TABLES $n = 4p+3$ (*continued*).

	a_1 2	a_2 3	a_3 1	a_4 $2p-1$	a_{4p-1} 1
a_{2p+1} 2	1	3	1	$(2p-1)^2$	1^{2p-1}
a_{p+1} 3		2	1	$2p-1$	1^p
a_p 1			1	$2p-1$	1^{p-1}
a_1 $2p-1$				1	1
a_{2p+2} 1	2	3	1^2	$(2p-1)^2$	1^{2p}

	a_1 $2p-1$	a_p 1	a_{p+1} 3	a_{2p+1} 2	a_{2p+2} 1
a_4 $2p-1$	1	1	3	2^2	1^2
a_3 1		1	3	2	1^2
a_2 3			2	2	1
a_1 2				1	1
a_{4p-1} 1	$2p-1$	1^{p-1}	3^p	2^{2p-1}	1^{2p}

	a_1 5	a_3 2	a_4 $2p-1$	a_{4p-1} 1
a_{p+1} 5	1	2	$2p-1$	1^p
a_p 2		1	$2p-1$	1^{p-1}
a_1 $2p-1$			1	1
a_{3p+2} 1	5	2^2	$(2p-1)^2$	1^{3p-1}

	a_1 $2p-1$	a_p 2	a_{p+1} 5	a_{3p+2} 1
a_4 $2p-1$	1	2	5	1^3
a_3 2		1	5	1^2
a_1 5			1	1
a_{4p-1} 1	$2p-1$	2^{p-1}	5^p	1^{3p-1}

On an Algebraical Form, and the Geometry of its Dual Connexion with a Polygon, plane or spherical. By T. COTTERILL, M.A.

[*Read February 8th*, 1872.]

The following is a *résumé* of some of the results and the method employed in a paper on this subject, read before the Society, Feb. 8, 1872. A portion of the MS. having been mislaid, and some improvements having suggested themselves, the whole paper will be given hereafter.

A polygon, plane or spherical, can be denoted in two ways—either by its angular points, or its lines equal in number (say m), taken in a definite order of sequence. The m lines intersect again in $\frac{m(m-3)}{2}$ points, called polygon points of the curve, or, for shortness, P points, and determine a curve of the order $(m-3)$ passing through them, the order curve of the polygon. In the same manner, the $\frac{m(m-3)}{2}$ diagonals determine the class curve of the polygon. The two figures being correlative, it is sufficient to consider only one case—for instance, the order curve. The lines are denoted by numerals or italics, or both; and when placed within a ( ) in a fixed order, represent a polygon or its corresponding curve. Thus $(123abc) = (23abc1) =$ &c. is a hexagon with consecutive sides 1, 2, 3, a, b, c, or the unique cubic which passes through the nine intersections of its non-consecutive lines. A symbol, as (a, x), is the point of intersection of the line-pair a and x; such P points are divided into sets, a point of the first set being the intersection of two lines separated by one line, and of the second set when it is separated by two lines, and so on. The minor of a curve of a polygon is the curve obtained by striking out any number of consecutive lines of the original polygon.

Thus (1234) is the right line connecting the P points (1, 3) and (2, 4).

The curve (12345) of a pentagon is the conic through the five P points (1, 3), (2, 4), (3, 5), (4, 1), and (5, 2). It follows, from Pascal's Theorem, that the tangent at a point, as (2, 4), passes through the intersection of the lines 3 and (1425).

Cubics through the P points of the conic (12345), cutting the lines 2, 3, 4 on the line 6, form a faisceau touching the conic at the point (2, 4). The cubic of this system, which passes through the point (1, 5), is the cubic of the hexagon (123456), and touches a minor conic, as (34561), at (4, 6), a point of the first set. The tangent at a point of the second set, as (2, 5), passes through the intersection of the lines (2516) and (2534). It is shown in the analysis that these constructions suffice for determining the tangent at any P point of the curve of a polygon. The three pairs of points (1, 3), (4, 6) and (2, 4), (5, 1) and (3, 5), (6, 2) correspond, in the meaning attached to the words by Prof. Cayley; so that the cubic is the locus of points at which the three pairs (as well as any other pair) of corresponding points subtend angles in involution.

The three P points (6, 4), (4, 1), (1, 5), on four consecutive lines 4, 5, 6, 1, form a triplet such that one conic of the reseau through them is inscribed in the cubic, and any other cuts the cubic again in a triplet of points of the same nature. The opposite of any such triplet and a fourth point on the curve, as d, is d', the corresponding point to d. From these and other properties suggested by the analysis there is no difficulty in tracing the curve.

The nine P points of the cubic are on any four consecutive lines, as 2, 3, 4, 5. Taking the four intersections of these lines with a fresh line 7, we have thirteen points lying on two broken quartics [the cubic (123456) and line 7] and (the four lines 2, 3, 4, 5). Any other quartic of the faisceau through these thirteen points touches the cubic in the three points (4, 2), (2, 5), (5, 3), and amongst them is the quartic of the heptagon (1234567), determined by passing through the point (1, 6), touched by each minor conic in one point and by each minor cubic in three points.

Proceeding in this manner, we see that the $\frac{n(n+3)}{3}$ P points of a polygon of $(n+3)$ lines determine one curve (order n), passing through the $\frac{r(r+3)}{2}$ P points of each minor curve (order r), and touching it in $\frac{r(r-1)}{2}$ of these points.

An inflexion or multiplicity at a P point is obtained by supposing one or more of the lines of the polygon to pass through it, besides the pair which determine it. Other properties are easily obtained geometrically; but the clearest insight into these curves, and their connexion with certain associated curves, is gained from the following theorem in determinants, which gives the quantic of the curve of a polygon referred to its lines in many different forms, as well as those of the associated curves just alluded to:—

Theorem in Determinants.

If a triad of letters in ( ), as (abc), represent the determinant of the third order $\begin{vmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \\ a_3 b_3 c_3 \end{vmatrix}$, the suffix numbers being the same in each determinant, then

$$\frac{1}{(txy)}\left\{\frac{(tab)}{(axy)(bxy)}+\frac{(tbc)}{(bxy)(cxy)}+\ldots\ldots+\frac{(tmn)}{(mxy)(nxy)}+\frac{(tna)}{(nxy)(axy)}\right\}$$

is independent of t, so that this function $\frac{1}{(txy)}\Sigma\frac{(tab)}{(axy)(bxy)}$ has a constant value, which can be obtained by substituting for t one of the fixed letters $a, b \ldots m, n$, in which case two of the terms disappear. Let this be denoted by I. Then $\mathrm{I}.(txy)=\Sigma\frac{(tab)}{(axy)(bxy)}$.

If I vanish, then $\Sigma\frac{(tab)}{(axy)(bxy)}=0$ identically. But if I is finite, and $\Sigma\frac{(tab)}{(axy)(bxy)}=0$, then $(txy)=0$.

The form $\Sigma\frac{(tab)}{(axy)(bxy)}$ has evidently the same properties as the expression for the area of a plane polygon in terms of the areas of the triangles and polygons into which it can be divided, and denoted in the usual manner by means of their vertices.

Substituting p for t, it is at once manifest that in the form $\frac{1}{(pxy)}\Sigma\frac{(pab)}{(axy)(bxy)}$ the triads may represent the areas of the triangles formed from the vertices of the polygon and the three points p, x, y. But in order to obtain the order curve of the polygon, we suppose, in the form $\frac{1}{(txy)}\Sigma\frac{(tab)}{(axy)(bxy)}$, the italics to represent lines, and a triad, as (tab), the alternating and therefore cyclical function correlative to twice the area denoted by the product of the sine of any angle of a triangle and its distance from the opposite side. In this case, if an italic by itself be the perpendicular from a point of the plane on the line denoted by the italic, we shall have $\frac{1}{t}\Sigma\frac{(tab)}{a.b}$ independent of the position of t in the plane. Let Π denote the product of the symbols $a, b \dots m, n$; then $\frac{\Pi}{t}\Sigma\frac{(tab)}{a.b} = \text{U} = (ab \dots mn)$, the simple multiform quantic of the order curve of the polygon, assuming as many shapes as the polygon can be divided into its own triangles and polygons. Thus, if the lines of the polygon be a single line a and p consecutive lines P, a single line x and q consecutive lines Q, and P and Q by themselves denote the perpendiculars from a point on the consecutive lines, then the identity $(a\text{P}x\text{Q}) = (a\text{P}x).\text{Q}+(x\text{Q}a).\text{P}$ proves the properties of the minors already mentioned, and that two complementary minors $(a\text{P}x)$, $(x\text{Q}a)$ intersect on the curve of the polygon. An important variety of the form is $(ta\text{P}x)$, in which t is variable, forming a reseau through p^2-p+1 points; any two of the curves intersecting again in $(p-1)$ points collinear with (a, x), and determined by two of the curves breaking up and containing curves of the order $(p-1)$. From this form we also find sets of $\frac{p(p-1)}{2}$ points on a curve of a polygon order n, through which a curve order $(p-1)$ can pass touching the primitive at the points; the sets of $\frac{n(n-1)}{2}$ points being such that a curve of the order $(n-1)$ through them cuts the primitive again in $\frac{n(n-1)}{2}$ points of the same nature.

Again, $t.\text{U} = \Pi\Sigma\frac{(tab)}{a.b}$, the quantic of the curve and an arbitrary line. Thus, $t.(a\text{P}x\text{Q}) = (ta\text{P}x).\text{Q}+(tx\text{Q}a).\text{P}$, showing that, if we take a line t, the curves $(ta\text{P}x)$ and $(tx\text{Q}a)$, having the P point (a, x) in common, generate the primitive curve by their intersections.

Geometrical meanings can sometimes be attached to the forms containing more terms.

The fundamental expression is also proved by comparing the area of a triangle with the corresponding correlative function of the polars of its vertices to a conic of the plane, and in a form showing the analogue in any space possessing an absolute. The Author is happy to add that the subject in space has attracted the attention of Prof. Clifford, who has laid the results of his investigations before the Society.

The following presents have been received in the recess:—

"Memorie del Reale Istituto Lombardo," vol. xii. della serie iii., fasc. ii., iii., iv., 1872.

"Rendiconti del R. I. Lombardo," serie ii., vol. iii., fasc. i.—viii. and xvi.—xx.; vol. iv., fasc. i.—xx.; vol. v., fasc. i.—vii.

"Crelle," Band 74, drittes und viertes Heft; Band 75, erstes Heft, 1872.

"Monatsbericht," April, 1872.

"Annali di Matematica," serie ii., tom. v., fasc. ii., May, 1872.

"Bulletin des Sciences," vol. iii., July to October, 1872; and Index to tom. ii.

"Smithsonian Report," 1870. (Washington, 1871.)

"Rappresentazione piana di alcune superficie algebriche dotate di curve cuspidali," nota del Prof. L. Cremona, 1872; and "Le figure reciproche nella statica grafica," L. Cremona, Milano, 1872: both from the Author.

"Le proprietà cardinali degli strumenti ottici anche non centrati," F. Casorati: from the Author.