# BRISM and the Born Rule in Continuity with Established Structural Theorems of Quantum Mechanics: U(1) Symmetry, Measure Uniqueness, POVM Dilation and Spectral Stability

Swen C. Heinze, 10.02.2026
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**Abstract**

The present work develops four formal bridges that anchor the interface structure introduced in January 2026 (BRISM) fully within the established Hilbert-space formalism: the phase-neutral mapping of complex bulk amplitudes into real, normalized measurement statistics on the brane. The BRISM architecture is fully compatible with the minimal statistical interpretation: the state vector represents an ensemble, and the "bulk" is identified with the mathematically necessary Naimark/Stinespring dilation space rather than an additional ontology. The interface projection reorganizes, but does not modify, the established Hilbert-space formalism. (i) **U(1) symmetry & Noether**: The global phase invariance of Schrodinger dynamics enforces norm and probability conservation and singles out the quadratic form under phase-neutral interface mappings. (ii) **Gleason/Busch**: Noncontextuality, positivity, and $\sigma$-additivity imply the uniqueness of the measure structure of all POVM-induced probabilities. (iii) **Naimark/Stinespring dilation**: Every POVM is the projection of a projective measurement in an enlarged Hilbert space; this dilation space identifies the "bulk" as a mathematically necessary extension of the brane. (iv) **Spectral stability (new)**: Phase-neutral, local, and additive interface mappings are stable only for quadratic densities $|\psi(x,t)|^2$; alternative local power forms violate normalization or additivity requirements. Thus, the Born quadratic form does not appear as a postulate but as a structural necessity of the interface.

## 1 Setup and Assumptions

**A1 (Hilbert space and dynamics).** The bulk is a separable complex Hilbert space $\mathcal{H}_{\text{bulk}}$. Its internal dynamics are unitary:

$$U_T = e^{-iH_{\text{bulk}}T}, \tag{1}$$

and

$$i\hbar\,\partial_T\Psi(T) = H_{\text{bulk}}\,\Psi(T). \tag{2}$$

**A2 (Interface operator).** For each brane-time $t$ let

$$\Pi_t : \mathcal{H}_{\text{bulk}} \to L^2(\mathbb{R}^3) \tag{3}$$

be linear. The resulting brane wavefunction is

$$\psi(x,t) = (\Pi_t\Psi)(x). \tag{4}$$

**Remark.** Note that the equation (4) corresponds to the standard representation of the abstract state $\Psi$ in the position basis of $L^2(\mathbb{R}^3)$. No additional structure beyond the usual Hilbert-space representation is assumed.

**A3 (Extraction of real measurement quantities).** Measurable brane densities arise through a mapping

$$F:\ L^2(\mathbb{R}^3) \to \mathbb{R}_{\geq 0}, \tag{5}$$

which is *phase-neutral*, *positive*, *σ-additive*, and *properly normalised*:

$$F(e^{i\phi}\psi) \;=\; F(\psi). \tag{6}$$

**A4 (POVM induction).** $\Pi_t$ induces a POVM $E_t(\mathrm{d}x)$ on $\mathcal{H}_{\text{bulk}}$ with

$$P(\mathrm{d}x) \;=\; \langle \Psi \mid E_t(\mathrm{d}x) \mid \Psi \rangle. \tag{7}$$

**Interpretation-neutrality.** BRISM does not assume collapse or ontic status of the state vector. It remains fully compatible with the minimal statistical (ensemble) interpretation. The term "bulk" denotes the Naimark/Stinespring dilation space required to represent any POVM on the brane. Thus the model introduces no additional ontology but reorganises the existing structure of the quantum formalism.

**Remark on the time parameters.** BRISM does not introduce a second physical time. There is a single external time parameter, as required by the standard Schrödinger–von Neumann formalism and by Pauli's theorem which forbids a self-adjoint time observable. The bulk parameter $T$ is merely the evolution parameter of the unitary family $U(T) = e^{-iHT}$ in the abstract Hilbert space. It is not an observable and carries no physical time flow. The brane parameter $t$ labels the measurement context through the interface operator $\Pi_t$ and denotes the laboratory time at which a POVM is applied. Thus $T$ and $t$ serve distinct *roles* of the same physical time: $T$ parametrizes the unitary Schrödinger evolution, while $t$ parametrizes the instant of readout on the brane. No commutation relation such as $[t, H]$ is implied, and no additional dynamics or second time flow is introduced.

# 2 Structural Bridges I–IV

## 2.1 Bridge I — U(1), Noether and Quadrature

**Definition 1** (U(1) invariance & phase neutrality)**.** *The Schrodinger equation is invariant under $\Psi \mapsto e^{i\alpha}\Psi$; the interface is phase-neutral:*

$$F(e^{i\phi}\psi) \;=\; F(\psi). \tag{8}$$

**Theorem 1** (Noether norm conservation)**.** *From U(1) symmetry it follows that*

$$\frac{d}{dt}\int_{\mathbb{R}^3} |\psi(x,t)|^2 \,\mathrm{d}x \;=\; 0. \tag{9}$$

**Lemma 1** (Quadrature uniqueness)**.** *Under the assumptions of positivity, phase neutrality, homogeneity, and additivity of orthogonal contributions we have:*

$$F(\psi) \;=\; c\,|\psi|^2 \qquad \textit{(almost everywhere)}, \tag{10}$$

*with $c > 0$ a constant (normalization fixes $c = 1$).*

**Quadrature uniqueness clarified.** The combination of phase-neutrality, $\sigma$-additivity and homogeneity excludes all functional forms except the quadratic one. Any alternative $F(\psi) = |\psi|^\alpha$ with $\alpha \neq 2$ breaks either additivity or normalisation. Thus the Born rule is the unique phase-invariant measure compatible with the POVM structure.

**Corollary 1** (Born as interface necessity)**.** *Under A1–A3:*

$$P(x,t) \;=\; |\psi(x,t)|^2. \tag{11}$$

### 2.2 Bridge II — Gleason/Busch measure uniqueness

**Theorem 2** (Gleason/Busch compatibility)**.** *For* $\dim \mathcal{H}_{\text{bulk}} \geq 3$*, all POVM-induced probabilities take the form*

$$P(\mathrm{d}x) \;=\; \langle \Psi \mid E_t(\mathrm{d}x) \mid \Psi \rangle. \tag{12}$$

*The BRISM quadrature is fully compatible with the Gleason/Busch measure structure.*

### 2.3 Bridge III — Naimark/Stinespring Dilation

**Theorem 3** (POVM dilation)**.** *For every POVM* $E_t(\mathrm{d}x)$ *there exist a Hilbert space* $K \supset \mathcal{H}_{\text{bulk}}$*, an isometry* $\iota : \mathcal{H}_{\text{bulk}} \to K$*, and a projective measure* $P(\mathrm{d}x)$ *on* $K$ *such that*

$$\langle \Psi \mid E_t(\mathrm{d}x) \mid \Psi \rangle \;=\; \langle \iota\Psi \mid P(\mathrm{d}x) \mid \iota\Psi \rangle. \tag{13}$$

Thus the “bulk” is not a new postulate, but the necessary dilation space of the POVM formalism.

**Dilation-space identification.** The enlarged Hilbert space $K$ appearing in the Naimark/ Stinespring dilation is not an interpretive addition but the mathematically necessary space from which all POVMs arise. BRISM simply names this dilation space the “bulk”.

### 2.4 Bridge IV — Spectral stability of the projection

**Definition 2** (Stably projectable spectral component)**.** *Let*

$$\Psi \;=\; \int \tilde{\Psi}(\lambda)\, \mathrm{d}\mu(\lambda) \tag{14}$$

*be a spectral representation and* $\Pi_\lambda$ *the corresponding spectral projector. A spectral component* $\lambda$ *is called stably projectable if*

$$\phi_\lambda(x,t) \;=\; (\Pi_t\, U_T\, \Pi_\lambda \Psi)(x) \tag{15}$$

*exists and* $F(\phi_\lambda)$ *remains phase-neutral, positive, locally normalizable and* $\sigma$*-additive.*

**Theorem 4** (Quadrature stability)**.** *Under A1–A3 and the definition above: Only quadratic densities*

$$P(x,t) \;=\; |\psi(x,t)|^2 \tag{16}$$

*are spectrally stable. Any alternative local power form*

$$F(\psi) \;=\; |\psi|^\alpha, \qquad \alpha \neq 2, \tag{17}$$

*violates either normalization stability or additivity.*

**Corollary 2** (Selection principle)**.** *The brane displays only those bulk spectral components that appear stably quadratic under* $\Pi_t$*. Non-stable components remain invisible.*

## 3 Consequences and Positioning

BRISM reorganizes the standard formalism and makes the Born quadratic rule comprehensible through U(1) symmetry, positivity, additivity, and spectral stability. No new building blocks of quantum mechanics are introduced; instead, the existing structure of the Hilbert-space formalism is reorganized along the bulk–brane interface. The architecture of the theory thereby becomes more precise without requiring new dynamics or additional ontological assumptions. The approach remains fully compatible with POVM theory, rigged Hilbert spaces, and unitary time evolution, and uses the mathematically necessary dilation space as the “bulk”.

**Statistical compatibility.** The BRISM architecture therefore remains compatible with the statistical (ensemble) interpretation: probabilities arise solely from the POVM-induced measure on the brane, while the dilation space supplies the structural completion required by Naimark/Stinespring.

## References to Prior Work

This paper is directly connected to two earlier works by the author, which introduced and motivated the conceptual foundations of the BRISM model:

- **BRISM Base Paper** Introduction of the bulk–brane separation, the interface operator, and phase-neutral extraction as the foundation of the model. Available at: DOI.org/10.5281/zenodo.18391944
- **BRISM Onepager (Short Version)** Compact presentation of the central idea, including the interpretation-neutral formulation and the Born structure as an interface property. Available at: DOI.org/10.5281/zenodo.18491724

These two documents form the conceptual and motivational basis on which the present mathematical bridge paper builds. The present work extends these ideas by formally deriving and anchoring the Born rule and the interface structure through established theorems of quantum mechanics (Gleason/Busch, Naimark/Stinespring, U(1) symmetry, and spectral stability).

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This work was authored solely by Swen C. Heinze. Large Language Models (AI) were used as auxiliary tools for formalizing mathematical structures, improving linguistic clarity, and ensuring consistency with established theoretical results. All conceptual developments, structural assumptions, and the final validation of the logical derivations are entirely the responsibility of the author.

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