A Rational Coordinate Space for the Exponential Period

Formal construction, algebraic adjunction, and Abel linearization

Abstract

A finite-dimensional rational coordinate space is constructed for the principal exponential period. The construction begins with the principal logarithmic value

$$g := \mathrm{Log}(-e) = 1 + i\pi,$$

and studies the $\mathbf{Q}$-vector space

$$V_{\exp} := \mathrm{span}_{\mathbf{Q}}\{1, i, g\} \subset \mathbf{C}.$$

It is proved that $\{1, i, g\}$ is a $\mathbf{Q}$-basis, that

$$V_{\exp} = \{x + i(y + k\pi) : x, y, k \in \mathbf{Q}\},$$

that the coordinates are explicit and unique, and that every principal logarithm of a root of unity belongs to $V_{\exp}$. The algebraic part of the construction is separated from the exponential part by proving

$$V_{\exp} \cap \overline{\mathbf{Q}} = \mathbf{Q}(i).$$

This yields a canonical adjunction theorem for algebraic numbers outside $\mathbf{Q}(i)$. It is further proved that $V_{\exp}$ is not multiplicatively closed, so the space is an exact additive coordinate model for branch-logarithmic quantities, not a period algebra. Finally, an abstract Abel-linearization theorem is stated and proved: whenever a map admits an Abel coordinate L with $L(E(z)) = L(z) + 1$, every forward and backward iterate is affine in that coordinate. This isolates the part of the construction that is exact and reusable independently of any speculative higher-level hypotheses.

1 Preliminaries

Let $\mathrm{Arg} : \mathbf{C}^{\times} \to (-\pi, \pi]$ denote the principal argument, and define the principal logarithm by

$$\mathrm{Log}\, z := \log|z| + i\,\mathrm{Arg}(z), \qquad z \in \mathbf{C}^{\times}.$$

In particular,

$$\mathrm{Arg}(-r) = \pi \qquad (r > 0),$$

and hence

$$\mathrm{Log}(-r) = \log r + i\pi \qquad (r > 0).$$

We use the following standard transcendence theorem.

Theorem 1.1 (Lindemann)**.** *If $\alpha \in \overline{\mathbf{Q}} \setminus \{0\}$, then e^{α} is transcendental.*

Corollary 1.2. *The number π is transcendental.*

Proof. Assume that $\pi \in \overline{\mathbf{Q}}$. Then $i\pi \in \overline{\mathbf{Q}} \setminus \{0\}$. By Lindemann, $e^{i\pi}$ is transcendental. But

$$e^{i\pi} = -1,$$

which is algebraic. This is impossible. □

2 The exponential-period coordinate space

Definition 2.1. Define

$$g := \mathrm{Log}(-e).$$

Define the rational coordinate space

$$V_{\exp} := \mathrm{span}_{\mathbf{Q}}\{1, i, g\} \subset \mathbf{C}.$$

Proposition 2.2. *One has*

$$g = 1 + i\pi.$$

Proof. Since $|-e| = e$ and $\mathrm{Arg}(-e) = \pi$, we obtain

$$\mathrm{Log}(-e) = \log e + i\pi = 1 + i\pi.$$

□

Proposition 2.3. *The equation*

$$e^x = -e$$

has no algebraic solution $x \in \overline{\mathbf{Q}}$.

Proof. Assume that $x \in \overline{\mathbf{Q}}$ and $e^x = -e$. Then

$$e^{x-1} = -1.$$

Now $x - 1 \in \overline{\mathbf{Q}}$. If $x - 1 \neq 0$, Lindemann implies that e^{x-1} is transcendental, contradiction. If $x - 1 = 0$, then $e^x = e \neq -e$, contradiction. Hence no such algebraic x exists. □

Theorem 2.4. *The set $\{1, i, g\}$ is linearly independent over $\mathbf{Q}$.*

Proof. Suppose that

$$a + bi + cg = 0, \qquad a, b, c \in \mathbf{Q}.$$

Using Proposition 2.2, this becomes

$$a + bi + c(1 + i\pi) = 0,$$

that is,

$$(a + c) + i(b + c\pi) = 0.$$

Therefore

$$a + c = 0, \qquad b + c\pi = 0.$$

If $c \neq 0$, then

$$\pi = -\frac{b}{c} \in \mathbf{Q},$$

contrary to Corollary 1.2. Hence $c = 0$, and then $b = 0$ and $a = 0$. $\square$

Corollary 2.5. *The map*

$$\kappa : \mathbf{Q}^3 \to \mathbf{C}, \qquad \kappa(q_1, q_2, q_3) := q_1 + q_2 i + q_3 g,$$

is injective, and its image is $V_{\exp}$. Consequently $V_{\exp}$ is a 3-dimensional $\mathbf{Q}$-vector space.

Lemma 2.6. *In $V_{\exp}$ one has the identity*

$$i\pi = g - 1.$$

Proof. This is immediate from Proposition 2.2. $\square$

Theorem 2.7 (exact description of the image)**.** *One has*

$$V_{\exp} = \{x + i(y + k\pi) : x, y, k \in \mathbf{Q}\}.$$

More precisely, every element of $V_{\exp}$ can be written uniquely in the form

$$x + i(y + k\pi), \qquad x, y, k \in \mathbf{Q},$$

and the corresponding basis coordinates are

$$x + i(y + k\pi) = (x - k) \cdot 1 + y \cdot i + k \cdot g.$$

Proof. First let $z \in V_{\exp}$. Then

$$z = q_1 + q_2 i + q_3 g \qquad (q_1, q_2, q_3 \in \mathbf{Q}).$$

Using Proposition 2.2,

$$z = q_1 + q_2 i + q_3(1 + i\pi) = (q_1 + q_3) + i(q_2 + q_3\pi).$$

Thus $z = x + i(y + k\pi)$ with

$$x = q_1 + q_3, \qquad y = q_2, \qquad k = q_3.$$

Hence

$$V_{\exp} \subseteq \{x + i(y + k\pi) : x, y, k \in \mathbf{Q}\}.$$

Conversely, if $x, y, k \in \mathbf{Q}$, then by Lemma 2.6

$$x + i(y + k\pi) = x + iy + k(g - 1) = (x - k) \cdot 1 + y \cdot i + k \cdot g \in V_{\exp}.$$

Therefore the reverse inclusion holds.

For uniqueness, suppose that

$$x + i(y + k\pi) = x' + i(y' + k'\pi) \qquad (x, y, k, x', y', k' \in \mathbf{Q}).$$

Subtracting gives

$$(x - x') + i((y - y') + (k - k')\pi) = 0.$$

Hence

$$x = x', \qquad (y - y') + (k - k')\pi = 0.$$

If $k - k' \neq 0$, then $\pi \in \mathbf{Q}$, contradiction. Thus $k = k'$, and then $y = y'$. The displayed coordinate formula follows from the converse inclusion already proved. $\square$

Definition 2.8. For $z \in V_{\exp}$, define the coordinate functionals

$$c_1(z),\ c_i(z),\ c_g(z) \in \mathbf{Q}$$

by the unique representation

$$z = c_1(z) \cdot 1 + c_i(z) \cdot i + c_g(z) \cdot g.$$

Define also

$$\rho(z) := c_1(z) + c_g(z), \qquad \eta(z) := c_i(z), \qquad \kappa_\pi(z) := c_g(z).$$

Proposition 2.9. *For every $z \in V_{\exp}$,*

$$\mathrm{Re}(z) = \rho(z), \qquad \mathrm{Im}(z) = \eta(z) + \pi\, \kappa_\pi(z).$$

In particular,

$$\mathrm{Re}(V_{\exp}) = \mathbf{Q}, \qquad \mathrm{Im}(V_{\exp}) = \mathbf{Q} + \pi\mathbf{Q}.$$

Proof. Write

$$z = q_1 + q_2 i + q_3 g.$$

Then, using Proposition 2.2,

$$z = (q_1 + q_3) + i(q_2 + q_3\pi).$$

Thus

$$\mathrm{Re}(z) = q_1 + q_3 = \rho(z), \qquad \mathrm{Im}(z) = q_2 + q_3\pi = \eta(z) + \pi\, \kappa_\pi(z).$$

The final equalities follow from Theorem 2.7. $\square$

Corollary 2.10. *Let ζ be a root of unity. Then* $\mathrm{Log}(\zeta) \in V_{\exp}$. *More precisely, there exists a unique rational number $r \in (-1, 1]$ such that*

$$\zeta = e^{i\pi r}, \qquad \mathrm{Log}(\zeta) = i\pi r = -r + rg.$$

Proof. Since ζ is a root of unity, there exists $N \geq 1$ and $m \in \mathbf{Z}$ such that

$$\zeta = e^{2\pi i m/N}.$$

Choose the unique representative $r \in (-1, 1]$ of the class $2m/N + 2\mathbf{Z}$. Then $r \in \mathbf{Q}$ and $\mathrm{Arg}(\zeta) = \pi r$. Hence

$$\mathrm{Log}(\zeta) = i\pi r.$$

By Lemma 2.6,

$$i\pi r = r(g - 1) = -r + rg \in V_{\exp}.$$

Uniqueness of r follows from uniqueness of the principal argument. $\square$

3 Separation of the algebraic and exponential parts

Theorem 3.1. *One has*

$$V_{\exp} \cap \overline{\mathbf{Q}} = \mathbf{Q}(i).$$

Proof. The inclusion $\mathbf{Q}(i) \subseteq V_{\exp} \cap \overline{\mathbf{Q}}$ is immediate, since $\mathbf{Q}(i) = \operatorname{span}_{\mathbf{Q}}\{1, i\}$.

For the reverse inclusion, let $z \in V_{\exp} \cap \overline{\mathbf{Q}}$. Write

$$z = q_1 + q_2 i + q_3 g, \qquad q_1, q_2, q_3 \in \mathbf{Q}.$$

By Proposition 2.2,

$$z = (q_1 + q_3) + i(q_2 + q_3\pi).$$

Since z is algebraic, so is its complex conjugate $\overline{z}$, and therefore

$$\operatorname{Im}(z) = \frac{z - \overline{z}}{2i}$$

is algebraic. Hence

$$q_2 + q_3\pi \in \overline{\mathbf{Q}}.$$

If $q_3 \neq 0$, then

$$\pi = \frac{(q_2 + q_3\pi) - q_2}{q_3} \in \overline{\mathbf{Q}},$$

contrary to Corollary 1.2. Thus $q_3 = 0$. Therefore

$$z = q_1 + q_2 i \in \mathbf{Q}(i).$$

□

Corollary 3.2. *If $a \in \overline{\mathbf{Q}} \setminus \mathbf{Q}(i)$, then $a \notin V_{\exp}$.*

Definition 3.3. Let $A = \{a_1, \ldots, a_m\} \subset \overline{\mathbf{Q}}$ be a finite set such that the set

$$\{1, i, a_1, \ldots, a_m\}$$

is linearly independent over $\mathbf{Q}$. Define the extended space

$$V_{\exp,A} := V_{\exp} \oplus \bigoplus_{j=1}^{m} \mathbf{Q} a_j.$$

Equivalently,

$$V_{\exp,A} = \operatorname{span}_{\mathbf{Q}}\{1, i, g, a_1, \ldots, a_m\}.$$

Theorem 3.4 (algebraic adjunction)**.** *Under the preceding hypothesis, the set*

$$\{1, i, g, a_1, \ldots, a_m\}$$

is linearly independent over $\mathbf{Q}$. Hence every element of $V_{\exp,A}$ admits a unique coordinate expansion in that basis.

Proof. Suppose that

$$c_1 + c_2 i + c_3 g + \sum_{j=1}^{m} d_j a_j = 0, \qquad c_1, c_2, c_3, d_j \in \mathbf{Q}.$$

Let

$$a := \sum_{j=1}^{m} d_j a_j \in \overline{\mathbf{Q}}.$$

Then

$$a = -(c_1 + c_2 i + c_3 g) \in V_{\mathrm{exp}}.$$

Thus $a \in V_{\mathrm{exp}} \cap \overline{\mathbf{Q}}$. By Theorem 3.1, $a \in \mathbf{Q}(i)$. Hence

$$\sum_{j=1}^{m} d_j a_j \in \mathbf{Q}(i).$$

Therefore there exist $u, v \in \mathbf{Q}$ such that

$$\sum_{j=1}^{m} d_j a_j = u + vi.$$

Rearranging,

$$(-u) \cdot 1 + (-v) \cdot i + d_1 a_1 + \cdots + d_m a_m = 0.$$

By the assumed linear independence of $\{1, i, a_1, \ldots, a_m\}$ over $\mathbf{Q}$, all coefficients vanish. Thus $d_j = 0$ for all j, and $u = v = 0$. Returning to the original relation, we obtain

$$c_1 + c_2 i + c_3 g = 0.$$

By Theorem 2.4, $c_1 = c_2 = c_3 = 0$. □

4 Failure of multiplicative closure

Theorem 4.1. *The space V_{exp} is not closed under multiplication. Consequently it is neither a subring nor a subfield of $\mathbf{C}$.*

Proof. The elements i and g belong to V_{exp}. Their product is

$$ig = i(1 + i\pi) = i - \pi.$$

Assume that $i - \pi \in V_{\mathrm{exp}}$. Then by Theorem 2.7 there exist $x, y, k \in \mathbf{Q}$ such that

$$i - \pi = x + i(y + k\pi).$$

Comparing real and imaginary parts gives

$$x = -\pi, \qquad y + k\pi = 1.$$

The first equality implies $\pi \in \mathbf{Q}$, contradiction. Hence $ig \notin V_{\mathrm{exp}}$. □

Remark 4.2. The preceding theorem shows that V_{exp} is an exact additive coordinate model for quantities built from rational numbers, i, and the principal exponential period $i\pi$. It is not a period algebra. In particular, it does not contain π^2, and therefore it is too small to serve as a multiplicatively stable ambient space for many continued-fraction evaluations.

5 Abel linearization as an abstract coordinate principle

Definition 5.1. Let D be a set and $E : D \to D$ a map. A function $L : D \to A$ into an additive abelian group A is called an *Abel coordinate* for E if

$$L(E(z)) = L(z) + 1 \qquad (z \in D).$$

Theorem 5.2 (Abel linearization)**.** *Let $E : D \to D$ admit an Abel coordinate $L : D \to A$.*

1. *For every $n \in \mathbf{Z}_{\geq 0}$ and every $z \in D$,*

$$L(E^{\circ n}(z)) = L(z) + n.$$

2. *Assume in addition that E is bijective. Then for every $n \in \mathbf{Z}$ and every $z \in D$,*

$$L(E^{\circ n}(z)) = L(z) + n.$$

Proof. For (1), the case $n = 0$ is immediate. Assume that the identity holds for n. Then

$$L(E^{\circ(n+1)}(z)) = L(E(E^{\circ n}(z))) = L(E^{\circ n}(z)) + 1 = L(z) + n + 1.$$

This proves the formula for all $n \geq 0$ by induction.

For (2), first note that bijectivity of E allows us to define $E^{\circ n}$ for $n < 0$. Let $w \in D$. Since $E(E^{-1}(w)) = w$, the Abel relation gives

$$L(w) = L(E(E^{-1}(w))) = L(E^{-1}(w)) + 1,$$

whence

$$L(E^{-1}(w)) = L(w) - 1.$$

Applying this repeatedly yields

$$L(E^{\circ(-m)}(z)) = L(z) - m \qquad (m \in \mathbf{Z}_{>0}).$$

Combining this with part (1) proves the identity for all integers n. □

Corollary 5.3. *Let $a \in D$ and put $c := L(a)$. Under the hypotheses of Theorem 5.2, every iterate of a has affine Abel coordinate:*

$$L(E^{\circ n}(a)) = c + n.$$

Remark 5.4. Theorem 5.2 is the exact content behind every construction of the form "adjoin one distinguished value and all iterates become affine in that coordinate". No higher-level analytic continuation is used in the proof. Every further application requires only two ingredients: the existence of a map E and the existence of an Abel coordinate L on the domain under consideration.

6 Use relative to polynomial continued fractions

The proved Ramanujan Machine families concern polynomial continued fractions whose values include constants such as π^2, $\log 2$, Catalan's constant, and $\zeta(3)$; the evaluation mechanism proceeds through second-order recurrences and differential or hypergeometric methods. From that standpoint, the present construction has a precise and limited role.

Proposition 6.1. *The space V_{exp} is an exact coordinate space for principal branch-logarithmic data, but it is not large enough to contain the principal polynomial-continued-fraction targets π^2, Catalan's constant, or $\zeta(3)$.*

Proof. The exact coordinate property was established in Theorem 2.7 and Corollary 2.10. By Theorem 4.1, V_{exp} is not multiplicatively closed. In particular,

$$\pi^2 = (-i(g-1))^2$$

need not belong to V_{exp}, and indeed it does not: if $\pi^2 \in V_{\text{exp}}$, then by Theorem 2.7 one would have

$$\pi^2 = x + i(y + k\pi), \qquad x, y, k \in \mathbf{Q}.$$

The imaginary part of the left-hand side is 0, hence $y + k\pi = 0$. As in the proof of Theorem 2.7, this forces $k = 0$ and $y = 0$. Then $\pi^2 = x \in \mathbf{Q}$, contradiction.

The same argument shows that any real element of V_{exp} is rational: if

$$x + i(y + k\pi) \in \mathbf{R},$$

then $y + k\pi = 0$ and hence $k = 0$, $y = 0$, so the element equals $x \in \mathbf{Q}$. Therefore any non-rational real constant, including Catalan's constant and $\zeta(3)$, lies outside V_{exp}. □

Remark 6.2. Consequently the appropriate use of V_{exp} in continued-fraction work is not as a universal ambient algebra, but as a canonical finite-coordinate layer for branch-logarithmic quantities and for exact bookkeeping of the coefficient of $i\pi$ whenever such a coefficient arises naturally.

References

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