Quantum Prime Spectral Theory

A Canonical Spectral Framework for the Hilbert–Pólya Paradigm

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Summary

Abstract

This work introduces a self-adjoint operator

$$K_{\text{full}} = K_{\text{arit}} \otimes I + I \otimes K_{\text{arq}},$$

combining an arithmetic component with discrete spectrum and an archimedean component with continuous spectrum. We show that the Riemann explicit formula arises canonically as an identity of regularized traces associated with the resolvent of K_{full}, in the classical Titchmarsh–Mellin formulation.

Within this framework, the arithmetic operator produces the von Mangoldt weights through the logarithmic derivative of its spectral determinant, while the archimedean operator yields the Gamma factor via finite-part regularization of its continuous spectrum. The non-trivial zeros of the Riemann zeta function appear as spectral singularities of the resolvent, rather than as genuine eigenvalues.

The self-adjointness of K_{full} induces a spectral symmetry corresponding to the functional equation of the completed zeta function and singles out the critical line as the distinguished axis in the spectral variable. No new claim regarding the Riemann Hypothesis is made; instead, the results provide a structural operator-theoretic interpretation of the explicit formula, clarifying the essential role of continuous spectrum and spectral regularization.

0. Introduction

The Riemann explicit formula lies at the heart of analytic number theory, expressing a deep duality between prime numbers and the zeros of the Riemann zeta function. Since its original formulations by Riemann and von Mangoldt, and later by Weil and Titchmarsh, the explicit formula has appeared in many analytically equivalent forms, yet its structural interpretation remains an object of ongoing interest.

A long-standing motivation in this direction is the Hilbert–Pólya philosophy, which suggests that the non-trivial zeros of the zeta function should admit a spectral interpretation in terms of a suitable self-adjoint operator. Despite many partial insights, a fully canonical operator-theoretic framework capturing both the arithmetic and archimedean aspects of the explicit formula while preserving self-adjointness has remained elusive.

The purpose of this paper is not to realize the zeta zeros as discrete eigenvalues, but to present a canonical operator-theoretic construction in which the Riemann explicit formula itself arises as an identity of regularized traces. The central object is a global self-adjoint operator

$$K_{\mathrm{full}} = K_{\mathrm{arit}} \otimes I + I \otimes K_{\mathrm{arq}},$$

whose resolvent encodes the explicit formula in a precise and intrinsic manner.

The arithmetic component K_{arit} has discrete spectrum given by $\log n$, $n \geq 1$, reflecting the multiplicative structure of the integers, while the archimedean component K_{arq}, defined in terms of the generator of dilations, carries continuous spectrum corresponding to the real place. Neither component alone suffices to produce the explicit formula. The arithmetic part yields the von Mangoldt weights through the logarithmic derivative of its spectral determinant, whereas the continuous archimedean spectrum contributes the Gamma factor via finite-part regularization. Within this framework, the non-trivial zeros of the zeta function emerge naturally as spectral singularities of the resolvent, rather than as genuine eigenvalues.

The main result shows that the Riemann explicit formula can be recovered as a regularized trace identity for the resolvent of K_{full}, in the classical Titchmarsh–Mellin form. No new claim concerning the Riemann Hypothesis is made. Rather, the present work clarifies the structural constraints imposed by self-adjointness, continuous spectrum, and spectral regularization, and explains their inevitable role in the formulation of the explicit formula.

The paper is organized as follows. Sections 1 and 2 introduce the arithmetic and archimedean operators. Section 3 constructs the global operator K_{full} and establishes its self-adjointness. Section 4 derives the explicit formula as a regularized trace identity. Section 5 discusses the spectral interpretation of the zeta zeros.

1 The Arithmetic block K_{arit}

This section isolates the *pure arithmetic structure* underlying the Euler product, prior to any spectral linearization or logarithmic parametrization. No additive scale, self–adjoint generator, or trace–level comparison is introduced here. The purpose is to identify the canonical *multiplicative degrees of freedom* that will later interact with the Archimedean component.

1.1 Prime atomicity of the multiplicative monoid

The multiplicative monoid of positive integers $(\mathbb{N}, \cdot)$ is a free commutative monoid generated by the prime numbers. Every integer $n \in \mathbb{N}$ admits a unique factorization

$$n \;=\; \prod_{p\in\mathcal{P}} p^{v_p(n)}, \qquad v_p(n) \in \mathbb{N}_0,$$

with only finitely many non–zero exponents.

Under multiplication, the exponents add:

$$v_p(mn) = v_p(m) + v_p(n).$$

Equivalently, $(\mathbb{N}, \cdot)$ identifies with the set of finitely supported families $(v_p)_{p\in\mathcal{P}}$ with $v_p \in \mathbb{N}_0$, endowed with pointwise addition. In this sense, one has a canonical decomposition

$$(\mathbb{N}, \cdot) \;\cong\; \bigoplus_{p\in\mathcal{P}}^{\mathrm{fin}} (\mathbb{N}_0, +),$$

where the direct sum denotes finitely supported sequences and each prime label defines an independent primary component. This decomposition is purely algebraic and precedes any analytic or spectral input.

At this level:

- no logarithmic parametrization is introduced;
- no additive real scale is present;
- only counting of prime powers is involved.

1.2 Canonical Hilbert–space realization without logarithms

To represent the above multiplicative structure faithfully, we use the *restricted infinite tensor product in the sense of von Neumann*

$$\mathcal{H}_{\mathrm{arit}} \;:=\; \bigotimes_{p\in\mathcal{P}} \big(\ell^2(\mathbb{N}_0), e_0\big),$$

taken with respect to the distinguished vacuum vector $e_0 \in \ell^2(\mathbb{N}_0)$. Concretely, $\mathcal{H}_{\mathrm{arit}}$ is the Hilbert space completion of finite linear combinations of elementary tensors that differ from e_0 in only finitely many tensor factors.

This Hilbert space has a canonical orthonormal basis given by

$$\bigotimes_{p\in\mathcal{P}} e_{p,v_p}, \qquad (v_p) \text{ finitely supported,}$$

which is naturally indexed by $\mathbb{N}$ via prime factorization.

For each prime p, define the number operator

$$N_p\, e_{p,k} \;=\; k\, e_{p,k},$$

acting as the identity on all other tensor factors. The operators N_p are self–adjoint, commute strongly for distinct primes, and encode only the *multiplicity of prime powers*. Importantly, the numerical size of the prime p does not enter at this stage: only its role as a label of an independent degree of freedom is retained.

The multiplicative monoid acts by commuting *isometric shifts*

$$U(n) \;=\; \bigotimes_{p\in\mathcal{P}} U_p^{\,v_p(n)},$$

where U_p denotes the unilateral shift on $\ell^2(\mathbb{N}_0)$ acting in the p–th tensor factor. This defines a faithful representation of the monoid $(\mathbb{N},\cdot)$ on $\mathcal{H}_{\mathrm{arit}}$, depending solely on the exponent data $v_p(n)$.

Consequently:

- the spectrum of each number operator N_p is $\{k\}_{k\in\mathbb{N}_0}$;
- this spectrum is identical for all primes;
- no arithmetic scale, ordering, or metric structure on the set of primes is encoded.

1.3 Absence of spectral linearization

At this stage, the arithmetic block contains *no additive spectral information.* In particular:

- there is no self–adjoint generator with real–valued spectrum;
- no comparison between different primes is available;
- no trace–level expression resembling the explicit formula can arise.

The arithmetic structure realized here corresponds to the Euler product *prior* to passing to logarithmic coordinates. Any additive spectral interpretation necessarily requires an additional linearization step, which will be introduced later and shown to be rigid.

1.4 The Euler product as a cyclic matrix coefficient

We now show that the Euler product is already fully encoded at the representation–theoretic level of the arithmetic block, without any spectral linearization or logarithmic parametrization.

Let

$$\mathcal{H}_{\mathrm{arit}} = \bigotimes_{p\in\mathcal{P}} (\ell^2(\mathbb{N}_0), e_{p,0})$$

be the arithmetic Hilbert space introduced above, and define the canonical vacuum vector

$$\Omega := \bigotimes_{p\in\mathcal{P}} e_{p,0}.$$

The vector Ω is cyclic for the isometric representation $U\colon (\mathbb{N},\cdot) \to \mathcal{B}(\mathcal{H}_{\mathrm{arit}})$, and one has

$$U(n)\Omega = \bigotimes_{p\in\mathcal{P}} e_{p,v_p(n)}.$$

Let $(\alpha_p)_{p\in\mathcal{P}}$ be a family of complex numbers such that $\sum_p |\alpha_p|^2 < \infty$, and for each prime p define

$$\psi_p := \sum_{k=0}^{\infty} \alpha_p^k\, e_{p,k} \ \in\ \ell^2(\mathbb{N}_0).$$

The restricted tensor product

$$\Psi := \bigotimes_{p\in\mathcal{P}} \psi_p$$

is then a well–defined vector in $\mathcal{H}_{\mathrm{arit}}$.

Proposition 1.1. *For every $n \in \mathbb{N}$, the matrix coefficient of the arithmetic representation satisfies*

$$\langle \Psi,\, U(n)\Omega\rangle = \prod_{p\in\mathcal{P}} \alpha_p^{v_p(n)}.$$

Proof. Using the tensor product structure and the fact that only finitely many $v_p(n)$ are non–zero, one computes

$$\langle \Psi,\, U(n)\Omega\rangle = \prod_{p\in\mathcal{P}} \langle \psi_p,\, e_{p,v_p(n)}\rangle = \prod_{p\in\mathcal{P}} \alpha_p^{v_p(n)}.$$

□

In particular, choosing $\alpha_p = p^{-s}$ with $\mathrm{Re}(s) > 1$ gives

$$\langle \Psi_s,\, U(n)\Omega\rangle = n^{-s}, \qquad \Psi_s := \bigotimes_{p} (1-p^{-s})^{1/2} \sum_{k\geq 0} p^{-ks} e_{p,k},$$

and summing over the orbit of Ω yields the Euler product

$$\sum_{n\geq 1} \langle \Psi_s,\, U(n)\Omega\rangle = \sum_{n\geq 1} n^{-s} = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}}.$$

Thus the Euler product arises as a cyclic matrix coefficient of the purely multiplicative representation $(\mathbb{N},\cdot) \curvearrowright \mathcal{H}_{\mathrm{arit}}$, prior to any additive linearization.

1.5 Rigidity of spectral linearization

The arithmetic block constructed in the previous section encodes the full multiplicative structure of the Euler product without introducing any additive spectral scale. At this level, only prime multiplicities are represented, and no comparison between different primes is available.

In order to pass from a purely multiplicative description to a spectral one, it is necessary to introduce a linearization map transforming multiplication into addition. Concretely, this amounts to postulating the existence of a map

$$E : (\mathbb{N}, \cdot) \longrightarrow (\mathbb{R}, +)$$

satisfying

$$E(mn) = E(m) + E(n),$$

which can be interpreted as assigning a real-valued spectral parameter to each integer.

Such a map cannot be arbitrary. If the resulting parameter is to deserve the name of a spectral scale, it must be compatible with a continuous notion of multiplicative dilation. We therefore require that E admit an extension

$$\widetilde{E} : \mathbb{R}_+^\times \longrightarrow \mathbb{R}$$

which is a homomorphism of multiplicative groups and satisfies a minimal regularity condition, such as measurability or continuity.

Lemma 1.2 (Rigidity of additive linearization)**.** *Let $\widetilde{E} : \mathbb{R}_+^\times \to \mathbb{R}$ be a measurable (or continuous) homomorphism satisfying*

$$\widetilde{E}(xy) = \widetilde{E}(x) + \widetilde{E}(y).$$

Then there exists a constant $c \in \mathbb{R}$ such that

$$\widetilde{E}(x) = c \log x \qquad \textit{for all } x > 0.$$

In particular, the restriction $E = \widetilde{E}|_{\mathbb{N}}$ satisfies $E(n) = c \log n$.

Proof. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) := \widetilde{E}(e^t)$. Then f satisfies the additive Cauchy equation

$$f(t + s) = f(t) + f(s),$$

and inherits measurability (or continuity) from $\widetilde{E}$. By the classical rigidity theorem for the additive Cauchy equation, there exists $c \in \mathbb{R}$ such that $f(t) = ct$, hence $\widetilde{E}(x) = c \log x$ for all $x > 0$. □

This result shows that the logarithm is not a modeling choice but the unique additive linearization of multiplication compatible with a continuous spectral scale. Any attempt to represent the arithmetic block by a self–adjoint operator with real spectrum necessarily assigns eigenvalues proportional to $\log n$.

Accordingly, once the existence of a genuine spectral parameter is required, the arithmetic generator is forced to be logarithmic. In the next subsection we use this rigidity to construct the canonical arithmetic operator K_{arit}.

1.6 Canonical logarithmic arithmetic generator

Having established that any additive spectral linearization compatible with a continuous notion of scale is necessarily logarithmic, we now construct the corresponding arithmetic generator.

We work on the purely multiplicative arithmetic Hilbert space

$$\mathcal{H}_{\mathrm{arit}} := \bigotimes_{p\in\mathcal{P}} \big(\ell^2(\mathbb{N}_0), e_{p,0}\big)$$

introduced in Section 1. For each prime p, let

$$N_p e_{p,k} = k\, e_{p,k}, \qquad k \in \mathbb{N}_0,$$

denote the number operator acting on the p-th tensor factor and as the identity on all others. The family $\{N_p\}_{p\in\mathcal{P}}$ consists of mutually strongly commuting self–adjoint operators.

By the rigidity result of the previous subsection, any self–adjoint operator implementing a genuine spectral scale for the arithmetic block must assign eigenvalues proportional to $\log p$ to each prime generator. Up to an overall normalization, this uniquely determines the arithmetic Hamiltonian.

Definition 1.3 (Arithmetic generator)**.** The canonical arithmetic generator is the self–adjoint operator

$$K_{\mathrm{arit}} := \sum_{p\in\mathcal{P}} (\log p)\, N_p,$$

defined on the natural domain

$$D(K_{\mathrm{arit}}) = \left\{ \psi \in \mathcal{H}_{\mathrm{arit}} : \sum_{p\in\mathcal{P}} (\log p)^2\, \|N_p\psi\|^2 < \infty \right\}.$$

Since only finitely many tensor components of any basis vector differ from the vacuum, the sum defining K_{arit} is well defined on a dense core, and standard results for sums of commuting self–adjoint operators ensure that K_{arit} is self–adjoint.

On the canonical basis vectors indexed by integers $n = \prod_p p^{v_p(n)}$, one has

$$K_{\mathrm{arit}} \bigotimes_p e_{p,v_p(n)} = \Big(\sum_p v_p(n)\, \log p\Big) \bigotimes_p e_{p,v_p(n)} = (\log n) \bigotimes_p e_{p,v_p(n)}.$$

Thus the spectrum of K_{arit} is

$$\sigma(K_{\mathrm{arit}}) = \{\log n : n \in \mathbb{N}\},$$

with multiplicities determined by the prime factorization.

This operator provides the unique logarithmic spectral realization of the purely multiplicative arithmetic block. No alternative assignment of spectral values is possible without abandoning the requirement of a continuous additive scale.

2 The Archimedean block K_{arq}

This section constructs the Archimedean block

$$K_{\mathrm{arq}} = \frac{1}{2\pi} \log |A|, \qquad A := -i\Big(x\partial_x + \tfrac{1}{2}\Big),$$

in a self–contained and operator–theoretic manner, and records the canonical steps leading to this definition. The argument uses only Stone's theorem, the Mellin transform, and the spectral theorem for self–adjoint operators [9, 10].

2.1 Step 1: the canonical symmetry on $\mathbb{R}_+$ is dilation

The Archimedean place is intrinsically continuous and multiplicative. Accordingly, the canonical one–parameter symmetry on $\mathbb{R}_+ = (0, \infty)$ is the dilation flow

$$x \mapsto e^t x, \qquad t \in \mathbb{R}.$$

On the Hilbert space $L^2(\mathbb{R}_+, dx)$, the unique unitary implementation of this flow is given by the *unitary dilation group*

$$(U_t f)(x) \;:=\; e^{t/2}\, f(e^t x), \qquad f \in L^2(\mathbb{R}_+),\ t \in \mathbb{R}. \tag{1}$$

Lemma 2.1 (Unitarity and strong continuity)**.** *The family $(U_t)_{t\in\mathbb{R}}$ defined in* (1) *is a strongly continuous one–parameter unitary group on $L^2(\mathbb{R}_+)$.*

Proof. For unitarity, compute by the change of variables $y = e^t x$:

$$\|U_t f\|_2^2 = \int_0^\infty |e^{t/2} f(e^t x)|^2\, dx = e^t \int_0^\infty |f(e^t x)|^2\, dx = \int_0^\infty |f(y)|^2\, dy = \|f\|_2^2.$$

Thus U_t is unitary and $U_{t+s} = U_t U_s$ is immediate. Strong continuity follows from density of $C_c^\infty(\mathbb{R}_+)$ in L^2 and standard approximation arguments for translation–type groups; see [9, Thm. VIII.7]. □

2.2 Step 2: the generator is forced, and the shift $\frac{1}{2}$ is not optional

By Stone's theorem, every strongly continuous unitary group has a unique self–adjoint generator.

Theorem 1 (Stone)**.** *There exists a unique self–adjoint operator A on $L^2(\mathbb{R}_+)$ such that $U_t = e^{itA}$ for all $t \in \mathbb{R}$. Moreover, for f in the domain $\mathcal{D}(A)$,*

$$Af = \frac{1}{i} \frac{d}{dt}\Big|_{t=0} U_t f \quad \textit{(strong derivative in } L^2\textit{)}.$$

Proposition 2.2 (Explicit form of the dilation generator)**.** *On the core $C_c^\infty(\mathbb{R}_+)$, the generator A acts as*

$$A = -i\Big(x\partial_x + \tfrac{1}{2}\Big).$$

In particular, the shift $+\frac{1}{2}$ is forced by unitarity of the dilation action on $L^2(\mathbb{R}_+)$.

Proof. Let $f \in C_c^\infty(\mathbb{R}_+)$. Differentiate (1):

$$\frac{d}{dt} U_t f(x) = \frac{1}{2} e^{t/2} f(e^t x) + e^{t/2} \, (x e^t) \, f'(e^t x).$$

At $t = 0$ this gives

$$\left. \frac{d}{dt} \right|_{t=0} U_t f(x) = \frac{1}{2} f(x) + x f'(x) = (x\partial_x + \tfrac{1}{2}) f(x).$$

Stone's theorem yields $Af = \frac{1}{i} \left. \frac{d}{dt} \right|_{t=0} U_t f = -i(x\partial_x + \frac{1}{2}) f$ on this core.

Finally, note that the factor $e^{t/2}$ in (1) is exactly what makes U_t unitary in $L^2(dx)$ (Lemma 2.1). Removing it would destroy unitarity, hence would change the generator. Thus the shift is not a convention: it is imposed by the Hilbert space structure. □

2.3 Step 3: diagonalization via Mellin transform

The dilation generator becomes diagonal after passing to logarithmic coordinates. Define the unitary map

$$(Vf)(u) := e^{u/2} f(e^u), \qquad u \in \mathbb{R},$$

which sends $L^2(\mathbb{R}_+, dx)$ unitarily onto $L^2(\mathbb{R}, du)$. A direct computation shows

$$(V U_t V^{-1} g)(u) = g(u + t),$$

so the dilation group becomes the translation group on $L^2(\mathbb{R})$. Now apply the unitary Fourier transform on $L^2(\mathbb{R})$,

$$(\mathcal{F} g)(\lambda) = \int_{\mathbb{R}} g(u) e^{-2\pi i \lambda u} \, du,$$

with inverse

$$g(u) = \int_{\mathbb{R}} (\mathcal{F} g)(\lambda) e^{2\pi i \lambda u} \, d\lambda.$$

Lemma 2.3 (Spectral representation of A)**.** *Let $\mathcal{M} := \mathcal{F} \circ V$. Then $\mathcal{M} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ is unitary and*

$$(\mathcal{M} U_t \mathcal{M}^{-1} h)(\lambda) = e^{2\pi i \lambda t} h(\lambda), \qquad t \in \mathbb{R}.$$

Consequently,

$$(\mathcal{M} A \mathcal{M}^{-1} h)(\lambda) = 2\pi \, \lambda \, h(\lambda)$$

on the natural domain $\{h \in L^2(\mathbb{R}) : \lambda h(\lambda) \in L^2(\mathbb{R})\}$.

Proof. The intertwining $V U_t V^{-1}$ with translations is immediate from the definitions. The Fourier transform diagonalizes translations: $\mathcal{F}(g(\cdot + t))(\lambda) = e^{2\pi i \lambda t} \mathcal{F} g(\lambda)$. Thus $\mathcal{M} U_t \mathcal{M}^{-1}$ is multiplication by $e^{2\pi i \lambda t}$. By Stone's theorem applied in the spectral representation, the generator is multiplication by $2\pi\lambda$. □

Remark (Mellin viewpoint). The unitary map $\mathcal{M} = \mathcal{F} \circ V$ is a standard unitary Mellin transform in logarithmic coordinates. Lemma 2.3 is the spectral theorem for the dilation generator.

2.4 Step 4: the logarithmic Archimedean block via functional calculus

The arithmetic block is logarithmic in nature (frequencies $k \log p$), so the continuous analogue is obtained by applying a logarithm to the canonical continuous generator. Since A is self–adjoint, the Borel functional calculus defines $\varphi(A)$ for any Borel function $\varphi : \mathbb{R} \to \mathbb{R}$. In particular, $\log |A|$ is defined by choosing $\varphi(t) = \log |t|$.

Definition 2.4 (Logarithm of the dilation generator)**.** Let A be the self–adjoint dilation generator. Define the self–adjoint operator $\log |A|$ by spectral calculus:

$$\log |A| := \int_{\mathbb{R}} \log |\mu| \, dE_A(\mu),$$

where E_A is the spectral resolution of A. Its domain is

$$\mathcal{D}(\log |A|) = \left\{ f \in L^2(\mathbb{R}_+) : \int_{\mathbb{R}} (\log |\mu|)^2 \, d\langle E_A(\mu) f, f \rangle < \infty \right\}.$$

Proposition 2.5 (Diagonal form of $\log |A|$)**.** *In the spectral representation of Lemma 2.3,*

$$(\mathcal{M} \, \log |A| \, \mathcal{M}^{-1} h)(\lambda) = \log |2\pi\lambda| \, h(\lambda),$$

with domain $\{h \in L^2(\mathbb{R}) : (\log |2\pi\lambda|) h(\lambda) \in L^2(\mathbb{R})\}$.

Proof. This is the functional calculus applied to the multiplication operator $2\pi\lambda$ from Lemma 2.3: $\varphi(2\pi\lambda) = \log |2\pi\lambda|$. □

2.5 Step 5: definition of K_{arq} and the standard $1/(2\pi)$ normalization

We now define the Archimedean block used in the main text.

Definition 2.6 (Archimedean block)**.** Let

$$A := -i\left(x\partial_x + \tfrac{1}{2}\right)$$

be the self–adjoint generator of the unitary dilation group on $L^2(\mathbb{R}_+)$. Define

$$K_{\text{arq}} := \frac{1}{2\pi} \log |A|$$

with domain $\mathcal{D}(K_{\text{arq}}) = \mathcal{D}(\log |A|)$.

Remark (Why the factor $1/(2\pi)$ is standard)*.* Under the unitary spectral representation $\mathcal{M}$, the generator A corresponds to multiplication by $2\pi\lambda$ (Lemma 2.3). Thus $\log |A|$ corresponds to multiplication by $\log |2\pi\lambda|$ (Proposition 2.5). The factor $1/(2\pi)$ in Definition 2.6 is the conventional Fourier normalization attached to the generator parameterization: it separates the universal 2π factor of the translation character $e^{2\pi i\lambda t}$ from the intrinsic logarithmic dependence. No asymptotic input is used here; the normalization is fixed at the level of the unitary diagonalization of the dilation group.

2.6 Core, self–adjointness, and positivity properties

For completeness we record standard operator facts used implicitly.

Proposition 2.7 (Core and self–adjointness)**.** *The space $C_c^\infty(\mathbb{R}_+)$ is a core for A, and A is essentially self–adjoint on $C_c^\infty(\mathbb{R}_+)$. Consequently, $\log|A|$ and K_{arq} are self–adjoint operators defined by functional calculus of A.*

Proof. Under the unitary map V, the operator A corresponds to $-i\partial_u$ on $L^2(\mathbb{R})$ (up to the Fourier convention), whose essential self–adjointness on $C_c^\infty(\mathbb{R})$ is classical [10, Sec. X.3]. Transporting back by V gives essential self–adjointness of A on $C_c^\infty(\mathbb{R}_+)$. Functional calculus preserves self–adjointness of $\log|A|$ and K_{arq}. $\square$

Remark (On positivity)*.* The operator A is not positive (its spectrum is $\mathbb{R}$), but $\log|A|$ is self–adjoint and unbounded from above and below. This is consistent with its interpretation as a logarithmic generator: it encodes scale rather than energy positivity. In the main text, the full undeformed operator $K_{\mathrm{full}} = K_{\mathrm{arit}} \oplus K_{\mathrm{arq}}$ is self–adjoint; no positivity assumption is required at this stage.

2.7 Summary of the construction

The Archimedean block is determined by three canonical steps:

(i) the dilation symmetry on $\mathbb{R}_+$ has a unique unitary realization on $L^2(\mathbb{R}_+)$, forcing the generator $A = -i(x\partial_x + \frac{1}{2})$;

(ii) the logarithmic nature of the arithmetic frequencies suggests applying $\log|\cdot|$ to this continuous generator;

(iii) the standard Fourier/Mellin normalization fixes the factor $1/(2\pi)$. This yields $K_{\mathrm{arq}} = \frac{1}{2\pi}\log|A|$ as in Definition 2.6.

3 The full operator K_{full}

Having constructed separately the arithmetic block K_{arit} and the Archimedean block K_{arq}, we now combine them into a single operator acting on the full Hilbert space. No additional structure is introduced at this stage: the form of the total operator is completely determined by the independence of the arithmetic and Archimedean sectors and by the canonical constructions of Sections 1 and 2.

3.1 The full Hilbert space

We consider the tensor product Hilbert space

$$\mathcal{H}_{\mathrm{full}} := \mathcal{H}_{\mathrm{arit}} \mathbin{\widehat{\otimes}} \mathcal{H}_{\mathrm{arq}},$$

where

$$\mathcal{H}_{\mathrm{arit}} = \bigotimes_{p\in\mathcal{P}} (\ell^2(\mathbb{N}_0), e_0) \quad \text{and} \quad \mathcal{H}_{\mathrm{arq}} = L^2(\mathbb{R}_+, dx).$$

The tensor product is taken in the standard Hilbert–space sense. Vectors in $\mathcal{H}_{\mathrm{full}}$ represent states carrying both arithmetic (prime multiplicity) and Archimedean (continuous scale) degrees of freedom. No coupling between these sectors is assumed at the kinematical level.

3.2 Definition of the full operator

Definition 3.1 (Full operator)**.** The full operator is defined as the direct sum

$$K_{\text{full}} := K_{\text{arit}} \otimes I_{\text{arq}} \; + \; I_{\text{arit}} \otimes K_{\text{arq}},$$

with domain

$$D(K_{\text{full}}) = D(K_{\text{arit}} \otimes I) \cap D(I \otimes K_{\text{arq}}).$$

Here I_{arit} and I_{arq} denote the identity operators on $\mathcal{H}_{\text{arit}}$ and $\mathcal{H}_{\text{arq}}$, respectively.

3.3 Self–adjointness and basic properties

Since K_{arit} and K_{arq} are self–adjoint operators acting on independent tensor factors, standard results on sums of commuting self–adjoint operators imply:

Proposition 3.2. *The operator K_{full} is self–adjoint on $D(K_{\text{full}})$.*

On elementary tensor states of the form

$$\Psi_n \otimes \phi \;\in\; \mathcal{H}_{\text{arit}} \otimes \mathcal{H}_{\text{arq}},$$

where Ψ_n is the canonical arithmetic basis vector indexed by $n \in \mathbb{N}$, one has

$$K_{\text{full}}(\Psi_n \otimes \phi) = (\log n)\, \Psi_n \otimes \phi \; + \; \Psi_n \otimes K_{\text{arq}}\phi.$$

Thus, K_{full} acts additively on the logarithmic arithmetic label and on the continuous Archimedean scale variable, in complete analogy with the logarithmic linearization of multiplicative structures.

3.4 Interpretation

The operator K_{full} encodes the total logarithmic scale of the system, combining:

- discrete arithmetic contributions $\log p$ arising from prime multiplicities,
- continuous Archimedean contributions arising from dilation symmetry on $\mathbb{R}_+$.

Importantly, no interaction term between these sectors is introduced. Any coupling or spectral deformation must therefore arise from additional dynamical or trace–level input, not from the kinematical definition of the operator itself.

In particular, the construction of K_{full} is entirely canonical once the requirements of multiplicative covariance and logarithmic linearization are imposed. The operator provides the natural starting point for spectral and trace–formula considerations in the Hilbert–Pólya framework.

4 The resolvent of K_{full} and the explicit formula

In this section we derive the Riemann explicit formula from the operator

$$K_{\text{full}} = K_{\text{arit}} \otimes I + I \otimes K_{\text{arq}},$$

by interpreting it as an identity of *regularized traces* of the resolvent. The key point is that the correct spectral object is not a naive trace, but the logarithmic derivative of a regularized spectral determinant.

4.1 Test function and Mellin transform

Let

$$g \in C_c^\infty(\mathbb{R}), \qquad g \text{ even}.$$

We introduce the shifted bilateral Laplace–Mellin transform

$$\Phi(s) := \int_{\mathbb{R}} g(u)\, e^{(s-\frac{1}{2})u}\, du.$$

Since g has compact support, Φ is an entire function that decays rapidly on vertical strips. The parity of g implies the symmetry

$$\Phi(s) = \Phi(1-s),$$

which will be crucial when shifting contours.

4.2 The global spectral object

We consider the global spectral function

$$\Xi(s) := -\frac{d}{ds} \log \det\nolimits_{\text{reg}}(s + K_{\text{full}}), \qquad \operatorname{Re} s > 1.$$

Using the Laplace representation of the resolvent,

$$(s + K_{\text{full}})^{-1} = \int_0^\infty e^{-ts}\, e^{-tK_{\text{full}}}\, dt,$$

and the tensor-sum decomposition of K_{full}, we obtain the factorization

$$e^{-tK_{\text{full}}} = e^{-tK_{\text{arit}}} \otimes e^{-tK_{\text{arq}}}.$$

Thus the logarithmic derivative of the regularized determinant splits into the arithmetic and archimedean contributions.

Regularized determinants. Throughout this section, the symbol $\det_{\text{reg}}$ denotes a *regularized spectral determinant* defined via the Laplace–Mellin representation of the resolvent,

$$(s + K)^{-1} = \int_0^\infty e^{-t(s+K)}\, dt,$$

together with finite–part subtraction for the continuous spectrum. In particular, the determinant is *not* defined as a convergent product of eigenvalues, but rather through its logarithmic derivative, which is uniquely determined after removal of the Weyl (volume) divergences. Different choices of regularization scheme modify $\log \det_{\text{reg}}$ at most by linear functionals of the test function g, and therefore do not affect the structure of the explicit formula.

4.3 Arithmetic contribution

The arithmetic operator K_{arit} acts diagonally by

$$K_{\mathrm{arit}}\Psi_n = (\log n)\Psi_n, \qquad n \geq 1.$$

Therefore,

$$\mathrm{Tr}\big(e^{-tK_{\mathrm{arit}}}\big) = \sum_{n\geq 1} e^{-t\log n} = \sum_{n\geq 1} n^{-t} = \zeta(t), \qquad \mathrm{Re}\, t > 1.$$

Hence,

$$\mathrm{Tr}\big((s+K_{\mathrm{arit}})^{-1}\big) = \int_0^\infty e^{-ts}\,\zeta(t)\,dt, \qquad \mathrm{Re}\, s > 1,$$

and, by analytic continuation,

$$-\frac{d}{ds}\log\det_{\mathrm{reg}}(s+K_{\mathrm{arit}}) = -\frac{\zeta'}{\zeta}(s).$$

To evaluate its action on the test function, consider

$$I_{\mathrm{arit}}(g) = \frac{1}{2\pi i}\int_{(c)}\Big(-\frac{\zeta'}{\zeta}(s)\Big)\Phi(s)\,ds, \qquad c > 1.$$

For $\mathrm{Re}\, s > 1$,

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n\geq 1}\frac{\Lambda(n)}{n^s},$$

so absolute convergence allows us to interchange sum and integral:

$$I_{\mathrm{arit}}(g) = \sum_{n\geq 1}\Lambda(n)\frac{1}{2\pi i}\int_{(c)}\Phi(s)\,n^{-s}\,ds.$$

Using the definition of Φ and Mellin inversion,

$$\frac{1}{2\pi i}\int_{(c)} e^{(s-\frac{1}{2})u}n^{-s}\,ds = e^{-u/2}\,\delta(u-\log n),$$

we obtain

$$\boxed{I_{\mathrm{arit}}(g) = \sum_{n\geq 1}\Lambda(n)\,g(\log n).}$$

Analytic justification. All exchanges of summation, integration and contour shifts appearing below are justified by the choice of test function $g \in C_c^\infty(\mathbb{R})$. Indeed, compact support of g implies rapid decay of its Mellin transform $\Phi(s)$ on vertical strips, ensuring absolute convergence of the Dirichlet series for $\mathrm{Re}\, s > 1$, as well as uniform integrability of the Laplace–Mellin representations. Contour shifts are justified by the functional equation of the completed zeta function together with the symmetry $\Phi(s) = \Phi(1-s)$.

4.4 Archimedean contribution

The archimedean operator is defined by

$$K_{\mathrm{arq}} = \frac{1}{2\pi}\log|A|,$$

which diagonalizes spectrally as multiplication by $\log|2\pi\lambda|$. Hence,

$$e^{-tK_{\mathrm{arq}}} \quad \longleftrightarrow \quad |2\pi\lambda|^{-t/(2\pi)}.$$

The corresponding trace diverges due to spectral volume growth. We therefore adopt a finite-part regularization with cutoff in λ, as discussed previously. The regularized contribution is equivalent to zeta-regularization and yields

$$-\frac{d}{ds}\log\det_{\mathrm{reg}}(s+K_{\mathrm{arq}}) = \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big) - \frac{1}{2}\log\pi + \mathrm{const.}$$

We thus consider

$$I_\Gamma(g) = \frac{1}{2\pi i}\int_{(c)}\Big(-\frac{1}{2}\psi\Big(\frac{s}{2}\Big)\Big)\Phi(s)\,ds,$$

where $\psi = \Gamma'/\Gamma$.

For $\operatorname{Re} z > 0$,

$$\psi(z) = -\gamma + \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right)dt.$$

Substituting $z = s/2$ and interchanging integrals, we obtain

$$I_\Gamma(g) = C_g + \frac{1}{2}\int_0^\infty \frac{1}{1-e^{-t}}\left(\frac{1}{2\pi i}\int_{(c)}\Phi(s)e^{-(s/2)t}\,ds\right)dt.$$

The inner integral is computed explicitly:

$$\begin{aligned}\frac{1}{2\pi i}\int_{(c)}\Phi(s)e^{-(s/2)t}\,ds &= \int_{\mathbb{R}} g(u)e^{-u/2}\left(\frac{1}{2\pi i}\int_{(c)} e^{s(u-\frac{t}{2})}\,ds\right)du \\ &= g(t/2)\,e^{-t/4}.\end{aligned}$$

Thus,

$$\boxed{I_\Gamma(g) = C_g + \int_0^\infty g(x)\,\frac{e^{-x/2}}{1-e^{-2x}}\,dx.}$$

The archimedean continuous spectrum. The continuous spectrum of the archimedean operator K_{arq} is not an artifact of the construction, but an intrinsic feature of the real place. The divergence of the naive trace reflects the Weyl law for dilations on $\mathbb{R}_+^\times$, and the finite–part subtraction precisely removes this volume contribution. The remaining finite term is exactly the logarithmic derivative of the Gamma factor, showing that the archimedean correction in the explicit formula arises naturally from regularized spectral theory.

4.5 Trivial poles

The factor $s(s-1)$ in the completed zeta contributes

$$-\frac{d}{ds}\log(s(s-1)) = -\frac{1}{s} - \frac{1}{s-1}.$$

Shifting the contour to $\operatorname{Re} s < 0$, the residues at $s = 0$ and $s = 1$ yield

$$I_{\text{pol}}(g) = -\Phi(0) - \Phi(1).$$

Since g is even, $\Phi(0) = \Phi(1)$, hence

$$\boxed{I_{\text{pol}}(g) = -2\Phi(0).}$$

4.6 Spectral side and contour shift

The completed zeta function $\xi(s)$ is entire and satisfies $\xi(s) = \xi(1-s)$. By the symmetry $\Phi(s) = \Phi(1-s)$, the contribution from the left vertical line cancels after shifting the contour. The remaining residues come from the non-trivial zeros ρ of $\zeta(s)$, each contributing $\Phi(\rho)$.

Therefore,

$$\boxed{\sum_\rho \Phi(\rho) = I_{\text{arit}}(g) + I_\Gamma(g) + I_{\text{pol}}(g).}$$

4.7 The explicit formula

Collecting all terms, we obtain the explicit formula

$$\boxed{\sum_\rho \Phi(\rho) = \sum_{n\geq 1} \Lambda(n)\, g(\log n) + \int_0^\infty g(x)\, \frac{e^{-x/2}}{1-e^{-2x}}\, dx - 2\Phi(0) + C_g.}$$

This is precisely the Riemann explicit formula in Titchmarsh form, derived as an identity of regularized traces of the resolvent of

$$K_{\text{full}} = K_{\text{arit}} \otimes I + I \otimes K_{\text{arq}}.$$

Interpretation. The explicit formula thus appears as a trace identity for the regularized resolvent of K_{full}. The arithmetic component produces the von Mangoldt weights through the logarithmic derivative of the zeta function, while the archimedean component contributes the Gamma factor via finite–part regularization of its continuous spectrum. In this sense, primes and zeros arise from different spectral decompositions of the same global operator.

5 Self-adjointness and spectral interpretation

In this section we discuss the spectral meaning of the operator

$$K_{\text{full}} = K_{\text{arit}} \otimes I + I \otimes K_{\text{arq}},$$

and explain how the non-trivial zeros of the Riemann zeta function arise naturally from its resolvent.

5.1 Self-adjointness of the global operator

Both components of K_{full} admit natural self-adjoint realizations. The arithmetic operator K_{arit} is self-adjoint on its canonical domain, being defined as a sum of commuting self-adjoint number operators. The archimedean operator

$$K_{\text{arq}} = \frac{1}{2\pi} \log |A|$$

is self-adjoint by functional calculus, since the generator of dilations A is essentially self-adjoint on a natural dense core.

As the two operators act on independent tensor factors, they commute strongly. Standard results on sums of commuting self-adjoint operators therefore imply that the tensor sum

$$K_{\text{full}} = K_{\text{arit}} \otimes I + I \otimes K_{\text{arq}}$$

admits a self-adjoint realization on the intersection of the natural domains. Throughout the paper, K_{full} is understood in this sense.

5.2 Zeros as spectral singularities

The non-trivial zeros of the Riemann zeta function do not correspond to discrete eigenvalues of K_{full}. Instead, they appear as *spectral singularities* of the resolvent

$$(s + K_{\text{full}})^{-1},$$

or, equivalently, as poles of the logarithmic derivative of the regularized spectral determinant.

This phenomenon is analogous to the appearance of resonances in scattering theory for non-compact or partially continuous systems. The arithmetic component contributes a discrete spectrum, while the archimedean component carries a continuous spectrum. The zeros arise from the interaction between these two parts and should therefore be viewed as global resonances rather than point spectrum.

5.3 Symmetry and the critical line

Since K_{full} is self-adjoint, its spectrum is real in the natural spectral parameter. Under the Mellin transform used throughout this paper, this reality corresponds to symmetry with respect to the line $\operatorname{Re} s = \frac{1}{2}$.

The functional equation of the completed zeta function is reflected at the operator level by this spectral symmetry, and the critical line emerges as the natural axis of symmetry for the resolvent. In this framework, the Riemann Hypothesis appears as the statement that all spectral singularities compatible with self-adjointness occur exactly on this axis.

While no new claim concerning the Riemann Hypothesis is made here, the present construction provides a precise sense in which the critical line is forced by self-adjointness and symmetry considerations alone.

6 Conclusion and perspective

The construction developed in this work was not guided by the search for a naïve realization of the Hilbert–Pólya conjecture, according to which the non-trivial zeros of the Riemann zeta function should arise as discrete eigenvalues of a self-adjoint operator. On the contrary, the results obtained here point in the opposite direction. Within a fully canonical framework, based solely on the multiplicative structure of the integers, the archimedean symmetry of dilations, and the requirement of self-adjointness, the zeros emerge inevitably as spectral singularities of the resolvent rather than as part of the point spectrum.

A central feature of this construction is the absence of arbitrary choices or tunable parameters. The logarithmic form of the arithmetic generator, the presence of continuous spectrum in the archimedean component, and the tensorial structure of the global operator are rigid consequences of the structural hypotheses imposed from the outset. In this sense, the present work suggests that any realization of Hilbert–Pólya type aiming to identify the zeros as genuine eigenvalues would necessarily have to abandon, modify, or extend at least one of these canonical structures, introducing additional dynamical ingredients not present at the purely kinematical level considered here.

Rather than proposing a candidate for a *Hilbert–Pólya operator*, this work therefore delineates precisely what such an operator cannot be. It clarifies the unavoidable role of continuous spectrum and spectral regularization in the formulation of the explicit formula, and delineates the structural constraints that any operator-theoretic approach to the zeta zeros must confront.

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