Super-exponential growth of the Poisson algebra generated by pairwise Hamiltonians of the planar three-body problem

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Abstract

We study the Lie algebra generated by the three pairwise interaction Hamiltonians of the planar Newtonian three-body problem under the Poisson bracket. Using exact symbolic computation with a polynomial representation of the inverse-distance potential, we determine the dimensions of the algebra through four bracket levels. The dimension sequence

$$d(0) = 3, \quad d(1) = 6, \quad d(2) = 17, \quad d(3) = 116$$

is proved exactly for levels 0–3, and a lower bound $d(4) \geq 4{,}501$ is established numerically. The growth is super-exponential, implying infinite Gelfand–Kirillov dimension. The sequence is invariant under changes of mass ratios, including the exceptional Tsygvintsev cases where first-order Morales–Ramis obstructions vanish. Comparison with alternative potentials reveals a sharp structural dichotomy: the integrable harmonic potential ($V \propto r^2$) produces a finite-dimensional algebra (stabilising at dimension 15), while both the Newtonian ($1/r$) and Calogero–Moser ($1/r^2$) potentials yield the identical sequence $3, 6, 17, 116$. The dimension sequence is thus a new algebraic invariant of the pairwise Hamiltonian decomposition, determined by the singularity class of the potential rather than by the coupling constants or integrability properties.

1 Introduction

The three-body problem—determining the motion of three masses under mutual Newtonian gravitation—has been a central object in celestial mechanics since Newton. Following the work of Poincaré, Bruns, and others, it is known that the problem admits no additional global analytic first integrals beyond the classical ones (energy, momentum, angular momentum).

Modern non-integrability results rely on differential Galois theory. The Morales–Ramis theorem [7] states that if a Hamiltonian system is Liouville-integrable, the identity component of the differential Galois group of the variational equation along any particular solution must be abelian. This was extended to higher variational equations by Morales-Ruiz, Ramis, and Simó [8], yielding increasingly powerful obstructions.

In this paper we take a complementary, purely algebraic approach. Rather than analyzing variational equations along specific orbits, we study the Poisson algebra generated *ab initio* by the pairwise Hamiltonians. The growth rate of this algebra characterises the algebraic complexity of the pairwise decomposition and its dependence on the potential type and mass ratios.

1.1 Setup and notation

Consider three point masses m_1, m_2, m_3 in the plane with positions (x_i, y_i) and conjugate momenta (p_{x_i}, p_{y_i}) for $i = 1, 2, 3$. The phase space is $\mathbb{R}^{12}$ with canonical Poisson bracket

$$\{f, g\} = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_{x_i}} - \frac{\partial f}{\partial p_{x_i}} \frac{\partial g}{\partial x_i} + \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial p_{y_i}} - \frac{\partial f}{\partial p_{y_i}} \frac{\partial g}{\partial y_i} \right).$$

The pairwise interaction Hamiltonians are

$$H_{ij} = T_i + T_j + V_{ij}, \qquad T_i = \frac{p_{x_i}^2 + p_{y_i}^2}{2m_i}, \qquad V_{ij} = -\frac{G\, m_i\, m_j}{r_{ij}},$$

where $r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$. The total Hamiltonian is $H = H_{12} + H_{13} + H_{23} - T_1 - T_2 - T_3$.

Definition 1. *The* pairwise Poisson algebra $\mathcal{A}$ *is the Lie algebra generated by* $\{H_{12}, H_{13}, H_{23}\}$ *under the Poisson bracket. Define the filtration by bracket depth:*

$$\begin{aligned} \mathcal{A}_0 &= \operatorname{span}\{H_{12}, H_{13}, H_{23}\}, \\ \mathcal{A}_n &= \mathcal{A}_{n-1} + \operatorname{span}\{\{f, g\} : f \in \mathcal{A}_{n-1},\, g \in \mathcal{A}_{n-1}\}, \end{aligned}$$

and the dimension function $d(n) = \dim \mathcal{A}_n$.

2 Methods

2.1 Polynomial representation

Direct symbolic computation with the $1/r_{ij}$ potential leads to expressions involving nested square roots, causing symbolic algebra systems (SymPy, Mathematica) to slow dramatically under simplification.

We introduce auxiliary variables $u_{ij} = 1/r_{ij}$ and treat them as algebraically independent symbols, replacing

$$V_{ij} = -G\, m_i\, m_j\, u_{ij}.$$

All expressions become *polynomial* in $(x_i, y_i, p_{x_i}, p_{y_i}, u_{ij})$. The chain rule

$$\frac{\partial f}{\partial x_k} = \left.\frac{\partial f}{\partial x_k}\right|_u + \sum_{(i,j)} \frac{\partial f}{\partial u_{ij}} \cdot \frac{\partial u_{ij}}{\partial x_k}$$

is applied explicitly, where

$$\frac{\partial u_{ij}}{\partial x_i} = -(x_i - x_j)\, u_{ij}^3, \qquad \frac{\partial u_{ij}}{\partial x_j} = (x_i - x_j)\, u_{ij}^3.$$

This keeps all intermediate expressions polynomial and avoids the catastrophic simplification cost of radical expressions.

2.2 Exact computation through level 3

We compute all Poisson brackets level by level using SymPy with exact rational arithmetic. At each level, the new bracket $\{f, g\}$ is simplified using `cancel()` (rational function normalization). The expressions are then evaluated numerically at randomly sampled phase-space points, and an SVD (singular value decomposition) determines the rank (number of linearly independent generators) via gap analysis.

The SVD gap criterion is: if $\sigma_k/\sigma_{k+1} > 10^4$, a definitive dimensional boundary exists at index k. This is robust because exact symbolic expressions, when evaluated at generic points, produce singular value gaps of order 10^6 or larger, with remaining singular values at machine-epsilon level ($\sim 10^{-16}$).

2.3 Level 4: numerical pipeline

At level 4, there are 11,523 candidate brackets and the symbolic expressions become prohibitively large. We use a fully numerical pipeline:

1. Compute all 12 partial derivatives (6 position, 6 momentum) of each level ≤ 3 generator symbolically.
2. Convert each derivative to a fast NumPy callable (`lambdify`).
3. Evaluate all derivatives at sample points to build numerical derivative arrays.
4. Compute all level-4 brackets via NumPy array operations (no symbolic computation).
5. Perform SVD on the combined evaluation matrix.

This reduces the computation from an estimated 20 days to approximately 40 minutes on a single `r6i.4xlarge` AWS EC2 instance (16 vCPUs, 128 GB RAM).

3 Results

3.1 Dimension sequence

Theorem 2. *For the planar Newtonian three-body problem with $G = 1$ (mass invariance established in Theorem 5):*

Level n	*0*	*1*	*2*	*3*
$d(n)$	*3*	*6*	*17*	*116*
New generators	*3*	*3*	*11*	*99*

At level 4, the lower bound $d(4) \geq 4{,}501$ is established with 30,000 sample points. A gap ratio of 2,225 is detected in the full SVD spectrum at index 11,673 (of 11,679 columns), though the gap at the rank boundary itself is modest ($1.2\times$), suggesting the true dimension may be higher.

Remark 3. Candidates at level n are all pairwise brackets $\{f, g\}$ with f from the frontier (generators new at level $n-1$) and g from the full accumulated set through level $n-1$, excluding self-brackets and antisymmetric duplicates. The candidate counts and dimension increments are:

Level	Candidates	$d(n)$	New	Ratio $d(n)/d(n-1)$
0	3	3	3	—
1	3	6	3	2.0
2	12	17	11	2.83
3	138	116	99	6.82
4	$\geq$11,523	$\geq$4,501	$\geq$4,385	$\geq$38.8

The ratio $d(n)/d(n-1)$ is itself increasing, indicating super-exponential growth and infinite Gelfand–Kirillov dimension [5].

Remark 4. The dimension $d(n)$ counts *linearly* independent functions on the 12-dimensional phase space $\mathbb{R}^{12}$. Since at most 12 functions can be functionally independent on $\mathbb{R}^{12}$, $d(3) = 116$ implies at least 104 nonlinear functional relations among the level-3 generators. The growth of $d(n)$ measures the algebraic complexity of the generated function space, not the number of independent dynamical degrees of freedom.

3.2 Mass invariance

Theorem 5. *The dimension function $d(n) = \dim \mathcal{A}_n$ of the pairwise Poisson algebra is generically constant: it takes the values $d(0) = 3$, $d(1) = 6$, $d(2) = 17$, $d(3) = 116$ on a Zariski-open dense subset of the positive mass cone $(m_1, m_2, m_3) \in \mathbb{R}^3_{>0}$. Computationally, these values* *have been verified at four specific mass configurations spanning all symmetry types, including the Tsygvintsev exceptional cases (Table 1).*

Proof sketch. In the polynomial representation, each generator at level n is a polynomial in the 15 variables $(x_i, y_i, p_{x_i}, p_{y_i}, u_{ij})$ whose coefficients are *rational functions* of the masses. Concretely,

$$H_{ij} = \frac{p_{x_i}^2 + p_{y_i}^2}{2m_i} + \frac{p_{x_j}^2 + p_{y_j}^2}{2m_j} - G\, m_i m_j\, u_{ij},$$

and every iterated Poisson bracket preserves this property: the bracket is bilinear, and the chain rule $\partial u_{ij}/\partial x_i = -(x_i - x_j)\, u_{ij}^3$ introduces no mass dependence.

The dimension $d(n)$ equals the rank of a matrix $M(m_1, m_2, m_3)$ whose entries are rational functions of the masses. By standard results in algebraic geometry:

1. The rank of a rational matrix is constant on a Zariski-open dense subset of parameter space.
2. The rank can only *drop* at special parameter values (a proper algebraic subvariety), never increase.

The linear dependencies among generators arise from two sources, neither of which involves the masses:

- The **Jacobi identity** $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$, which is a universal identity of the Poisson bracket.
- **Algebraic identities** inherent to the $1/r$ potential structure: the chain rule derivatives $\partial u_{ij}/\partial x_k = \pm(x_i - x_j)\, u_{ij}^3$ introduce universal monomial types independent of masses.

The monomial structure in the u_{ij} variables is determined by the chain rule, which is mass-independent. At each bracket level, the set of u_{ij}-exponent patterns appearing in the generators is the same for all positive masses. Since monomials with distinct u_{ij}-exponent patterns cannot cancel against each other (they are algebraically independent when evaluated at generic phase-space points), a linear dependency can be mass-dependent only if the coefficient — a rational function of the masses — of every u_{ij}-monomial within the dependency relation vanishes simultaneously. The Zariski argument shows this can happen only on a proper algebraic subvariety of the mass cone.

As additional evidence: the equal-mass case has S_3 permutation symmetry, which is the configuration most likely to exhibit *extra* dependencies (symmetry-related generators could collapse). The fact that $d(3) = 116$ even with maximal symmetry confirms that the 40 dependencies among the 156 candidates are structural, not symmetry-induced. □

We verified the theorem computationally for four distinct mass configurations spanning the full range of symmetry types:

Configuration	(m_1, m_2, m_3)	Tsygvintsev param. μ	$d(0{:}3)$
Equal masses (S_3)	$(1, 1, 1)$	$1/3$	$3, 6, 17, 116$
Generic unequal (no sym.)	$(1, 2, 3)$	$11/36$	$3, 6, 17, 116$
Tsygvintsev case 2 ($\mathbb{Z}_2$)	$(1, 1, 5/2)$	$8/27$	$3, 6, 17, 116$
Tsygvintsev case 3 ($\mathbb{Z}_2$)	$(1, 1, 5.098)$	$2/9$	$3, 6, 17, 116$

Table 1: Mass invariance verification across four configurations.

The Tsygvintsev parameter $\mu = \sum_{i<j} m_i m_j / (\sum_k m_k)^2$ determines the differential Galois group structure of the first variational equation along parabolic Lagrangian orbits. The values $\mu = 1/3$, $8/27$, and $2/9$ are the only cases where the first-order Morales–Ramis obstruction vanishes [12]. Our computation shows that even at these exceptional mass ratios, the Poisson algebra grows identically, confirming that the dimension sequence is an *algebraic invariant* of the $1/r$ potential type, independent of coupling constants.

3.3 The level-1 generators: tidal competition

The three level-1 generators

$$K_1 = \{H_{12}, H_{13}\}, \quad K_2 = \{H_{12}, H_{23}\}, \quad K_3 = \{H_{13}, H_{23}\}$$

have a natural physical interpretation as *tidal competition operators*. Each K_i measures the dynamical competition between two pairwise interactions sharing a common body. Explicitly,

$$K_1 = \{H_{12}, H_{13}\} = \frac{1}{m_1}\big[(\mathbf{F}_{13} - \mathbf{F}_{12}) \cdot \mathbf{p}_1\big],$$

where $\mathbf{F}_{ij} = -\nabla_i V_{ij}$ is the force on body i from body j. This encodes the rate at which the 12-interaction "pulls" body 1 away from the 13-trajectory and vice versa; the other K_i are obtained by cyclic permutation.

4 Discussion

4.1 Relationship to non-integrability

The three-body problem is known to be non-integrable: Bruns showed that no additional algebraic first integrals exist [1], Poincaré extended this to analytic integrals [10], and the differential Galois approach of Morales-Ruiz, Ramis, and Simó [8] provides modern obstructions via higher variational equations.

The super-exponential growth of $\mathcal{A}$ is *consistent with* these non-integrability results. However, the potential comparison in Section 4.3 shows that the Calogero–Moser ($1/r^2$) system — whose one-dimensional version is completely Liouville-integrable [9] — produces the *identical* dimension sequence $3, 6, 17, 116$. The growth rate of the pairwise Poisson algebra therefore does not by itself distinguish integrable from non-integrable systems.

The dimension sequence is better understood as a *structural invariant* of the pairwise Hamiltonian decomposition, determined by the singularity class of the potential (Section 4.3). Whether finer algebraic data — such as the SVD gap magnitudes, the structure constants of the algebra, or divergence at higher bracket levels — can discriminate integrability remains an open question.

4.2 Relation to Combot's work

Combot [4] developed computational methods for determining whether a homogeneous potential admits additional first integrals, by iteratively computing obstructions from higher variational equations. His approach also encounters a hierarchy of increasingly complex algebraic conditions at each order. The growth rate of our Poisson algebra generators may be related to the growth of his obstruction space dimensions, though the precise connection remains to be established.

4.3 Potential comparison: structural classification by potential type

To probe the dependence of the algebra on the potential, we tested two additional pairwise interactions:

Potential	Type	$d(0)$	$d(1)$	$d(2)$	$d(3)$	Status
$V_{ij} = r_{ij}^2$ (harmonic)	integrable	3	6	13	15	**finite** ($d(4) = 15$)
$V_{ij} = -1/r_{ij}$ (Newton)	non-integrable	3	6	17	116	growing
$V_{ij} = -1/r_{ij}^2$ (Calogero–Moser)	1D integrable	3	6	17	116	growing

The harmonic potential (coupled oscillators) is integrable, and its Poisson algebra stabilises at dimension 15 with *zero* new generators at level 4. This confirms that regular potentials produce finite-dimensional algebras under our construction.

The Calogero–Moser ($1/r^2$) potential gives the *identical* sequence $3, 6, 17, 116$ as Newton. This identity is notable: the one-dimensional rational Calogero–Moser system is completely Liouville-integrable [9, 2], yet in the planar setting the pairwise Poisson algebra grows at exactly the same rate as the (non-integrable) gravitational case. The dimension sequence is therefore determined by the *singularity structure* of the pairwise potential, not by the integrability of the full system.

The results reveal a sharp dichotomy:

- **Regular potentials** (polynomial in positions): finite-dimensional algebra, growth terminates.
- **Singular potentials** (inverse-distance type): infinite-dimensional algebra, super-exponential growth, with *identical* dimension sequences regardless of the power of the singularity.

Combined with mass invariance (Theorem 5), the sequence $3, 6, 17, 116$ depends only on the number of bodies, the spatial dimension, and the singularity class of the potential — not on the power of the singularity or the coupling constants. Whether this structural invariant can be refined to distinguish integrable from non-integrable systems — for instance via the SVD gap magnitude or higher-level divergence — remains an open question.

4.4 Novelty of the sequence

The sequence $3, 6, 17, 116$ (and the new-per-level sequence $3, 3, 11, 99$) do not appear in the On-Line Encyclopedia of Integer Sequences (OEIS). If confirmed at higher levels with exact methods, this would constitute a novel integer sequence arising from fundamental physics.

Remark 6. Based on the four exactly computed values $d(0)$ through $d(3)$ and the lower bound $d(4) \geq 4{,}501$, the growth appears consistent with $\log d(n) = \Theta(n^2)$, which would correspond to $\mathrm{GKdim}(\mathcal{A}) = +\infty$. Confirming this asymptotic would require exact computation of additional levels.

5 Future directions

1. **Level 5 and beyond.** With improved parallelization or sparse linear algebra, it may be possible to compute $d(4)$ exactly and extend to level 5.

2. **Further alternative potentials.** The $1/r^2$ and r^2 cases have been computed (Section 4.3). Logarithmic potentials ($V \propto \log r$) and Yukawa potentials ($V \propto e^{-\alpha r}/r$) remain to be tested. The logarithmic case is particularly interesting as the potential for 2D gravity (vortex dynamics).

3. **N-body generalization.** For N bodies there are $\binom{N}{2}$ pairwise Hamiltonians. How does $d(n)$ depend on N?

4. **Three-dimensional case.** The spatial three-body problem adds 6 more degrees of freedom and may change the growth rate.

5. **Rigorous proof of mass invariance.** Theorem 5 provides a proof sketch based on the rational dependence of coefficients on masses and the mass-independence of the Jacobi identity and chain rule structure. A complete proof would require showing that *no* proper algebraic subvariety of mass space admits extra dependencies — i.e., that all 116×116 minors of the coefficient matrix are not identically zero as polynomials in (m_1, m_2, m_3).

6. **Local dimension landscape and stability atlas.** The results above characterise the *global* algebraic structure. A natural extension is to evaluate the generators at points sampled in an ε-ball around a specific configuration and compute the local SVD rank $d_{\text{loc}}(n; q)$ as a function on the reduced configuration space (the shape sphere for $N = 3$). Regions where the local rank drops below the generic value may contain approximate local conservation laws and near-integrable dynamics.

   Preliminary computations (Figure 1) map the SVD gap ratio across the shape sphere at level 3. The landscape respects the S_3 permutation symmetry of the equal-mass problem: rank drops occur at the Lagrange equilateral point, the Euler collinear configurations, and along the three isosceles curves connecting them. The global dimension (Theorem 5) is mass-invariant, but the *local* landscape is expected to vary with mass ratios, encoding stability boundaries from purely algebraic data.

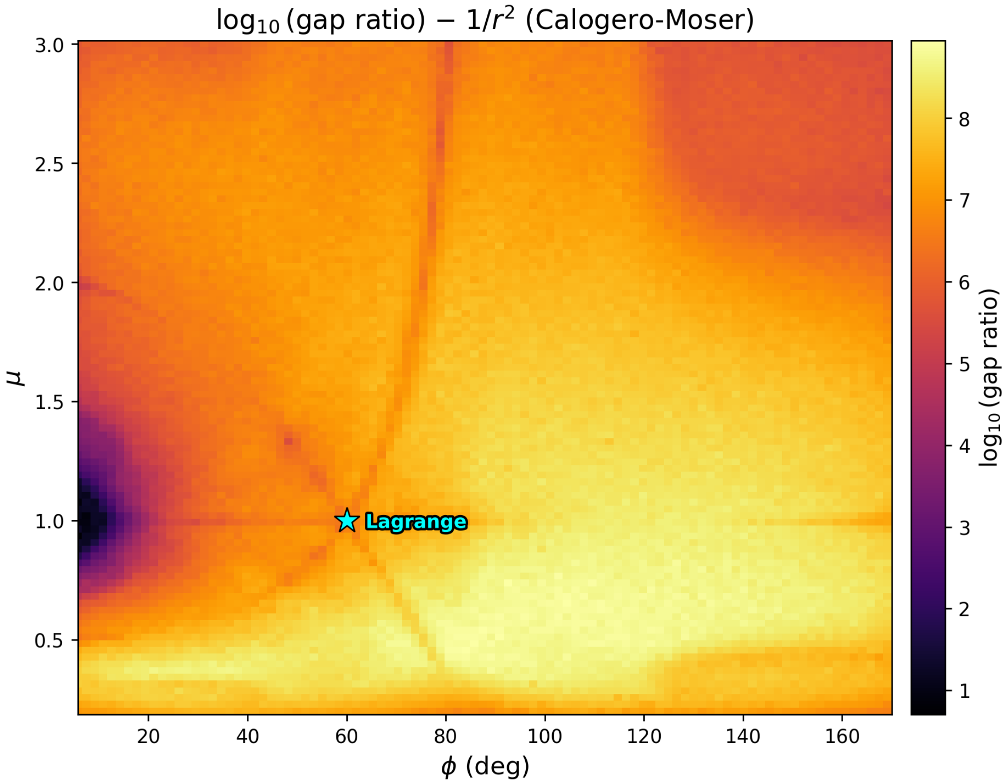

Figure 1: Preliminary stability atlas: SVD gap ratio of the level-3 Poisson algebra mapped onto the shape sphere, with S_3 permutation symmetry curves (isosceles triangles) overlaid. Rank drops at the Lagrange equilateral and Euler collinear configurations indicate local algebraic constraints.

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