \mathbf{RP}^1$$

which preserves the projection to the visible complex value. The algebraic part of the construction is separated from the exponential part by proving

$$V_{\exp} \cap \overline{\mathbf{Q}} = \mathbf{Q}(i).$$

This yields a canonical adjunction theorem for algebraic numbers outside $\mathbf{Q}(i)$. It is further proved that $V_{\exp}$ is not multiplicatively closed, so the space is an exact additive coordinate model for branch-logarithmic quantities, not a period algebra. Finally, an abstract Abel-linearization theorem is stated and proved: whenever a map admits an Abel coordinate L with $L(E(z)) = L(z) + 1$, every forward and backward iterate is affine in that coordinate.

1 Preliminaries

Let $\operatorname{Arg} : \mathbf{C}^\times \to (-\pi, \pi]$ denote the principal argument, and define the principal logarithm by

$$\operatorname{Log} z := \log |z| + i \operatorname{Arg}(z), \qquad z \in \mathbf{C}^\times.$$

In particular,

$$\operatorname{Arg}(-r) = \pi \qquad (r > 0),$$

and hence

$$\operatorname{Log}(-r) = \log r + i\pi \qquad (r > 0).$$

We use the following standard transcendence theorem.

Theorem 1.1 (Lindemann). *If $\alpha \in \overline{\mathbf{Q}} \setminus \{0\}$, then e^α is transcendental.*

Corollary 1.2. *The number π is transcendental.*

Proof. Assume that $\pi \in \overline{\mathbf{Q}}$. Then $i\pi \in \overline{\mathbf{Q}} \setminus \{0\}$. By Lindemann, $e^{i\pi}$ is transcendental. But

$$e^{i\pi} = -1,$$

which is algebraic. This is impossible. □

2 The exponential-period coordinate space

Definition 2.1. Define

$$g := \operatorname{Log}(-e).$$

Define the rational coordinate space

$$V_{\exp} := \operatorname{span}_{\mathbf{Q}}\{1, i, g\} \subset \mathbf{C}.$$

Proposition 2.2. *One has*

$$g = 1 + i\pi.$$

Proof. Since $|-e| = e$ and $\operatorname{Arg}(-e) = \pi$, we obtain

$$\operatorname{Log}(-e) = \log e + i\pi = 1 + i\pi.$$

□

Proposition 2.3. *The equation*

$$e^x = -e$$

has no algebraic solution $x \in \overline{\mathbf{Q}}$.

Proof. Assume that $x \in \overline{\mathbf{Q}}$ and $e^x = -e$. Then

$$e^{x-1} = -1.$$

Now $x - 1 \in \overline{\mathbf{Q}}$. If $x - 1 \neq 0$, Lindemann implies that e^{x-1} is transcendental, contradiction. If $x - 1 = 0$, then $e^x = e \neq -e$, contradiction. Hence no such algebraic x exists. □

Theorem 2.4. *The set* $\{1, i, g\}$ *is linearly independent over* $\mathbf{Q}$.

Proof. Suppose that

$$a + bi + cg = 0, \qquad a, b, c \in \mathbf{Q}.$$

Using Proposition 2.2, this becomes

$$a + bi + c(1 + i\pi) = 0,$$

that is,

$$(a + c) + i(b + c\pi) = 0.$$

Therefore

$$a + c = 0, \qquad b + c\pi = 0.$$

If $c \neq 0$, then

$$\pi = -\frac{b}{c} \in \mathbf{Q},$$

contrary to Corollary 1.2. Hence $c = 0$, and then $b = 0$ and $a = 0$. □

Corollary 2.5. *The map*

$$\kappa : \mathbf{Q}^3 \to \mathbf{C}, \qquad \kappa(q_1, q_2, q_3) := q_1 + q_2 i + q_3 g,$$

is injective, and its image is $V_{\exp}$. *Consequently* $V_{\exp}$ *is a* 3*-dimensional* $\mathbf{Q}$*-vector space.*

Lemma 2.6. *In* $V_{\exp}$ *one has the identity*

$$i\pi = g - 1.$$

Proof. This is immediate from Proposition 2.2. □

Theorem 2.7 (exact description of the image)**.** *One has*

$$V_{\exp} = \{x + i(y + k\pi) : x, y, k \in \mathbf{Q}\}.$$

More precisely, every element of $V_{\exp}$ *can be written uniquely in the form*

$$x + i(y + k\pi), \qquad x, y, k \in \mathbf{Q},$$

and the corresponding basis coordinates are

$$x + i(y + k\pi) = (x - k) \cdot 1 + y \cdot i + k \cdot g.$$

Proof. First let $z \in V_{\exp}$. Then

$$z = q_1 + q_2 i + q_3 g \qquad (q_1, q_2, q_3 \in \mathbf{Q}).$$

Using Proposition 2.2,

$$z = q_1 + q_2 i + q_3(1 + i\pi) = (q_1 + q_3) + i(q_2 + q_3\pi).$$

Thus $z = x + i(y + k\pi)$ with

$$x = q_1 + q_3, \qquad y = q_2, \qquad k = q_3.$$

Hence

$$V_{\exp} \subseteq \{x + i(y + k\pi) : x, y, k \in \mathbf{Q}\}.$$

Conversely, if $x, y, k \in \mathbf{Q}$, then by Lemma 2.6

$$x + i(y + k\pi) = x + iy + k(g - 1) = (x - k) \cdot 1 + y \cdot i + k \cdot g \in V_{\exp}.$$

Therefore the reverse inclusion holds.

For uniqueness, suppose that

$$x + i(y + k\pi) = x' + i(y' + k'\pi) \qquad (x, y, k, x', y', k' \in \mathbf{Q}).$$

Subtracting gives

$$(x - x') + i((y - y') + (k - k')\pi) = 0.$$

Hence

$$x = x', \qquad (y - y') + (k - k')\pi = 0.$$

If $k - k' \neq 0$, then $\pi \in \mathbf{Q}$, contradiction. Thus $k = k'$, and then $y = y'$. The displayed coordinate formula follows from the converse inclusion already proved. □

Definition 2.8. For $z \in V_{\exp}$, define the coordinate functionals

$$c_1(z),\ c_i(z),\ c_g(z) \in \mathbf{Q}$$

by the unique representation

$$z = c_1(z) \cdot 1 + c_i(z) \cdot i + c_g(z) \cdot g.$$

Define also

$$\rho(z) := c_1(z) + c_g(z), \qquad \eta(z) := c_i(z), \qquad \kappa_\pi(z) := c_g(z).$$

Proposition 2.9. *For every $z \in V_{\exp}$,*

$$\mathrm{Re}(z) = \rho(z), \qquad \mathrm{Im}(z) = \eta(z) + \pi\,\kappa_\pi(z).$$

In particular,

$$\mathrm{Re}(V_{\exp}) = \mathbf{Q}, \qquad \mathrm{Im}(V_{\exp}) = \mathbf{Q} + \pi\mathbf{Q}.$$

Proof. Write

$$z = q_1 + q_2 i + q_3 g.$$

Then, using Proposition 2.2,

$$z = (q_1 + q_3) + i(q_2 + q_3\pi).$$

Thus

$$\mathrm{Re}(z) = q_1 + q_3 = \rho(z), \qquad \mathrm{Im}(z) = q_2 + q_3\pi = \eta(z) + \pi\,\kappa_\pi(z).$$

The final equalities follow from Theorem 2.7. □

Corollary 2.10. *Let ζ be a root of unity. Then $\operatorname{Log}(\zeta) \in V_{\exp}$. More precisely, there exists a unique rational number $r \in (-1, 1]$ such that*

$$\zeta = e^{i\pi r}, \qquad \operatorname{Log}(\zeta) = i\pi r = -r + rg.$$

Proof. Since ζ is a root of unity, there exists $N \geq 1$ and $m \in \mathbf{Z}$ such that

$$\zeta = e^{2\pi i m/N}.$$

Choose the unique representative $r \in (-1, 1]$ of the class $2m/N + 2\mathbf{Z}$. Then $r \in \mathbf{Q}$ and $\operatorname{Arg}(\zeta) = \pi r$. Hence

$$\operatorname{Log}(\zeta) = i\pi r.$$

By Lemma 2.6,

$$i\pi r = r(g-1) = -r + rg \in V_{\exp}.$$

Uniqueness of r follows from uniqueness of the principal argument. □

3 Intrinsic coordinate model and the hidden period direction

Definition 3.1. Define the intrinsic rational coordinate model

$$E_{\mathbf{Q}} := \mathbf{Q}^3$$

with evaluation map

$$\varepsilon_{\mathbf{Q}} : E_{\mathbf{Q}} \to \mathbf{C}, \qquad \varepsilon_{\mathbf{Q}}(x, y, k) := x + i(y + \pi k).$$

Define its realification by

$$E_{\mathbf{R}} := \mathbf{R} \otimes_{\mathbf{Q}} E_{\mathbf{Q}} \cong \mathbf{R}^3,$$

and extend the evaluation map to the real-linear map

$$\varepsilon_{\mathbf{R}} : E_{\mathbf{R}} \to \mathbf{C}, \qquad \varepsilon_{\mathbf{R}}(x, y, k) := x + i(y + \pi k).$$

Proposition 3.2. *The map $\varepsilon_{\mathbf{Q}}$ is a $\mathbf{Q}$-linear isomorphism of $E_{\mathbf{Q}}$ onto $V_{\exp}$.*

Proof. By Theorem 2.7, every element of $V_{\exp}$ has a unique expression of the form

$$x + i(y + k\pi) \qquad (x, y, k \in \mathbf{Q}).$$

This is exactly the image of (x, y, k) under $\varepsilon_{\mathbf{Q}}$. Hence $\varepsilon_{\mathbf{Q}}$ is bijective onto $V_{\exp}$. □

Theorem 3.3 (realified exact sequence)**.** *The sequence of real vector spaces*

$$0 \longrightarrow K \longrightarrow E_{\mathbf{R}} \xrightarrow{\varepsilon_{\mathbf{R}}} \mathbf{C} \longrightarrow 0$$

is exact, where

$$K := \ker(\varepsilon_{\mathbf{R}}) = \mathbf{R}(0, -\pi, 1).$$

In particular, $\varepsilon_{\mathbf{R}}$ is surjective and $\dim_{\mathbf{R}} K = 1$.

Proof. Surjectivity is immediate: for every $u + iv \in \mathbf{C}$,

$$\varepsilon_{\mathbf{R}}(u, v, 0) = u + iv.$$

Now

$$\varepsilon_{\mathbf{R}}(x, y, k) = 0$$

if and only if

$$x = 0, \qquad y + \pi k = 0.$$

Hence

$$(x, y, k) = k(0, -\pi, 1),$$

so

$$\ker(\varepsilon_{\mathbf{R}}) = \mathbf{R}(0, -\pi, 1).$$

Since this kernel is one-dimensional over $\mathbf{R}$, the stated exact sequence follows. $\square$

Theorem 3.4 (rational transversality)**.** *One has*

$$E_{\mathbf{Q}} \cap K = \{0\}.$$

Proof. Let $(x, y, k) \in E_{\mathbf{Q}} \cap K$. Since $(x, y, k) \in K$, there exists $t \in \mathbf{R}$ such that

$$(x, y, k) = t(0, -\pi, 1).$$

Therefore

$$x = 0, \qquad y = -\pi t, \qquad k = t.$$

Since $k \in \mathbf{Q}$, one has $t \in \mathbf{Q}$. Since also $y \in \mathbf{Q}$, the equality $y = -\pi t$ implies $\pi t \in \mathbf{Q}$. By irrationality of π, this forces $t = 0$. Hence $(x, y, k) = (0, 0, 0)$. $\square$

Corollary 3.5 (uniqueness of rational lifts)**.** *For every $z \in \mathbf{C}$, the fiber $\varepsilon_{\mathbf{R}}^{-1}(z)$ contains at most one point of $E_{\mathbf{Q}}$. For every $z \in V_{\exp}$, the fiber contains exactly one point of $E_{\mathbf{Q}}$.*

Proof. If $u, v \in E_{\mathbf{Q}}$ satisfy $\varepsilon_{\mathbf{R}}(u) = \varepsilon_{\mathbf{R}}(v)$, then

$$u - v \in E_{\mathbf{Q}} \cap K.$$

By Theorem 3.4, this implies $u - v = 0$, so $u = v$. This proves uniqueness. If $z \in V_{\exp}$, Proposition 3.2 gives existence of a point of $E_{\mathbf{Q}}$ mapping to z. $\square$

Definition 3.6. Define the distinguished kernel vector

$$\kappa_K := (0, -\pi, 1) \in K.$$

Define the canonical section

$$s : \mathbf{C} \to E_{\mathbf{R}}, \qquad s(u + iv) := (u, v, 0).$$

Proposition 3.7 (splitting along the hidden direction)**.** *Every $u \in E_{\mathbf{R}}$ admits a unique decomposition*

$$u = s(\varepsilon_{\mathbf{R}}(u)) + t\kappa_K$$

with $t \in \mathbf{R}$.

Proof. Let $u = (x, y, k) \in E_{\mathbf{R}}$. Then

$$\varepsilon_{\mathbf{R}}(u) = x + i(y + \pi k),$$

so

$$s(\varepsilon_{\mathbf{R}}(u)) = (x, y + \pi k, 0).$$

Therefore

$$u - s(\varepsilon_{\mathbf{R}}(u)) = (0, -\pi k, k) = k(0, -\pi, 1) = k\kappa_K.$$

Thus the stated decomposition exists with $t = k$. Uniqueness follows because $K = \mathbf{R}\kappa_K$ is one-dimensional. $\square$

Corollary 3.8. *The map*

$$\Phi : \mathbf{C} \times \mathbf{R} \to E_{\mathbf{R}}, \qquad \Phi(z, t) := s(z) + t\kappa_K,$$

is a real-linear isomorphism. Explicitly,

$$\Phi(u + iv, t) = (u, v - \pi t, t),$$

and its inverse is

$$\Phi^{-1}(x, y, k) = (x + i(y + \pi k), k).$$

Consequently,

$$E_{\mathbf{R}} \cong \mathbf{C} \times \mathbf{R}$$

as a split real extension of $\mathbf{C}$ *by one hidden coordinate.*

Proof. The formula for Φ follows from the definitions of s and κ_K. The formula for Φ^{-1} is obtained by direct substitution. Proposition 3.7 implies that these maps are mutually inverse. $\square$

Remark 3.9. The quotient map $\varepsilon_{\mathbf{R}} : E_{\mathbf{R}} \to \mathbf{C}$ forgets the hidden coordinate, while Corollary 3.5 shows that the rational lattice $E_{\mathbf{Q}}$ meets each fiber in at most one point. Hence the pair $(E_{\mathbf{R}}, E_{\mathbf{Q}})$ carries strictly more linear information than $\mathbf{C}$ without introducing rational ambiguity in the visible value.

4 Compactification of the realified coordinate model

Definition 4.1. The isotropic one-point compactification of the affine real space $E_{\mathbf{R}} \cong \mathbf{R}^3$ is denoted by

$$E_{\mathbf{R}}^+ \cong S^3.$$

Relative to the splitting of Corollary 3.8, define the fiberwise compactification by

$$\overline{E}_{\text{fib}} := \widehat{\mathbf{C}} \times \mathbf{RP}^1.$$

Proposition 4.2. *There is no continuous map*

$$\widetilde{\varepsilon} : E_{\mathbf{R}}^+ \to \widehat{\mathbf{C}}$$

whose restriction to $E_{\mathbf{R}}$ *equals* $\varepsilon_{\mathbf{R}}$.

Proof. Let $\infty \in E_{\mathbf{R}}^{+} \setminus E_{\mathbf{R}}$ denote the unique added point. Since $t\kappa_K \to \infty$ in $E_{\mathbf{R}}^{+}$ as $t \to +\infty$ and

$$\varepsilon_{\mathbf{R}}(t\kappa_K) = 0 \qquad (t > 0),$$

continuity would imply

$$\widetilde{\varepsilon}(\infty) = 0.$$

On the other hand, if $w := (1, 0, 0)$, then $tw \to \infty$ in $E_{\mathbf{R}}^{+}$ and

$$\varepsilon_{\mathbf{R}}(tw) = t \to \infty$$

in $\widehat{\mathbf{C}}$. Therefore continuity would also imply

$$\widetilde{\varepsilon}(\infty) = \infty,$$

a contradiction. □

Proposition 4.3 (fiberwise extension)**.** *Under the identification $E_{\mathbf{R}} \cong \mathbf{C} \times \mathbf{R}$ of Corollary 3.8, the map $\varepsilon_{\mathbf{R}}$ extends uniquely to the continuous projection*

$$\overline{\varepsilon} : \overline{E}_{\text{fib}} \to \widehat{\mathbf{C}}, \qquad \overline{\varepsilon}(z, \lambda) = z.$$

For every $z \in \widehat{\mathbf{C}}$, the fiber $\overline{\varepsilon}^{-1}(z)$ is canonically isomorphic to $\mathbf{RP}^1$.

Proof. In the coordinates of Corollary 3.8, one has

$$\varepsilon_{\mathbf{R}}(z, t) = z \qquad ((z, t) \in \mathbf{C} \times \mathbf{R}).$$

The formula

$$\overline{\varepsilon}(z, \lambda) = z \qquad ((z, \lambda) \in \widehat{\mathbf{C}} \times \mathbf{RP}^1)$$

defines a continuous extension to the compactified space. Uniqueness is immediate because $\mathbf{C} \times \mathbf{R}$ is dense in $\widehat{\mathbf{C}} \times \mathbf{RP}^1$. The fiber over a fixed z is

$$\{z\} \times \mathbf{RP}^1 \cong \mathbf{RP}^1.$$

□

Definition 4.4. Write

$$\mathbf{RP}^1 = \mathbf{R} \cup \{\infty_K\}.$$

Define the hidden boundary section by

$$\Sigma_\infty := \widehat{\mathbf{C}} \times \{\infty_K\} \subset \overline{E}_{\text{fib}}.$$

Proposition 4.5. *The hidden boundary section satisfies*

$$\Sigma_\infty \cong \widehat{\mathbf{C}}.$$

Moreover, for each $z \in \widehat{\mathbf{C}}$, the compactified hidden fiber over z is the projective line

$$\overline{\varepsilon}^{-1}(z) = \{z\} \times \mathbf{RP}^1.$$

Thus the added boundary retains one distinguished hidden point at infinity over every visible complex value.

Proof. The first statement follows from the definition of Σ_∞ as a product with the singleton $\{\infty_K\}$. The second statement is exactly the fiber description established in Proposition 4.3. $\square$

Proposition 4.6 (global projective collapse)**.** *In the projective compactification* $\mathbf{RP}^3$ *of the affine space* $E_{\mathbf{R}} \cong \mathbf{R}^3$*, every affine line parallel to the hidden direction* K *meets the hyperplane at infinity at the same point*

$$[0 : 0 : -\pi : 1].$$

Proof. Let $u \in E_{\mathbf{R}}$ and consider the affine line

$$\ell_u := u + \mathbf{R}\kappa_K.$$

In homogeneous coordinates $[X_0 : X_1 : X_2 : X_3]$ on $\mathbf{RP}^3$, with the affine chart $X_0 = 1$ identified with $E_{\mathbf{R}}$ by

$$(x, y, k) \longleftrightarrow [1 : x : y : k],$$

the line ℓ_u has direction vector $[0 : 0 : -\pi : 1]$. Therefore ℓ_u meets the hyperplane at infinity $X_0 = 0$ at the common point

$$[0 : 0 : -\pi : 1].$$

$\square$

Theorem 4.7 (general hidden-coordinate extension)**.** *Let* E *be a finite-dimensional real vector space, let*

$$\varepsilon : E \to \mathbf{C}$$

be a surjective real-linear map, and write

$$K := \ker(\varepsilon), \qquad m := \dim_{\mathbf{R}} K.$$

Let $L \subset E$ *be a* $\mathbf{Q}$*-vector subspace such that*

$$L \cap K = \{0\}.$$

Choose a real-linear splitting $E \cong \mathbf{C} \oplus K \cong \mathbf{C} \times \mathbf{R}^m$*. Then:*

1. *every fiber* $\varepsilon^{-1}(z)$ *is an affine translate of* K*;*
2. *each fiber contains at most one point of* L*;*
3. *the fiberwise projective compactification is*
$$\widehat{\mathbf{C}} \times \mathbf{RP}^m;$$
4. *for* $m = 1$*, one recovers the present compactification*
$$\widehat{\mathbf{C}} \times \mathbf{RP}^1.$$

Proof. After choosing a splitting, identify E with $\mathbf{C} \times K$ and write elements as (z, ξ). Then

$$\varepsilon(z, \xi) = z.$$

Hence the fiber over z is

$$\{z\} \times K,$$

which is an affine translate of K. If $u, v \in L$ satisfy $\varepsilon(u) = \varepsilon(v)$, then $u - v \in L \cap K = \{0\}$, so $u = v$. This proves (1) and (2). Projectively compactifying each hidden affine fiber $K \cong \mathbf{R}^m$ gives $\mathbf{RP}^m$, whence the fiberwise compactification is $\widehat{\mathbf{C}} \times \mathbf{RP}^m$. The case $m = 1$ is immediate. $\square$

5 Separation of the algebraic and exponential parts

Theorem 5.1. *One has*

$$V_{\exp} \cap \overline{\mathbf{Q}} = \mathbf{Q}(i).$$

Proof. The inclusion $\mathbf{Q}(i) \subseteq V_{\exp} \cap \overline{\mathbf{Q}}$ is immediate, since $\mathbf{Q}(i) = \mathrm{span}_{\mathbf{Q}}\{1, i\}$.

For the reverse inclusion, let $z \in V_{\exp} \cap \overline{\mathbf{Q}}$. Write

$$z = q_1 + q_2 i + q_3 g, \qquad q_1, q_2, q_3 \in \mathbf{Q}.$$

By Proposition 2.2,

$$z = (q_1 + q_3) + i(q_2 + q_3\pi).$$

Since z is algebraic, so is its complex conjugate $\overline{z}$, and therefore

$$\mathrm{Im}(z) = \frac{z - \overline{z}}{2i}$$

is algebraic. Hence

$$q_2 + q_3\pi \in \overline{\mathbf{Q}}.$$

If $q_3 \neq 0$, then

$$\pi = \frac{(q_2 + q_3\pi) - q_2}{q_3} \in \overline{\mathbf{Q}},$$

contrary to Corollary 1.2. Thus $q_3 = 0$. Therefore

$$z = q_1 + q_2 i \in \mathbf{Q}(i).$$

□

Corollary 5.2. *If $a \in \overline{\mathbf{Q}} \setminus \mathbf{Q}(i)$, then $a \notin V_{\exp}$.*

Definition 5.3. Let $A = \{a_1, \ldots, a_m\} \subset \overline{\mathbf{Q}}$ be a finite set such that the set

$$\{1, i, a_1, \ldots, a_m\}$$

is linearly independent over $\mathbf{Q}$. Define the extended space

$$V_{\exp,A} := V_{\exp} \oplus \bigoplus_{j=1}^{m} \mathbf{Q} a_j.$$

Equivalently,

$$V_{\exp,A} = \mathrm{span}_{\mathbf{Q}}\{1, i, g, a_1, \ldots, a_m\}.$$

Theorem 5.4 (algebraic adjunction)**.** *Under the preceding hypothesis, the set*

$$\{1, i, g, a_1, \ldots, a_m\}$$

is linearly independent over $\mathbf{Q}$. Hence every element of $V_{\exp,A}$ admits a unique coordinate expansion in that basis.

Proof. Suppose that

$$c_1 + c_2 i + c_3 g + \sum_{j=1}^{m} d_j a_j = 0, \qquad c_1, c_2, c_3, d_j \in \mathbf{Q}.$$

Let

$$a := \sum_{j=1}^{m} d_j a_j \in \overline{\mathbf{Q}}.$$

Then

$$a = -(c_1 + c_2 i + c_3 g) \in V_{\exp}.$$

Thus $a \in V_{\exp} \cap \overline{\mathbf{Q}}$. By Theorem 5.1, $a \in \mathbf{Q}(i)$. Hence

$$\sum_{j=1}^{m} d_j a_j \in \mathbf{Q}(i).$$

Therefore there exist $u, v \in \mathbf{Q}$ such that

$$\sum_{j=1}^{m} d_j a_j = u + vi.$$

Rearranging,

$$(-u) \cdot 1 + (-v) \cdot i + d_1 a_1 + \cdots + d_m a_m = 0.$$

By the assumed linear independence of $\{1, i, a_1, \ldots, a_m\}$ over $\mathbf{Q}$, all coefficients vanish. Thus $d_j = 0$ for all j, and $u = v = 0$. Returning to the original relation, we obtain

$$c_1 + c_2 i + c_3 g = 0.$$

By Theorem 2.4, $c_1 = c_2 = c_3 = 0$. $\square$

6 Failure of multiplicative closure

Theorem 6.1. *The space $V_{\exp}$ is not closed under multiplication. Consequently it is neither a subring nor a subfield of $\mathbf{C}$.*

Proof. The elements i and g belong to $V_{\exp}$. Their product is

$$ig = i(1 + i\pi) = i - \pi.$$

Assume that $i - \pi \in V_{\exp}$. Then by Theorem 2.7 there exist $x, y, k \in \mathbf{Q}$ such that

$$i - \pi = x + i(y + k\pi).$$

Comparing real and imaginary parts gives

$$x = -\pi, \qquad y + k\pi = 1.$$

The first equality implies $\pi \in \mathbf{Q}$, contradiction. Hence $ig \notin V_{\exp}$. $\square$

Remark 6.2. The preceding theorem shows that $V_{\exp}$ is an exact additive coordinate model for quantities built from rational numbers, i, and the principal exponential period $i\pi$. It is not a period algebra. In particular, it does not contain π^2, and therefore it is too small to serve as a multiplicatively stable ambient space for many continued-fraction evaluations.

7 Abel linearization as an abstract coordinate principle

Definition 7.1. Let D be a set and $E : D \to D$ a map. A function $L : D \to A$ into an additive abelian group A is called an *Abel coordinate* for E if

$$L(E(z)) = L(z) + 1 \qquad (z \in D).$$

Theorem 7.2 (Abel linearization)**.** *Let $E : D \to D$ admit an Abel coordinate $L : D \to A$.*

1. *For every $n \in \mathbf{Z}_{\geq 0}$ and every $z \in D$,*

$$L(E^{\circ n}(z)) = L(z) + n.$$

2. *Assume in addition that E is bijective. Then for every $n \in \mathbf{Z}$ and every $z \in D$,*

$$L(E^{\circ n}(z)) = L(z) + n.$$

Proof. For (1), the case $n = 0$ is immediate. Assume that the identity holds for n. Then

$$L(E^{\circ(n+1)}(z)) = L(E(E^{\circ n}(z))) = L(E^{\circ n}(z)) + 1 = L(z) + n + 1.$$

This proves the formula for all $n \geq 0$ by induction.

For (2), first note that bijectivity of E allows us to define $E^{\circ n}$ for $n < 0$. Let $w \in D$. Since $E(E^{-1}(w)) = w$, the Abel relation gives

$$L(w) = L(E(E^{-1}(w))) = L(E^{-1}(w)) + 1,$$

whence

$$L(E^{-1}(w)) = L(w) - 1.$$

Applying this repeatedly yields

$$L(E^{\circ(-m)}(z)) = L(z) - m \qquad (m \in \mathbf{Z}_{>0}).$$

Combining this with part (1) proves the identity for all integers n. □

Corollary 7.3. *Let $a \in D$ and put $c := L(a)$. Under the hypotheses of Theorem 7.2, every iterate of a has affine Abel coordinate:*

$$L(E^{\circ n}(a)) = c + n.$$

Remark 7.4. Theorem 7.2 is the exact content behind every construction of the form "adjoin one distinguished value and all iterates become affine in that coordinate". No higher-level analytic continuation is used in the proof. Every further application requires only two ingredients: the existence of a map E and the existence of an Abel coordinate L on the domain under consideration.

8 Use relative to polynomial continued fractions

The proved Ramanujan Machine families concern polynomial continued fractions whose values include constants such as π^2, $\log 2$, Catalan's constant, and $\zeta(3)$; the evaluation mechanism proceeds through second-order recurrences and differential or hypergeometric methods. From that standpoint, the present construction has a precise and limited role.

Proposition 8.1. *The space $V_{\exp}$ and its realified hidden-coordinate extension $E_{\mathbf{R}}$ provide exact additive models for principal branch-logarithmic data, but they are not large enough to contain the principal polynomial-continued-fraction targets π^2, Catalan's constant, or $\zeta(3)$ as elements of the visible complex image $V_{\exp}$.*

Proof. The exact coordinate property was established in Theorem 2.7 and Corollary 2.10. By Theorem 6.1, $V_{\exp}$ is not multiplicatively closed. In particular,

$$\pi^2 = (-i(g-1))^2$$

need not belong to $V_{\exp}$, and indeed it does not: if $\pi^2 \in V_{\exp}$, then by Theorem 2.7 one would have

$$\pi^2 = x + i(y + k\pi), \qquad x, y, k \in \mathbf{Q}.$$

The imaginary part of the left-hand side is 0, hence $y + k\pi = 0$. As in the proof of Theorem 2.7, this forces $k = 0$ and $y = 0$. Then $\pi^2 = x \in \mathbf{Q}$, contradiction.

The same argument shows that any real element of $V_{\exp}$ is rational: if

$$x + i(y + k\pi) \in \mathbf{R},$$

then $y + k\pi = 0$ and hence $k = 0$, $y = 0$, so the element equals $x \in \mathbf{Q}$. Therefore any non-rational real constant, including Catalan's constant and $\zeta(3)$, lies outside $V_{\exp}$. □

Remark 8.2. Consequently the appropriate use of $V_{\exp}$ and $E_{\mathbf{R}}$ in continued-fraction work is not as a universal ambient algebra, but as a canonical finite-coordinate layer for branch-logarithmic quantities and for exact bookkeeping of the hidden $i\pi$-direction whenever such a coefficient arises naturally.

References

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