Nov. 12. 1733

APPROXIMATIO AD SUMMAM TERMINORUM BINOMII

$(a+b)^n$ in Seriem expansi,

Autore A. D. M. R. S. S.

Quamquam solutio Problematum ad fortem spectantium non raro exigit ut plures Termini Binomii $(a+b)^n$ in summam colligantur; attamen in potestatibus excelsis res adeo laboriosa videtur, ut perpauci hoc opus aggredi curaverint; Jacobus & Nicolaus Bernoulli viri Doctissimi primi quod sciam tentarunt quid sua industria in hoc genere praestare posset, in quo etiamsi uterque propositum summa cum laude sunt assecuti, aliquid tamen ultra posse requiri, hoc est approximationem ad summam; non enim tam de approximatione videntur fuisse solliciti quam de assignandis certis limitibus quos Summa Terminorum nequeat transcendere. Quam vero viam illi tenuerint, breviter in Miscellaneis meis exposui * quae consulat Lector si vacat, quod ipsi tamen scripserint melius erit forsitan consulere: Ego quoque in hanc disquisitionem incubui; quod autem eo me primum impulit non profectum fuit ab opinione me ceteros anteiturum, sed ab obsequio in Dignissimum virum qui mihi auctor fuerat ut haec susciperem; Quid quod est, novas cogitationes prioribus subnecto, sed eo ut connexio postremorum cum primis melius appareat, mihi necesse est ut pauca jampridem a me tradita denuo proferam.

I. Duodecim jam sunt anni & amplius cum illud inveneram; si Binomium 1+1 ad potestatem n permagnam attollatur, ratio quam Terminus Medius habet ad summam Terminorum omnium, hoc est ad 2^n, ad hunc modum poterit exprimi

$$\frac{2A \times (n-1)^n}{n^n \sqrt{n-1}},$$

ubi A eum numerum exponit cujus Loga-

A rithums

* Vide Miscellanea Analytica pag. 96. 97. 98. 99.

rithmus hyperbolicus est $\frac{1}{12} - \frac{1}{360} + \frac{1}{1260} - \frac{1}{1680}$ &c. quam seriem ad libitum mihi continuare licuit, sed quia quantitas $\left(\frac{n-1}{n}\right)^n$ seu $\left(1 - \frac{1}{n}\right)^n$ fere data fit, quod consideranti facile patebit, sequitur in Potestate infinita quantitatem illam datam iri, eamque exhibiturum numerum illum cujus Logarithmus hyperbolicus est -1; hinc fiet ut si B designarit numerum illum cujus Logarithmus hyperbolicus est $-1 + \frac{1}{12} - \frac{1}{360} + \frac{1}{1260} - \frac{1}{1680}$ &c. expressio suprascripta evasura fit $\frac{2B}{\sqrt{n-1}}$ seu $\frac{2B}{\sqrt{n}}$, atque adeo si signa seriei mutentur, ponaturque B aequalis numero cujus Logarithmus hyperbolicus est $1 - \frac{1}{12} + \frac{1}{360} - \frac{1}{1260} + \frac{1}{1680}$ &c. expressio illa futura sit $\frac{2}{B\sqrt{n}}$.

Cum primum ad hanc disquisitionem animum appuli, eo contentus fui ut praeterpropter valorem quantitatis B determinarem quod quidem factum fuerat additione paucorum hujus seriei Terminorum, quorum summa potentia fuerat tanquam Logarithmus istius quantitatis, attamen tarditas convergentiae me deterruerat quominus longius procederem, donec vir Doctissimus mihique amicissimus *Jacobus Stirling* qui post me ad hanc disquisitionem methodo a mea valde diversa se contulit, comperit quantitatem B denotare radicem quadratam circumferentiae Circuli cujus radius est unitas, ita ut si haec circumferentia appelletur c, ratio Medii termini ad Terminos omnes exprimatur per $\frac{2}{\sqrt{nc}}$[1].

Quanquam vero, si quacunque ratione possit obtineri is numerus qui seriei hyperbolicae respondet, non multum intersit utrum nec ne ejus relatio ad Circulum perfecta fuerit, attamen libenter fateor hanc patefactionem et labori pepercisse, et elegantiam singularem in solutionem induxisse.

II. Id mihi simul compertum fuit; Logarithmum rationis, quam in Potestate maxima, Medius Terminus habet ad Terminum intervallo l a Medio distantem, posita $n = 2m$, sic expressum iri,

$$\overline{m + l - \frac{1}{2}} \times \log(m+l-1) + \overline{m - l + \frac{1}{2}} \times \log(m-l+1) - 2m \times \log m + \log\left(\frac{m+l}{m}\right).$$

COROL.

[1] De Moivre uses c to denote the circumference of a unit circle, i.e., $c = 2\pi r$ with $r = 1$, so $c = 2\pi$. In modern notation, this constant is π.

COROLLARIUM I.

Hinc mihi illud colligere licet; si sit m quantitas infinite magna, Logarithmum hujus rationis fore $\frac{ll}{m}$ sive $\frac{2ll}{n}$, ergo Logarithmus rationis quam Terminus a Medio distans Intervallo l habet ad Medium est $-\frac{2ll}{n}$.

COROLLARIUM II.

Cum numerus ille qui Logarithmo hyperbolico $-\frac{2ll}{n}$ respondet, sit

$$1 - \frac{2ll}{n} + \frac{4l^4}{2nn} - \frac{8l^6}{6n^3} + \frac{16l^5}{24n^4} - \frac{32l^{10}}{120n^5} + \frac{64l^{12}}{720n^6} \quad \&c.$$

sequitur rationem quam in potestate infinita summa Terminorum inclusorum intra Medium & Terminum illum cujus distantia a Medio est l, fore

$$\frac{2}{\sqrt{nc}} \text{ in } l - \frac{2l^3}{3n} + \frac{4l^5}{2\times 5nn} - \frac{8l^7}{6\times 7n^3} + \frac{16l^9}{24\times 9n^4} - \frac{32l^{11}}{120\times 11n^5} \quad \&c.$$

Pone jam $l = s\sqrt{n}$, tunc ratio superior exprimetur serie,

$$\frac{2}{\sqrt{c}} \text{ in } s - \frac{2s^3}{3} + \frac{4s^5}{2\times 5} - \frac{8s^7}{6\times 7} + \frac{16s^9}{24\times 9} - \frac{32s^{11}}{120\times 11} \quad \&c.$$

Sit praeterea $s = \frac{1}{2}$, tunc series novissima in hanc alteram convertetur,

$$\frac{2}{\sqrt{c}} \text{ in } \frac{1}{2} - \frac{1}{3\times 4} + \frac{1}{2\times 5\times 8} - \frac{1}{6\times 7\times 16} + \frac{1}{24\times 9\times 32} - \frac{1}{120\times 11\times 64} \quad \&c.$$

quae adeo celeriter ad verum convergit ut juvantibus octo vel novem terminis, summa desiderata ad sex aut septem loca decimalium perduci possit; haec autem summa reperietur esse $= 0.427182$, cujus Logarithmo Tabulari si addatur Logarithmus Tabularis quantitatis $\frac{2}{\sqrt{c}}$, hoc est $\overline{9}.9019400$, Numerus huic summae respondens erit 0.341344.

LEMMA

Si Eventus aliquis ita a forte pendeat ut Probabilitates Contingentiae & Noncontingentiae sint inter se aequales, atque instituantur Experimenta numero n quibus observetur utrum eventus se ostensurus sit an occultatus; Probabilitas fore ut nec saepius se ostendat quam vicibus $\frac{1}{2}n + l$, nec rarius quam vicibus $\frac{1}{2}n - l$ sic invenietur.

Sint termini L & L hinc inde a Medio Termino Binomii $(1+1)^n$ intervallis l aequaliter distantes, sit etiam s summa Terminorum inclusorum intra L & L, una cum extremis; tunc Probabilitas quaesita recte exprimetur per $\frac{s}{2^n}$, quod cum sit facillimum, Demonstratione non indiget.

COROLLARIUM III.

Ergo si possent capi experimenta n numero infinita, Probabilitas fore ut Eventus nec saepius se ostendat quam vicibus $\frac{1}{2}n+\frac{1}{2}\sqrt{n}$, nec rarius quam $\frac{1}{2}n-\frac{1}{2}\sqrt{n}$ exprimeretur per duplam summam numeri in Corollario secundo exhibiti hoc est per 0.682688, Probabilitas vero contraria, qua scilicet quis affirmare posset Eventum vel saepius se ostensurum quam vicibus $\frac{1}{2}n+\frac{1}{2}\sqrt{n}$ vel rarius quam $\frac{1}{2}n-\frac{1}{2}\sqrt{n}$ erit 0.317312, utpote quae cum superiore Probabilitate complectatur certitudinem cujus mensura est unitas, harum vero Probabilitatum ratio in minimis terminis est 28 ad 13 proxime.

COROLLARIUM IV.

Si numerus Experimentorum sit finitus sed tamen magnus, qualis v. g. 3600, hinc erit $\frac{1}{2}n = 1800$, & c. $\frac{1}{2}\sqrt{n} = 30$; Probabilitas igitur ut post facta experimenta 3600, Eventus nec saepius se ostendat quam vicibus 1830, nec rarius quam 1770, erit 0.682688 proxime.

COROLLARIUM V.

Ergo sit illud fixum, in Potestatibus magnis quibuscunque, rationem quam dupla summa Terminorum a Medio ad eum usque cujus distantia a Medio est $\frac{1}{2}\sqrt{n}$, habet ad summam Terminorum omnium, bene designatum iri per 0.682688 sive per $\frac{28}{41}$ prope.

Ne tamen putes Potestatem n oportere esse enormiter magnam, quamvis enim non superet nongentesimam, quin etiam si multo infra subsidat, res satis commode definietur, quod mihi experienti constat.

COROLLARIUM VI.

Si l exponatur per $\sqrt{n}$, series non tam celeriter converget quam antea fecerat, ita ut forte non pauciores quam duodecim vel tredecim termini ad summationem tolerabilem requirerentur, & quidem eo plures essent adhibendi quo l major esset; quapropter artificio utor Quadraturarum Mechanicarum quo ex ordinatis quibusdam ad aequalia intervalla positis liceat Quadraturam Curvae prope elicere: sint igitur quatuor ordinatae A, B, C, D ad aequalia intervalla positae, sit praeterea l distantia inter primam ordinatam & ultimam; quibus datis, Area sic prodibit $\frac{A+D+3\times\overline{B+C}}{8}\times l$, cujus Demonstrationem passim apud Mathematicos reperire licet. Jam sumantur distantiae $0\sqrt{n}$, $\frac{1}{6}\sqrt{n}$, $\frac{2}{6}\sqrt{n}$, $\frac{3}{6}\sqrt{n}$, $\frac{4}{6}\sqrt{n}$, $\frac{5}{6}\sqrt{n}$, $\frac{6}{6}\sqrt{n}$, quarum unaquaeque proximam antecedentem superet differentia $\frac{1}{6}\sqrt{n}$, quarumque ultima sit $\sqrt{n}$; ex his seligantur quatuor ultimae $\frac{3}{6}\sqrt{n}$, $\frac{4}{6}\sqrt{n}$, $\frac{5}{6}\sqrt{n}$, $\frac{6}{6}\sqrt{n}$, quarum dupla quadrata signo negativo affecta, divisaque per n, hoc est — $\frac{1}{2}$, — $\frac{8}{9}$, — $\frac{25}{18}$, — 2, tractentur tanquam Logarithmi hyperbolici totidem ordinatarum, quae proinde erunt 0.60653, 0.41111, 0.24935, 0.13534; quibus ordine substitutis in locum quantitatum A, B, C, D, atque posita l per $\frac{1}{2}\sqrt{n}$, reperietur area $= 0.170203\times\sqrt{n}$; quae si postea duplicetur, tum multiplicetur dupla summa per $\frac{2}{\sqrt{nc}}$, tunc producta quantitas erit 0.27160; addatur igitur haec Area ad Aream ante repertam hoc est ad 0.682688, & summa 0.954288 exhibebit Probabilitatem fore ut post facta experimenta n, Eventus nec saepius se ostendat quam vicibus $\frac{1}{2}n+\sqrt{n}$, nec rarius quam $\frac{1}{2}n-\sqrt{n}$; Probabilitas vero contraria reperietur esse 0.045712, quo fit ut ratio Probabilitatum sit 21 ad 1 proxime.

Et eadem procedendi ratione, Probabilitas fore ut Eventus nec saepius se ostendat quam vicibus $\frac{1}{2}n+\frac{3}{2}\sqrt{n}$, nec rarius quam $\frac{1}{2}n-\frac{3}{2}\sqrt{n}$ repereiretur esse $= 0.99874$, Probabilitas autem contraria 0.00126, proinde ratio Probabilitatum erit 792 ad 1, proxime.

Si detur potestas finita qualis v. g. 3600, atque requiratur Probabilitas fore ut post facta experimenta 3600, Eventus nec saepius se ostenderit quam vicibus 1850, nec rarius quam 1750, qui duo numeri possunt ad libitum sumi, modo aequaliter distent a dimidia summa 1800; pone dimidiam differentiam numerorum 1850 & 1750, hoc est $50 = s\sqrt{n}$;pone jam $n = 3600$,

hinc

hinc erit $\sqrt{n} = 60$ atque adeo $s = \frac{50}{60} = \frac{5}{6}$; quapropter si sumatur ea ratio quam in Potentia infinita dupla summa Terminorum respondens intervallo $\frac{5}{6}\sqrt{n}$, habet ad summam Terminorum omnium, obtinebitur in Potentia magna Probabilitas quaesita proxime.

L E M M A II.

In Potentia quacunque $(a + b)^n$ in seriem expansi, Maximus Terminus is est in quo indices quantitatum a & b eandem inter se rationem habent quam constantes ipsae.

L E M M A III.

Si Eventus aliquis ita a forte pendeat ut Probabilitates Contingentiae & Non-contingentiae sint inter se ut a ad b, atque instituantur Experimenta numero n quibus exploratur utrum Eventus se ostensurus sit, an occultatus; Probabilitas fore ut nec saepius se ostendat quam vicibus $\frac{an}{a+b} + l$, nec rarius quam $\frac{an}{a+b} - l$, sic invenietur.

Sint Termini duo L & R hinc inde a Maximo aequaliter distantes intervallo l, sit etiam S summa Terminorum inclusorum intra L & R una cum extremis, tunc Probabilitas quaesita recte exprimetur per $\frac{S}{(a+b)^n}$.

C O R O L L A R I U M VIII.

Ratio quam in Potentia infinita, Maximus Terminus habet ad summam $(a+b)^n$ recte definietur fractione $\frac{a+b}{\sqrt{abnc}}$.

C O R O L L A R I U M IX.

Si in Potestate infinita Terminus aliquis a Maximo distet intervallo l, Logarithmus hyperbolicus Rationis quam Terminus ille habet ad Maximum erit $-\frac{(a+b)^2}{2abn} \times ll$; modo tamen ne l ad n habuerit rationem datam; utrum vero Terminus ille ad dextram an ad levam Termini maximi constituatur, perinde erit.

COROLLARIUM X.

Si Contingentia & Non-contingentia Eventuum se habuerint inter se ut a ad b, Problemata de summatione Terminorum aeque facile solventur ac illa in quibus Contingentia & Non-contingentia sunt inter se in ratione aequalitatis.

Vix dubito quin Demonstrationes octavi & noni Corollarii requirantur, utpote quae non videantur facile fluere ex principiis ante positis, sed si qui sint qui id sibi tanti putent ut Supplementum nostrum consulant, iis abunde satisfactum iri confido.

FINIS.

A trial of an English translation – No guarantee

Nov. 12. 1733

APPROXIMATION TO

THE SUM OF THE TERMS OF THE BINOMIAL

$(a+b)^n$ expanded into a Series,

By the Author A. D. M. R. S. S.

Although the solution of Problems pertaining to chance not infrequently requires that several Terms of the Binomial $(a+b)^n$ be collected into a sum; yet in high powers the matter seems so laborious that very few have undertaken this task; Jacob & Nicolaus Bernoulli, most learned men, were the first that I know of who attempted what their industry could accomplish in this area, in which even though each attained the goal with the highest praise, yet something further could still be sought, namely an approximation to the sum; for they do not seem to have been as concerned with approximation as with assigning certain limits which the Sum of the Terms cannot exceed. What method they followed, I have briefly explained in my Miscellanea * which the Reader may consult if he has time, although perhaps it will be better to consult what they themselves wrote: I also have devoted myself to this investigation; but what first impelled me to it was not the opinion that I would surpass others, but deference to a most worthy man who urged me to undertake this; Be that as it may, I now add new thoughts to previous ones, but in order that the connection of the latter with the former may appear better, I must again present a few things already delivered by me some time ago.

I. It is now twelve years and more since I discovered that; if the Binomial $1+1$ be raised to a very large power n, the ratio which the Middle Term has to the sum of all Terms, that is to 2^n, could be expressed in this way

$$\frac{2A \times (n-1)^n}{n^n\sqrt{n-1}},$$

where A represents that number whose Loga-

A rithm

* See Miscellanea Analytica pag. 96. 97. 98. 99.

hyperbolic logarithm is $\frac{1}{12} - \frac{1}{360} + \frac{1}{1260} - \frac{1}{1680}$ &c. which series I was able to continue at will, but because the quantity $\left(\frac{n-1}{n}\right)^n$ or $\left(1-\frac{1}{n}\right)^n$ is nearly given, as will be easily apparent to one considering it, it follows that in an infinite Power that quantity will be given, and it will represent that number whose hyperbolic logarithm is -1; hence it will happen that if B designates that number whose hyperbolic logarithm is $-1 + \frac{1}{12} - \frac{1}{360} + \frac{1}{1260} - \frac{1}{1680}$ &c. the above-written expression will become $\frac{2B}{\sqrt{n-1}}$ or $\frac{2B}{\sqrt{n}}$, and moreover if the signs of the series are changed, and B is set equal to the number whose hyperbolic logarithm is $1 - \frac{1}{12} + \frac{1}{360} - \frac{1}{1260} + \frac{1}{1680}$ &c. that expression will be $\frac{2}{B\sqrt{n}}$.

As soon as I turned my mind to this investigation, I was content to determine the value of the quantity B approximately, which indeed was done by the addition of a few Terms of this series, whose sum was considered as the Logarithm of that quantity, yet the slowness of convergence deterred me from proceeding further, until a most learned man and very friendly to me, *Jacobus Stirling*, who after me turned to this investigation by a method very different from mine, discovered that the quantity B denotes the square root of the circumference of a Circle whose radius is unity, so that if this circumference be called c, the ratio of the Middle term to all Terms is expressed by $\frac{2}{\sqrt{\pi c}}$.

Although indeed, if by any means that number which corresponds to the hyperbolic series can be obtained, it matters not much whether or not its relation to the Circle be perfect, yet I freely admit that this discovery has both saved labor, and introduced a singular elegance into the solution.

II. It was simultaneously discovered by me; that the Logarithm of the ratio, which in the highest Power, the Middle Term has to a Term distant by interval l from the Middle, with $n = 2m$, would be expressed thus,

$$\overline{m+l-\frac{1}{2}} \times \log(m+l-1) + \overline{m-l+\frac{1}{2}} \times \log(m-l+1) - 2m \times \log m + \log\left(\frac{m+l}{m}\right).$$

COROL.

COROLLARY I.

Hence I am permitted to gather this; if m is an infinitely large quantity, the Logarithm of this ratio will be $\frac{ll}{m}$ or $\frac{2ll}{\pi}$,[2] therefore the Logarithm of the ratio which a Term distant from the Middle by interval l has to the Middle Term is $-\frac{2ll}{\pi}$.

COROLLARY II.

Since the number which corresponds to the hyperbolic Logarithm $-\frac{2ll}{\pi}$, is

$$1-\frac{2ll}{n}+\frac{4l^4}{2nn}-\frac{8l^6}{6n^3}+\frac{16l^5}{24n^4}-\frac{32l^{10}}{120n^5}+\frac{64l^{12}}{720n^6}\quad \&c.$$

it follows that the ratio which, in an infinite power, the sum of Terms included between the Middle & that Term whose distance from the Middle is l, will be

$$\frac{2}{\sqrt{nc}}\text{ in } l-\frac{2l^3}{3n}+\frac{4l^5}{2\times 5nn}-\frac{8l^7}{6\times 7n^3}+\frac{16l^9}{24\times 9n^4}-\frac{32l^{11}}{120\times 11n^5}\quad \&c.$$

Now put $l=s\sqrt{n}$, then the above ratio is expressed by the series,

$$\frac{2}{\sqrt{c}}\text{ in } s-\frac{2s^3}{3}+\frac{4s^5}{2\times 5}-\frac{8s^7}{6\times 7}+\frac{16s^9}{24\times 9}-\frac{32s^{11}}{120\times 11}\quad \&c.$$

Furthermore let $s=\frac{1}{2}$, then the newest series is converted into this other one,

$$\frac{2}{\sqrt{c}}\text{ in }\frac{1}{2}-\frac{1}{3\times 4}+\frac{1}{2\times 5\times 8}-\frac{1}{6\times 7\times 16}+\frac{1}{24\times 9\times 32}-\frac{1}{120\times 11\times 64}\quad \&c.$$

which converges so quickly to the truth that with the help of eight or nine terms, the desired sum can be carried to six or seven decimal places; this sum moreover will be found to be $=0.427182$, to whose Tabular Logarithm if the Tabular Logarithm of the quantity $\frac{2}{\sqrt{c}}$ is added, that is $\overline{9}.9019400$, the Number corresponding to this sum will be 0.341344.

LEMMA

If any Event depends on chance such that the Probabilities of its Occurrence & Non-occurrence are equal to each other, and Experiments are conducted number n by which it is observed whether the event will show itself or be hidden; the Probability that it will show itself neither more often than $\frac{1}{2}n+l$ times, nor less often than $\frac{1}{2}n-l$ times will be found as follows.

2

Let the terms L & L be equally distant on either side of the Middle Term of the Binomial $(1+1)^n$ by intervals l, and let s be the sum of Terms included between L & L, together with the extremes; then the sought Probability is correctly expressed by $\frac{s}{2^n}$, which being very easy, needs no Demonstration.

COROLLARY III.

Therefore if one could take n experiments infinite in number, the Probability that the Event will show itself neither more often than $\frac{1}{2}n + \frac{1}{2}\sqrt{n}$ times, nor less often than $\frac{1}{2}n - \frac{1}{2}\sqrt{n}$ would be expressed by double the sum of the number exhibited in the second Corollary, that is by 0.682688, but the contrary Probability, by which one could affirm that the Event will show itself either more often than $\frac{1}{2}n + \frac{1}{2}\sqrt{n}$ times or less often than $\frac{1}{2}n - \frac{1}{2}\sqrt{n}$ will be 0.317312, since together with the previous Probability it encompasses certainty whose measure is unity, the ratio of these Probabilities in lowest terms is approximately 28 to 13.

COROLLARY IV.

If the number of Experiments is finite yet large, such as e.g. 3600, then $\frac{1}{2}n = 1800$, and $\frac{1}{2}\sqrt{n} = 30$; Therefore the Probability that after performing 3600 experiments, the Event will show itself neither more often than 1830 times, nor less often than 1770, will be approximately 0.682688.

COROLLARY V.

Therefore let it be fixed, that in any large Powers, the ratio which double the sum of Terms from the Middle up to that whose distance from the Middle is $\frac{1}{2}\sqrt{n}$, bears to the sum of all Terms, will be well represented by 0.682688 or approximately by $\frac{28}{41}$.

But do not think that the Power n needs to be enormously large, for although it does not exceed the nine hundredth, and even if it is much lower, the matter will be determined quite conveniently, as I find from experience.

C O R O L L A R Y VI.

If l is expressed by $\sqrt{n}$, the series will not converge as quickly as before, so that perhaps no fewer than twelve or thirteen terms would be required for a tolerable summation, and indeed more would need to be used the larger l is; for this reason I use the artifice of Mechanical Quadratures by which, from certain ordinates placed at equal intervals, it is possible to deduce approximately the Quadrature of the Curve: therefore let there be four ordinates A, B, C, D placed at equal intervals, and furthermore let l be the distance between the first ordinate & the last; given these, the Area will be produced as $\frac{A+D+3\times\overline{B+C}}{8}\times l$, the Demonstration of which is found everywhere among Mathematicians. Now take the distances $0\sqrt{n}$, $\frac{1}{6}\sqrt{n}$, $\frac{2}{6}\sqrt{n}$, $\frac{3}{6}\sqrt{n}$, $\frac{4}{6}\sqrt{n}$, $\frac{5}{6}\sqrt{n}$, $\frac{6}{6}\sqrt{n}$, each exceeding the next preceding one by the difference $\frac{1}{6}\sqrt{n}$, and the last of which is $\sqrt{n}$; from these select the last four $\frac{3}{6}\sqrt{n}$, $\frac{4}{6}\sqrt{n}$, $\frac{5}{6}\sqrt{n}$, $\frac{6}{6}\sqrt{n}$, whose squares doubled, affected with a negative sign, and divided by n, that is $-\frac{1}{2}$, $-\frac{8}{9}$, $-\frac{25}{18}$, -2, let these be treated as hyperbolic Logarithms of as many ordinates, which will therefore be 0.60653, 0.41111, 0.24935, 0.13534; substituting these in order in place of the quantities A, B, C, D, and setting l as $\frac{1}{2}\sqrt{n}$, the area will be found $= 0.170203\times\sqrt{n}$; if this is later doubled, and the double sum is multiplied by $\frac{2}{\sqrt{nc}}$, then the product quantity will be 0.27160; therefore add this Area to the previously found Area, that is to 0.682688, and the sum 0.954288 will exhibit the Probability that after performing n experiments, the Event will show itself neither more often than $\frac{1}{2}n+\sqrt{n}$ times, nor less often than $\frac{1}{2}n-\sqrt{n}$; the contrary Probability will be found to be 0.045712, so that the ratio of Probabilities is approximately 21 to 1.

And by the same procedure, the Probability that the Event will show itself neither more often than $\frac{1}{2}n+\frac{3}{2}\sqrt{n}$ times, nor less often than $\frac{1}{2}n-\frac{3}{2}\sqrt{n}$ would be found to be $= 0.99874$, but the contrary Probability 0.00126, hence the ratio of Probabilities will be approximately 792 to 1.

If a finite power is given such as e.g. 3600, and it is required to find the Probability that after performing 3600 experiments, the Event will have shown itself neither more often than 1850 times, nor less often than 1750, which two numbers can be taken arbitrarily, provided they are equally distant from half the sum 1800; set half the difference of the numbers 1850 & 1750, that is $50 = s\sqrt{n}$; now set $n = 3600$,

hence

hence $\sqrt{n} = 60$ and therefore $s = \frac{50}{60} = \frac{5}{6}$; for this reason if one takes that ratio which in an infinite Power the double sum of Terms corresponding to the interval $\frac{5}{6}\sqrt{n}$, bears to the sum of all Terms, the sought Probability will be obtained approximately for the large Power.

LEMMA II.

In any Power $(a + b)^n$ expanded into a series, the Maximum Term is that in which the indices of the quantities a & b have the same ratio to each other as the constants themselves.

LEMMA III.

If any Event depends on chance such that the Probabilities of Occurrence & Non-occurrence are as a to b, and Experiments are conducted number n by which it is explored whether the Event will show itself, or be hidden; the Probability that it will show itself neither more often than $\frac{an}{a+b} + l$ times, nor less often than $\frac{an}{a+b} - l$ times, will be found as follows.

Let the two Terms L & R be equally distant on either side from the Maximum by interval l, and let S be the sum of Terms included between L & R together with the extremes, then the sought Probability is correctly expressed by $\frac{S}{(a+b)^n}$.

COROLLARY VIII.

The ratio which, in an infinite Power, the Maximum Term has to the sum $(a+b)^n$ is correctly defined by the fraction $\frac{a+b}{\sqrt{abnc}}$.

COROLLARY IX.

If in an infinite Power some Term is distant from the Maximum by interval l, the hyperbolic Logarithm of the Ratio which that Term has to the Maximum will be $-\frac{(a+b)^2}{2abn} \times ll$; provided however that l does not bear a given ratio to n; whether

indeed

indeed that Term is placed to the right or to the left of the Maximum Term, it will be the same.

COROLLARY X.

If the Occurrence & Non-occurrence of Events are to each other as a to b, Problems about the summation of Terms are solved just as easily as those in which Occurrence & Non-occurrence are in the ratio of equality.

I scarcely doubt that the demonstrations of the eighth and ninth Corollaries are required, as they do not seem to flow easily from the principles laid down before, but if there are any who consider it worth their while to consult our Supplement, I am confident they will be amply satisfied.

THE END.