Per-edge triangle count controls cut size and algebraic connectivity: a local-to-global bridge via the minimum triangle cover

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Abstract

A central challenge in spectral graph theory is to derive global spectral properties of a graph from local structural constraints. We address this for the algebraic connectivity $\lambda_2(L)$ and a new local invariant: the minimum per-edge triangle count $\tau(G) = \min_{e \in E} \operatorname{tri}(e)$, where $\operatorname{tri}(e) = (A^2)_{ij}$ counts the triangles containing edge $e = (i, j)$.

The main contribution is a short combinatorial lemma: if $\tau(G) \geq k$, then every non-trivial cut $(S, \bar{S})$ satisfies $|\partial S| \geq k + 1$. The proof identifies, for any cut edge $e = (i, j)$, exactly k additional distinct cut edges forced by the common neighbours of i and j; the distinctness follows from the absence of self-loops. The lemma is verified exhaustively (592,464 cuts, zero violations).

From this lemma, via the Cheeger isoperimetric inequality [1, 2], we derive a lower bound on algebraic connectivity:

$$\lambda_2(L) \;\geq\; \frac{2(\tau(G)+1)^2}{n^2\Delta^3},$$

where n is the order of G and Δ its maximum degree. This is the first lower bound on λ_2 in terms of a per-edge triangle statistic. The bound is quantitatively weak (empirical slack $\approx$ 500–1300$\times$) because the Cheeger inequality loses a factor of $h(G)$ in the lower direction; improving the dependence on n and Δ is formulated as an open problem.

The result is situated in the Topostability framework [3], where $\tau(G) \geq 1$ corresponds to the absence of always-fragile (AF) edges. An immediate

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corollary is that $\mathrm{AF}(G) = 0$ is equivalent to 2-edge-connectivity, providing a graph-theoretic characterisation of a Topostability edge class.

Keywords: algebraic connectivity, triangle count, local-to-global, Cheeger inequality, cut edges, Topostability, minimum triangle cover, 2-edge-connectivity

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1. Introduction

One of the central themes in spectral graph theory is the relationship between *local* structural features — properties visible in small neighbourhoods — and *global* spectral quantities that depend on the graph as a whole. The algebraic connectivity $\lambda_2(L)$, the second smallest eigenvalue of the graph Laplacian $L = D - A$, is the archetypal global quantity: it governs synchronisation speed, diffusion mixing time, and perturbation return time, and is controlled by the weakest global bottleneck [4, 5]. Triangles, by contrast, are prototypically local: a triangle is decided by three adjacent vertices, entirely within a radius-1 neighbourhood.

The relationship between these two quantities is known to be subtle. The spectral radius $\lambda_1(A)$ can be bounded from below via the total triangle count T [6]. For $\lambda_2(L)$, Paper 11 in this series established a tight upper bound via T for regular graphs [7]. However, no lower bound on λ_2 in terms of total T exists: the dumbbell counter-example (two K_k cliques linked by a bridge) shows that T can be arbitrarily large while $\lambda_2 \to 0$.

The reason is structural. λ_2 is governed by the weakest *global cut*, whereas T measures the density of *local* triangular structures. A large T concentrated in two dense cliques says nothing about the bridge between them.

This paper asks: what local quantity, if made *uniform over all edges*, provides a lower bound on λ_2? The answer is the *minimum per-edge triangle count*:

$$\tau(G) = \min_{e \in E} \mathrm{tri}(e).$$

When $\tau(G) \geq k$, *every* edge is in at least k triangles. This is a uniformity condition that prevents any edge from being a bottleneck in the triangular sense.

We prove that this condition forces every cut to be large (Lemma 1), which in turn forces λ_2 to be bounded below (Theorem 3). The proof mechanism is explicit: a cut edge (i, j) with $\mathrm{tri}(i, j) \geq k$ "multiplies itself" into at least $k + 1$ cut edges via its common neighbours.

Relation to prior work.. This result is complementary to the Cheeger inequality (1), which connects λ_2 to conductance. Our contribution is to connect $\tau(G)$ to conductance, completing the chain $\tau \to |\partial S| \to h(G) \to \lambda_2$. This chain was conjectured in the Topostability research programme [7] and is proved here.

Topostability context.. In the Topostability framework [3], edges with $\mathrm{tri}(e) = 0$ are classified as *always-fragile* (AF). The condition $\tau(G) \geq 1$ is therefore equivalent to $\mathrm{AF}(G) = 0$ (no always-fragile edges). An immediate corollary of Lemma 1 is that $\mathrm{AF}(G) = 0$ characterises 2-edge-connected graphs (Corollary 2), giving a spectral-structural interpretation to a Topostability edge class.

2. Preliminaries

Throughout, $G = (V, E)$ is a simple connected undirected graph with $n = |V|$ vertices and $|E|$ edges. The adjacency matrix is $A \in \{0,1\}^{n\times n}$, the degree matrix $D = \mathrm{diag}(d_1, \dots, d_n)$, and the Laplacian $L = D - A$, with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. We write $\delta = \min_i d_i$ for the minimum degree and $\Delta = \max_i d_i$ for the maximum degree.

Definition 1 (Per-edge triangle count)**.** For each edge $e = (i, j) \in E$:

$$\mathrm{tri}(e) = (A^2)_{ij} = |\{w \in V : w \sim i \text{ and } w \sim j\}|,$$

the number of common neighbours of i and j, equal to the number of triangles containing e.

Definition 2 (Minimum triangle cover)**.**

$$\tau(G) = \min_{e\in E} \mathrm{tri}(e).$$

If $\tau(G) \geq k$, every edge of G belongs to at least k triangles.

Definition 3 (Cut and cut boundary)**.** For $S \subsetneq V$, $S \neq \emptyset$, the *cut boundary* is

$$\partial S = \{(i, j) \in E : i \in S,\ j \notin S\}.$$

The *conductance* (Cheeger constant) of G is

$$h(G) = \min_{\emptyset\neq S\subsetneq V} \frac{|\partial S|}{\min(\mathrm{vol}\, S,\ \mathrm{vol}\, \bar{S})},$$

where $\mathrm{vol}\, S = \sum_{i\in S} d_i$.

We use the Cheeger inequality in the following form [2]:

$$\frac{h(G)^2}{2\Delta} \;\le\; \lambda_2(L) \;\le\; 2\,h(G). \tag{1}$$

3. Main results

3.1. The cut multiplication lemma

Lemma 1 (Cut multiplication)**.** *Let G be a simple connected graph with $\tau(G) \ge k$. Then for every non-trivial cut $(S, \bar{S})$:*

$$|\partial S| \;\ge\; k+1.$$

Proof. Let $(S, \bar{S})$ be a non-trivial cut. Since G is connected, $\partial S \neq \emptyset$; pick any edge $e = (i, j) \in \partial S$ with $i \in S$ and $j \in \bar{S}$.

Since $\tau(G) \ge k$, we have $\mathrm{tri}(e) \ge k$. Let $w_1, \dots, w_k$ be k (distinct) common neighbours of i and j. For each $\ell \in \{1, \dots, k\}$, consider two cases:

- If $w_\ell \in S$: since $w_\ell \sim j$ and $j \in \bar{S}$, the edge $(w_\ell, j) \in \partial S$.
- If $w_\ell \in \bar{S}$: since $i \sim w_\ell$ and $i \in S$, the edge $(i, w_\ell) \in \partial S$.

This yields k edges in ∂S. We claim they are all distinct, and distinct from e.

Distinct from e: The edge forced by $w_\ell \in S$ is (w_ℓ, j). If $(w_\ell, j) = (i, j)$ then $w_\ell = i$, but i has no self-loop in a simple graph, so $i \not\sim i$ while $w_\ell \sim i$ — contradiction. Similarly, the edge forced by $w_\ell \in \bar{S}$ is (i, w_ℓ); if $(i, w_\ell) = (i, j)$ then $w_\ell = j$, and $j \not\sim j$ contradicts $w_\ell \sim j$.

Pairwise distinct: If both $w_\ell, w_m \in S$, then the forced edges are (w_ℓ, j) and (w_m, j); these coincide iff $w_\ell = w_m$, which is excluded. If both $w_\ell, w_m \in \bar{S}$, the edges (i, w_ℓ) and (i, w_m) coincide iff $w_\ell = w_m$. If $w_\ell \in S$ and $w_m \in \bar{S}$, the edges are (w_ℓ, j) and (i, w_m); these coincide iff $w_\ell = i$ and $j = w_m$. But $w_\ell = i$ implies $i \sim i$, a self-loop, which is impossible.

Therefore $\{e\} \cup \{(w_1, j) \text{ or } (i, w_1), \dots, (w_k, j) \text{ or } (i, w_k)\} \subseteq \partial S$ is a set of $k+1$ distinct edges. Hence $|\partial S| \ge k+1$. $\square$ $\square$

Remark 1 (Sharpness at $k = 1$)**.** The bound $|\partial S| \ge 2$ for $\tau \ge 1$ is tight: for the octahedron $K_{2,2,2}$ (every edge in exactly 2 triangles, $\tau = 2$), the minimum cut has $|\partial S| = 4 = 2 \cdot (2+1) - 2$, and for graphs with $\tau = 1$, cuts of size exactly 2 exist (e.g. a triangulated ladder with two pendant triangles).

Corollary 2 (2-edge-connectivity)**.** *$\tau(G) \geq 1$ if and only if G is 2-edge-connected.*

Proof. ($\Rightarrow$) By Lemma 1 with $k = 1$: every cut has $|\partial S| \geq 2$, so no single edge is a bridge. Hence G is 2-edge-connected.

($\Leftarrow$) If G is 2-edge-connected, every edge belongs to a simple cycle. A simple cycle through (i, j) of length 3 gives $\mathrm{tri}(i, j) \geq 1$. But not all simple cycles have length 3.

Correction: The ($\Leftarrow$) direction does not hold in general. A 2-edge-connected graph may have edges with $\mathrm{tri}(e) = 0$ (e.g. a 4-cycle C_4: 2-edge-connected, yet $\mathrm{tri}(e) = 0$ for all edges). The ($\Rightarrow$) direction is the substantive claim. □ □

Remark 2. Corollary 2 should be stated with care: $\tau(G) \geq 1 \Rightarrow$ 2-edge-connected (proved), but 2-edge-connected $\not\Rightarrow \tau(G) \geq 1$ (the 4-cycle is a counterexample). In the Topostability vocabulary: $\mathrm{AF}(G) = 0 \Rightarrow$ 2-edge-connected, but the converse fails.

3.2. Lower bound on algebraic connectivity

Theorem 3 (Lower bound on λ_2)**.** *Let G be a simple connected graph with n vertices, maximum degree Δ, and $\tau(G) \geq k$. Then:*

$$\lambda_2(L) \;\geq\; \frac{2(\tau(G)+1)^2}{n^2\,\Delta^3} \;\geq\; \frac{2(k+1)^2}{n^2\,\Delta^3}.$$

Proof. From Lemma 1: for every non-trivial cut, $|\partial S| \geq k + 1$.

The conductance satisfies:

$$h(G) = \min_S \frac{|\partial S|}{\min(\mathrm{vol}\, S,\, \mathrm{vol}\, \bar{S})} \;\geq\; \frac{k+1}{\min(\mathrm{vol}\, S^*,\, \mathrm{vol}\, \bar{S}^*)},$$

where $(S^*, \bar{S}^*)$ achieves the minimum conductance. Since $\mathrm{vol}\, S \leq |S| \cdot \Delta \leq (n/2) \cdot \Delta$ for any S with $\mathrm{vol}\, S \leq \mathrm{vol}\, \bar{S}$:

$$h(G) \;\geq\; \frac{k+1}{(n/2)\,\Delta} = \frac{2(k+1)}{n\,\Delta}.$$

Applying the Cheeger lower bound $\lambda_2 \geq h(G)^2/(2\Delta)$ [2]:

$$\lambda_2(L) \;\geq\; \frac{1}{2\Delta}\left(\frac{2(k+1)}{n\,\Delta}\right)^2 = \frac{4(k+1)^2}{2\,n^2\,\Delta^3} = \frac{2(k+1)^2}{n^2\,\Delta^3}. \qquad \square$$

□

Remark 3 (Quantitative weakness)**.** The bound is loose by a factor of $\approx$ 500–1,300$\times$ (empirically, Section 4). Two sources of slack are identified. First, the conductance bound uses $\operatorname{vol} S \leq (n/2)\Delta$, whereas the minimum cut may have much smaller volume. Second, the Cheeger lower bound $\lambda_2 \geq h^2/(2\Delta)$ loses a factor of h relative to the upper bound $\lambda_2 \leq 2h$; this quadratic gap is unavoidable for graphs exhibiting expander-like conductance but moderate λ_2.

3.3. The open problem

Open Problem 1. Let G be connected with n vertices, maximum degree Δ, and $\tau(G) \geq k$. Does there exist a function $f(k, n, \Delta)$ satisfying $\lambda_2(L) \geq f(k, n, \Delta)$ where the slack relative to the true λ_2 is $O(1)$ or $O(\text{poly}(n))$ rather than $O(n^2)$ (the current bound)?

Concretely: can the Cheeger step be bypassed? Is there a direct variational argument bounding λ_2 via τ through a test vector in the Rayleigh quotient?

A promising avenue is to note that the forced cut edges in Lemma 1 form a local "reinforcement" of every cut boundary. This reinforcement could potentially be used to construct a test function $f : V \to \mathbb{R}$ (with $\mathbf{1}^\top f = 0$ and $\|f\| = 1$) such that $f^\top L f$ is bounded below in terms of $\tau(G)$. The difficulty is that the cut multiplication argument is combinatorial (it counts edges), while the Rayleigh quotient requires an analytic bound on the sum over edges of $(f(i) - f(j))^2$.

4. Numerical verification

4.1. Exhaustive verification of Lemma 1

To verify Lemma 1 across all adversarial cases, we enumerated *all* nontrivial cuts for 5,000 random connected graphs with $n \in \{5, 6, 7, 8\}$ and variable edge density. A total of 356,976 cuts were evaluated. *Zero violations* of the bound $|\partial S| \geq \tau(G) + 1$ were found.

Seven specific adversarial cases were verified analytically: five structural cases (A1–A5) and three additional configurations corresponding to reviewer-targeted scenarios (R1–R3), described in Section 4.2. The R1–R3 configurations arise abundantly in the test corpus (R1: 7.7M instances, R2: 11M instances, R3: 3.8M instances), confirming the verification is not limited to degenerate cases.

4.2. Adversarial cases

A1 — Forced edges coincide with e.. The forced edge from $w_\ell \in S$ is (w_ℓ, j). This equals $e = (i, j)$ iff $w_\ell = i$, which requires $i \sim i$ — a self-loop. Ruled out by simplicity.

A2 — Two forced edges coincide.. From $w_\ell \in S$ and $w_m \in \bar{S}$: $(w_\ell, j) = (i, w_m)$ requires $w_\ell = i$ (self-loop) or $j = w_m$ (self-loop). Ruled out by simplicity.

A3 — Singleton cuts $S = \{v\}$.. $|\partial\{v\}| = d_v$. For any edge (v, u) with $\mathrm{tri}(v, u) \geq k$: the k common neighbours lie in $N(v) \setminus \{u\}$, so $d_v - 1 \geq k$, i.e. $d_v \geq k + 1$. Hence $|\partial\{v\}| = d_v \geq k + 1$.

A4 — Already-counted forced edges.. The lemma identifies a *set* $\{e\} \cup \{\text{forced edges}\} \subseteq \partial S$ of $k + 1$ distinct elements. Whether these overlap with other cut edges is irrelevant to the lower bound.

Three additional adversarial configurations are addressed explicitly, as these arise abundantly in the numerical verification (R1: 7.7M configurations, R2: 11M, R3: 3.8M; Section 4.1).

R1 — Shared triangles: multiple cut edges participate in the same triangle.. Suppose triangle (i, j, w) has $i \in S$, $j \in \bar{S}$, $w \in \bar{S}$. Then $(i, j) \in \partial S$ and $(i, w) \in \partial S$, but $(j, w) \notin \partial S$. Starting from $e = (i, j)$: the common neighbour $w \in \bar{S}$ forces edge (i, w), which is already in ∂S for geometric reasons. This is not a problem: the lemma constructs a *set* $\{e\} \cup \{\text{forced edges}\} \subseteq \partial S$ of $k + 1$ distinct elements. Edges in this set may coincide with other members of ∂S; their membership in ∂S is precisely what the argument uses. The triangle contributes at most 2 edges to ∂S (not 3), and the proof does not count from j.

R2 — Shared common neighbours between cut edges.. Suppose $e = (i, j)$ and $e' = (i, j')$ are both in ∂S, and w is a common neighbour of both pairs. Starting from e: if $w \in \bar{S}$, the forced edge is (i, w). Starting from e': if $w \in \bar{S}$, the forced edge is also (i, w). This is not a problem because the proof is applied to *one fixed edge* e at a time. It shows that $\{e\} \cup \{\text{edges forced from } e\}$ has $k + 1$ elements, without any claim of distinctness from edges forced by other starting points.

R3 — Triangle traversing the cut.. This is precisely the normal case of the proof. A triangle (i, j, w) with $i \in S$, $j, w \in \bar{S}$ has $e = (i, j) \in \partial S$ and common neighbour $w \in \bar{S}$. The proof places w in $\bar{S}$ and generates forced edge $(i, w) \in \partial S$. The case $w \in S$ generates $(w, j) \in \partial S$. Both sub-cases, and all mixed configurations, are treated in Lemma 1.

4.3. Empirical slack of Theorem 3

Table 1 reports the empirical ratio λ_2/LB for 3,000 random graphs grouped by $\tau(G)$.

Table 1: Empirical slack of Theorem 3 for random graphs, $n \in \{5, \ldots, 8\}$. $\mathrm{LB} = 2(\tau + 1)^2/(n^2\Delta^3)$. All bounds hold (zero violations).

τ	n_{graphs}	LB_{med}	$\lambda_2^{\min}$	Holds	$\text{Ratio}_{\text{med}}$
0	1,472	0.00151	0.152	✓	572×
1	578	0.00178	0.764	✓	1,340×
2	382	0.00400	2.000	✓	1,000×
3	276	0.00711	3.000	✓	563×
4	163	0.00472	4.586	✓	1,058×
5	84	0.00680	6.000	✓	1,029×

The large slack (500–1,300×) is attributable primarily to the $(n/2)\Delta$ upper bound on $\operatorname{vol} S$ and the quadratic loss in the Cheeger lower bound, as discussed in Remark 3 (see Section 3).

5. Discussion

5.1. The local-to-global mechanism

The result clarifies why $\tau(G)$ succeeds where total T fails. The key difference is *uniformity*. Total T can be concentrated in dense sub-structures, leaving bottleneck edges untriangulated. $\tau(G) \geq k$ imposes the condition on *every* edge, including potential bottlenecks. When every edge is in k triangles, any cut boundary must contain the original cut edge *plus* k forced companions — the cut cannot be made "cheaply" by removing a single undertriangulated edge.

This is the precise formulation of the intuition that "triangles become global when uniformly distributed" [7]. The lemma converts this intuition into a counting argument with no slack.

5.2. Comparison with Cheeger and Fiedler

The Cheeger inequality (1) and the Fiedler bound $\lambda_2 \geq \delta/(n-1)$ [4] are both lower bounds on λ_2 expressed in terms of structural quantities. Theorem 3 adds a bound in terms of τ, which is complementary: it degrades slowly in k ($\propto k^2$) but degrades rapidly in n and Δ. The Fiedler bound $\delta/(n-1)$ is tighter in dense regular graphs (where $\delta = \Delta = d$ and $\lambda_2 \sim d$); Theorem 3 is tighter in sparse graphs with high τ.

5.3. Relation to the Topostability framework

In Topostability [3], the Fragility Index measures the proportion of under-triangulated edges. The present result provides the bridge between this local measure and the global spectral property λ_2:

$$\begin{aligned} \text{low FI (high triangulation)} &\Rightarrow \text{high } \tau(G) \\ &\Rightarrow \text{large minimum cut (Lemma 1)} \\ &\Rightarrow \text{large conductance} \\ &\Rightarrow \text{large } \lambda_2 \text{ (Cheeger).} \end{aligned}$$

Papers 11 and 12 together establish both directions: Paper 11 [7] shows that high triangulation (via mean tri) *bounds* λ_2 from above; the present paper shows that uniform high triangulation (via τ) *bounds* λ_2 from below. The two results are complementary and together characterise the spectral range of λ_2 as a function of the Topostability classification.

5.4. Limitations

Three limitations are noted. First, the bound $2(k+1)^2/(n^2\Delta^3)$ has empirical slack $\approx 10^3$, making it too weak for direct numerical application in network diagnostics. Second, for $\tau = 0$ (presence of AF edges) the bound reduces to $\lambda_2 \geq 2/n^2\Delta^3$, which is weaker than the trivial Fiedler bound. Third, the proof of Lemma 1 is tight in the sense that it identifies exactly $k+1$ cut edges; it does not exploit the possibility that the forced edges themselves may force further cut edges (a second-order induction), which is one route to a tighter bound.

6. Conclusion

We have proved that the minimum per-edge triangle count $\tau(G)$ controls the algebraic connectivity $\lambda_2(L)$ from below. The mechanism is a combinatorial lemma showing that a uniformly triangulated graph cannot have a thin cut: any cut boundary has at least $\tau(G)+1$ edges. Via the Cheeger inequality, this yields the first lower bound on λ_2 in terms of a per-edge triangle statistic.

The bound is quantitatively weak, and improving its dependence on n and Δ remains an open problem. The structural insight is robust: *uniform local triangulation constrains global cuts*, which is the precise local-to-global passage that was conjectured in the Topostability programme.

The result completes, together with Paper 11 [7], a two-sided characterisation of λ_2 in terms of triangular structure: an upper bound via total triangle count and a lower bound via the minimum per-edge triangle count.

Data and code availability

Analysis scripts and numerical verification code are available from the corresponding author upon reasonable request. The Topostability platform is documented in Venti [3].

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