THE ORDINAL RELATIONS OF THE TERMS OF A CONVERGENT SEQUENCE

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1. Suppose that we are given a sequence

$$(a_n) \quad (n = 1, 2, 3, \ldots) \tag{1}$$

and are able, when any two terms of the sequence, a_p and a_q say, are taken, to determine which of the relations

$$a_p \gtreqless a_q \tag{2}$$

holds between a_p and a_q. We may express this by saying that we know the *ordinal relations* (as regards order of magnitude) which hold between the various terms of the sequence.

This note is devoted to a consideration of the following question, which seems to me of some logical interest: *How far is it possible to discriminate, on the ground* simply *of the ordinal relations that hold between their terms, between sequences which converge to a limit and sequences which do not?* And, in so far as this *is* possible, what is the simplest expression of the ordinal relations which characterise convergent sequences?

2. It is easy to recognise certain limitations to which the answers to these questions must be subject. In the first place, it is clearly hopeless to attempt to discriminate in this way between *convergent* and *properly divergent* sequences. This appears at once from a consideration of the simplest case of all, that of *monotonic* (say increasing) sequences. In this case

$$a_p \geqslant a_q,$$

if $p \geqslant q$. A classical theorem asserts that (a_n) is convergent or divergent to $+\infty$. But we cannot distinguish between the two cases: the sequences $(1-1/n)$ and (n) are characterised by exactly the same ordinal relations.

Thus we are obliged to regard (as, indeed, it is often convenient to regard) sequences which diverge to $+\infty$ or to $-\infty$ as a special case of convergent sequences, and to distinguish only between convergent and oscillatory series.

But even when this reservation is made, it is easy to see that the first of our two questions cannot be answered with a simple affirmative. Consider, for example, the sequence (a_n), where $a_n = (\sin \frac{1}{2}n\pi)/n$. Here the limit is zero. And now consider the sequence defined by the equations

$$b_n = 1 + \frac{1}{n} \sin \tfrac{1}{2}n\pi \qquad (n = 4k+1),$$

$$b_n = \frac{1}{n} \sin \tfrac{1}{2}n\pi \qquad (n = 4k,\ 4k+2),$$

$$b_n = -1 + \frac{1}{n} \sin \tfrac{1}{2}n\pi \quad (n = 4k+3).$$

Each of the inequalities $a_p \gtreqless a_q$ involves the corresponding one of the inequalities $b_p \gtreqless b_q$: the ordinal relations of the terms of the two sequences are precisely the same; but (a_n) is convergent, while (b_n) has the *three* limiting values -1, 0, 1.

We shall see later that this last example is of genuine theoretical interest as being as simple an example as possible of the one type of oscillatory sequence that cannot be distinguished from a convergent sequence simply by means of its ordinal relations. The answer to our questions is, in fact, that the ordinal relations of convergent sequences do possess an exceedingly simple characteristic which distinguishes them from all other sequences except one very special type of oscillatory sequence; but that this special type of oscillatory sequence cannot be distinguished from a convergent sequence by *any* marks of its ordinal relations.

The Classes P_n, Q_n, R_n.

3. We choose a definite term a_n of the sequence (a_n) and divide the terms of the sequence (including a_n itself) into three classes P_n, Q_n, R_n, which comprise the terms a_m for which

$$a_m < a_n, \qquad a_m = a_n, \qquad a_m > a_n$$

respectively. Thus in a monotonic (increasing) sequence, no two terms of which are equal, P_n includes $a_1, a_2, \dots a_{n-1}$, Q_n includes a_n only, and R_n includes $a_{n+1}, a_{n+2}, \dots$.

Quasi-monotonic Sequences.

4. A *quasi-monotonic (increasing) sequence* is a sequence such that the classes P_n, Q_n are always finite (or such that P_n+Q_n is always finite).

THEOREM.—*The necessary and sufficient condition that a sequence should converge to a limit, greater than any of its terms, is that it should be quasi-monotonic (increasing).*

There are, of course, a corresponding definition and theorem for quasi-monotonic (decreasing) sequences.

Proof of the Theorem.—First, let the sequence converge to a limit greater than any a_n. If P_n+Q_n were infinite, for any particular value of n, we could find values of m as large as we please and such that $a_m \leqslant a_n$. This would imply $\lim a_m \leqslant a_n$, contrary to our hypothesis.

Secondly, let P_n+Q_n be always finite. Then, given n_1, we can choose n_2 so that

$$a_p > a_q \quad (p \geqslant n_2,\ q \leqslant n_1):$$

for the class formed by all the terms which belong to any of the classes P_q+Q_p $(q \leqslant n_1)$ is finite. Also we can suppose n_2 to be the *least* number satisfying this condition. Similarly we define n_3 as the least number such that

$$a_p > a_q \quad (p \geqslant n_3,\ q \leqslant n_2):$$

and so on. Thus we define a monotonic sequence (a_{n_ν}), which must tend to a limit a greater than any a_{n_ν} or to $+\infty$. If $a_{n_\nu} \to +\infty$, we have

$$a_{n_\nu} > G$$

by choice of ν, however large be G, and so

$$a_p > G \quad (p \geqslant n_{\nu+1}),$$

so that $a_n \to +\infty$.

On the other hand, if $a_{n_\nu} \to a$, a must be greater than any a_n. For, if $a_n \geqslant a$, and we choose ν so that $n_\nu > n$, we have $a_{n_{\nu+1}} > a$, which is impossible.

Finally, however small be ϵ, we can choose ν so that

$$a-a_{n_\nu} < \epsilon,$$

and so
$$a-a_n < a-a_{n_\nu} < \epsilon \quad (n \geqslant n_{\nu+1}).$$

Hence $a_n \to a$, and the theorem is established.

5. The proof may be shortened by the use of maximum and minimum limits of indetermination. If a_n does not converge to a limit, let G and H be its minimum and maximum limits: H may be $+\infty$, but neither can be $-\infty$, since only a finite number of terms are less than any a_n.

We can choose a subsequence (b_μ) whose limit is G, and a subsequence (c_ν) whose limit is H, and, as $G < H$, we can determine n_0 so that

$$b_\mu < c_\nu \quad (\mu, \nu \geqslant n_0).$$

Hence P_n is infinite for c_ν, in contradiction to our hypotheses.

6. The theorem of § 4 is verbally identical with one which I obtained some time ago,* but using a different definition. The definition of a quasi-monotonic (increasing) sequence that I adopted then is as follows: let ν be the largest number such that $a_1, a_2, \ldots, a_\nu$ are all less than a_n, so that ν is a function of n; then the sequence is quasi-monotonic (increasing) if $\nu \to \infty$ with n. It is easy to prove directly that the two definitions are equivalent. First, if the definition just stated is satisfied, then $P_n + Q_n$ is always finite. Otherwise we could, for some value of n, find an infinite number of terms $a_{m_1}, a_{m_2}, \ldots$, all less than or equal to a_n. But, by the definition, if ν is large enough,

$$a_m < a_{m_\nu} \quad (m \leqslant n),$$

which involves a contradiction.

Secondly, suppose that the definition just stated is *not* satisfied. Then we can find a number n_0 such that

$$a_{n_\nu} \geqslant a_{m_\nu},$$

where (m_ν) is an infinite ascending sequence, and $n_\nu \leqslant n_0$ for all values of ν. Since a_{n_ν} has at most n_0 possible values, there must be some one a_n $(n \leqslant n_0)$ which is not less than an infinity of the terms of the sequence (a_{m_ν}): and this shows that $P_n + Q_n$ is not always finite.

Quasi-monotonic Sequences in the wider sense.

7. A *quasi-monotonic (increasing) sequence in the wider sense* is a sequence such that P_n is always finite.

A quasi-monotonic increasing sequence in the wider sense has a limit not less than *any term of the sequence.* The proof is the same as that of the theorem of § 4: but there is one case in which the limit may be equal to terms of the sequence, viz., the trivial case in which all the terms of the sequence, from a certain rank onwards, are equal.

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* The result is stated in substance, but without proof, as an example in Bromwich's *Infinite Series* (Ex. 14, p. 393).

The converse is not true. For let us take a quasi-monotonic (in the stricter sense) increasing sequence. It has a limit l greater than any term. In this sequence interpolate in any manner an infinity of terms equal to l. The sequence has still the limit l, not less than any of its terms: but it is no longer quasi-monotonic, even in the wider sense; for, if $a_n = l$, P_n is infinite.

Apart from this case of exception, it is easy to see that the theorem obtained from that of § 4 by changing *greater* to *not less* and *quasi-monotonic* to *quasi-monotonic in the wider sense* is true.

Convergent Sequences in General.

8. Now let us consider any convergent sequence (a_n). If its limit is $+\infty$ or $-\infty$, it is a quasi-monotonic sequence, increasing or decreasing.

Let us then suppose that it converges to a finite limit a. If Q_n is infinite for any value of n, then $a_n = a$, and the sequence contains an infinity of terms equal to a. The remaining terms may be divided into two classes, those less than and those greater than a. Hence *any convergent sequence is formed of* (i) *a sequence of equal terms,* (ii) *a quasi-monotonic* (*increasing*) *sequence, and* (iii) *a quasi-monotonic* (*decreasing*) *sequence.* Any of these classes, of course, may be finite or entirely absent. The terms of (ii) and (iii) are characterised by the facts that $P_n + Q_n$ and $Q_n + R_n$ are respectively finite: for the terms of (i) the classes P_n, Q_n, R_n may all be infinite.

Conversely, suppose that a sequence (a_n) possesses the properties (i) *that for at most one value of a are there infinitely many terms equal to a,* (ii) *that for any term of the sequence not equal to a either P_n or R_n is finite.*

For the latter terms Q_n is finite, and so either $P_n + Q_n$ or $Q_n + R_n$. Let (b_μ) be the sequence formed by the terms for which $P_n + Q_n$ is finite, and (c_ν) the sequence formed by the terms for which $Q_n + R_n$ is finite. If P'_μ, Q'_μ, R'_μ are the classes corresponding to b_μ, but formed from terms of the sequence (b_μ) only, $P'_\mu + Q'_\mu$ is, *a fortiori*, finite, and so (b_μ) is a quasi-monotonic (increasing) sequence, and converges to a limit β greater than any of its terms. Similarly (c_ν) is a quasi-monotonic (decreasing) sequence, and converges to a limit γ less than any of its terms. Also

$$\beta \leqslant a \leqslant \gamma.$$

For it is obvious that, if $\beta > a$ or $\beta > \gamma$, then P_n would be infinite for some terms of (b_μ).

If a, β, γ are equal, say all equal to a, the sequence converges to a limit, viz. a. But it is quite possible that $\beta < a < \gamma$: thus, in the case of the sequence considered in § 2, $\beta = -1$, $a = 0$, $\gamma = 1$.

If $\beta < a < \gamma$, the sequence has precisely the same ordinal relations as the sequence (b_n) defined by the equations

$$b_n = a_n \qquad (a_n = a),$$

$$b_n = a_n + (a - \beta) \quad (a_n < a),$$

$$b_n = a_n - (\gamma - a) \quad (a_n > a);$$

a sequence which plainly converges to the limit a.

Thus *the properties* (i) *and* (ii) *above characterise convergent sequences, in so far as these sequences can be characterised by their ordinal relations only. Every convergent sequence possesses them: and any sequence which possesses them is either convergent or is an oscillatory sequence of the special type formed by adding a constant to all the terms of a convergent sequence which are greater than the limit, and subtracting a constant from all those which are less than the limit. And this class of oscillatory sequences is indistinguishable from a convergent sequence by any test dependent on ordinal relations only.*

Oscillatory Sequences.

9. If (a_n) oscillates there must be terms for which both P_n and R_n are infinite (except in the special case discussed above). All of P_n, Q_n, R_n may always be infinite, as in the sequence

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \tfrac{2}{6}, \tfrac{3}{6}, \ldots;$$

or, again, P_n and R_n may always be infinite, and Q_n consist of one term only, as in the sequence deduced from that above by rejecting all fractions which are not in their lowest terms.