Composing Music with the Pythagorean Table

Matrices for Writing Counterpoint and Canons for Any Number of Voices.

A Mathematical Reinterpretation of the Neapolitan Partimento Tradition

Luca Bianchini

Doctor of Musicology – University of Pavia

School of Paleography and Musical Philology

`luca.bianchini@italianopera.org`

February 5, 2026

Editorial note. This is the original English version of a paper also available in Italian, translated by the author and published on Zenodo under the title: *Comporre musica con la tavola pitagorica. Matrici che servono a scrivere contrappunti e canoni a qualsiasi numero di voci. Una rivisitazione matematica dei partimenti di scuola napoletana.*

Abstract

This article presents a new mathematical model for contrapuntal composition, based on the formalization of the concept of the *Abstract Seed* and on the use of transformational operators (keys and prolations). Unlike modern approaches, which are often complex and limited to a small number of voices, the proposed system introduces a new combinatorial logic, showing how the management of eight voices paradoxically makes the compositional process simpler than the management of two or three voices.

The article further illustrates how, according to this theory, all the principal types of canons can be realized at any interval, including inverse, retrograde, retrograde-inverse, augmented, and proportional canons. The method, validated through algebraic formalization and systematic application to historical partimenti, is applicable to every species of counterpoint and proposes a shift beyond traditional pedagogy in favor of an approach based on matrices and potentially infinite keys.

Contents

1 Introduction

1.1 The Historiographical and Pedagogical Problem

Traditional composition pedagogy, as well as musical historiography, describes the development of polyphony as a linear achievement. According to this evolutionary and progressive view, the student must first master the horizontal dimension (melody), then the vertical one (harmony), in order to finally attempt the synthesis of the two in counterpoint and, only at the end of this path, confront the complexity of the canon. This sequential approach carries with it an implicit and misleading message: that the dimensions of musical space are separate entities, to be combined with one another through a laborious process of artisanal assembly.

1.2 The Limits of the Linear View

The direct consequence of this approach is the idea that compositional difficulty increases linearly, or even exponentially, as the number of voices grows. In common practice, managing a two-voice canon requires a certain degree of local control over intervals; managing a four-voice canon is considered difficult; writing one for eight or more voices is seen as a virtuoso feat accessible to only a few, the result of an exhausting process of local interval-based *trial-and-error*. This perspective, however, clashes with a counterintuitive mathematical reality that this article aims to demonstrate: complexity does not reside in the number of parts, but in the method of generation.

1.3 An Alternative Hypothesis: Dimensional Simultaneity

Our hypothesis is that the horizontal, vertical, and diagonal dimensions (temporal imitation) are not successive layers, but coexisting properties already contained *in potentia* within a single abstract object: the **Pythagorean Triad**, understood not as a historical-tonal chord, but as a relational numerical structure. In the system presented here, the numerical sequence 1–3–5 is not treated as a mere chord, but as an **Abstract Seed**, in which all the information necessary to develop the entire polyphonic architecture is already encoded, rendering the distinction between melody and harmony a purely axial projection.

1.4 Genesis and Validation of the System

The theoretical model formalized here is not an abstract speculation, but the systematization of a compositional method developed by the author and privately transmitted in 2013 to Dr. Coreen Morsink, later documented in her doctoral dissertation at Goldsmiths College (University of London).[1]

In this research work, the author explicitly acknowledges Luca Bianchini as the originator of the matrix-based "prototype," including in the appendix the documentation of the private correspondence in which the theoretical foundations of the system based on Pythagorean numbers are discussed.

The validity of the system was therefore empirically demonstrated well before its present algebraic formalization: the practical application of the method made it possible to realize canons of up to 28 real voices,[2] a result impossible to achieve within a short time frame using traditional contrapuntal techniques, thereby demonstrating the computational efficiency of the matrix-based approach.

[1] C. E. R. Morsink, *The Composition of New Music Inspired by Music Philosophy and Musical Theoretical Writings from Ancient Greece*, PhD Diss., Goldsmiths, University of London, 2013.

[2] The incipit of this canon, composed by Coreen Morsink on the basis of the author's matrix theories, is reproduced on page 101 of her doctoral dissertation: Coreen Emmie Rose Morsink, Goldsmiths, University of London, PhD in Music, 2013.

1.5 The Ontological Thesis

If a single numerical structure (the **Abstract Seed**) already contains within itself melody (horizontal axis), harmony (vertical axis), and imitation (diagonal projection), a strong ontological thesis follows: polyphony is not a historical elaboration added to sound, but an intrinsic property of the numerical system itself. The task, therefore, is not to construct polyphony, but to activate it operationally through specific operators (the Keys).

1.6 Objectives of the Article

In the following pages, we translate this compositional theory into a rigorous mathematical formalism based on matrix algebra and modular arithmetic. The objective is threefold:

- To demonstrate mathematically the **Paradox of Complexity**: to show how, in a saturated system (when $n \to 8$), composition becomes paradoxically simpler and more deterministic than the management of a small number of voices.
- To define the canon not as a difficult musical form, but as a particular and automatic case of vector addition.
- To propose a shift in pedagogical paradigm: from trial-based assembly to controlled algorithmic generation.

This article does not aim to propose a new compositional style, but rather a formal generative model capable of reproducing and extending historical contrapuntal practices. The work is published as a preprint with the aim of establishing a complete and formalized theoretical model; the historical, pedagogical, and compositional implications of the system will be developed separately in a more accessible form.

2 Theoretical Foundations: The Abstract Seed

2.1 The Horizontal Dimension of Melody

In our model, the numerical sequence represented by the numbers **1**, **3**, and **5** defines, in a horizontal sense, a melodic sequence consisting of three distinct sounds. We assign to this succession the name **Abstract Seed**, since it does not merely describe a melody, but functions as a generating structure for the entire polyphonic architecture. As in nature, this Abstract Seed contains *in potentia* everything required to construct the musical complexity that will be examined below.

This notation is positional and relative, in that the numbers indicate that each note lies at a distance of a third from the preceding one.

The sequence 1–3–5 can in fact embody several concrete realizations:

- the melody *C–E–G*
- the melody *F–A–C*
- the melody *B–D–F*

At this level of abstraction, the specific quality of the interval (whether the third is major or minor) is irrelevant, because what the system encodes is the structural, or diatonic, distance between one sonic event and the next.

1	**3**	**5**

Table 1: Horizontal representation of the Abstract Seed.

2.2 The Vertical Dimension of Harmony

When arranged vertically within a column of the table, the same numbers 1–3–5 cease to represent a temporal succession and instead define a sonic simultaneity.

In this configuration:

- the number **1** represents the fundamental tone (bass);
- the number **3** indicates the third above the bass;
- the number **5** indicates the fifth.

The resulting object is the triad.[3]

5
3
1

Table 2: Vertical representation (chord).

[3]At this descriptive stage, we adopt a tabular representation in order to highlight the spatial disposition of intervals. In Section 4, these structures will be formalized as vectors and matrices to which algebraic transformers are applied.

2.3 The Diagonal Dimension: The Canon

In diagonal projection, by repeating the same numbers 1–3–5 with a positional offset, the system generates the first canon at the unison.

1	**3**	**5**		
	1	**3**	**5**	
		1	**3**	**5**

Table 3: Structure of the finite canon at the unison.

By contrast, when the same numbers 1–3–5 are repeated with a positional offset in a cyclical manner—either at the unison or transposed by the octave—the system generates the first infinite canon. By convention, the lowest row is placed one octave below.

1	**3**	**5**	**1**	**3**	**5**	**1**	**3**	**5**	**1**	**3**	...
	1	**3**	**5**	**1**	**3**	**5**	**1**	**3**	**5**	**1**	...
		1	**3**	**5**	**1**	**3**	**5**	**1**	**3**	**5**	...

Table 4: Development of the infinite canon at the unison.

2.4 Why the Numbers Cannot Be Random

If we wish to expand the canon to **four voices**, we must add a fourth number to the seed. Let us try doubling the number 1 at the beginning:

$$1 - 1 - 3 - 5$$

Musically, this means that each voice will sing the two notes 1 (C) one after the other, followed by 3 (E) and 5 (G), repeating the sequence indefinitely. Up to this point, everything functions correctly.

1	**1**	**3**	**5**	**1**	**1**	**3**	**5**	**1**	**1**	**3**	...
	1	**1**	**3**	**5**	**1**	**1**	**3**	**5**	**1**	**1**	...
		1	**1**	**3**	**5**	**1**	**1**	**3**	**5**	**1**	...
			1	**1**	**3**	**5**	**1**	**1**	**3**	**5**	...

Table 5: Development of the infinite canon at the unison with the seed 1135.

Matrice sonora

di un canone a 4 voci

Luca Bianchini

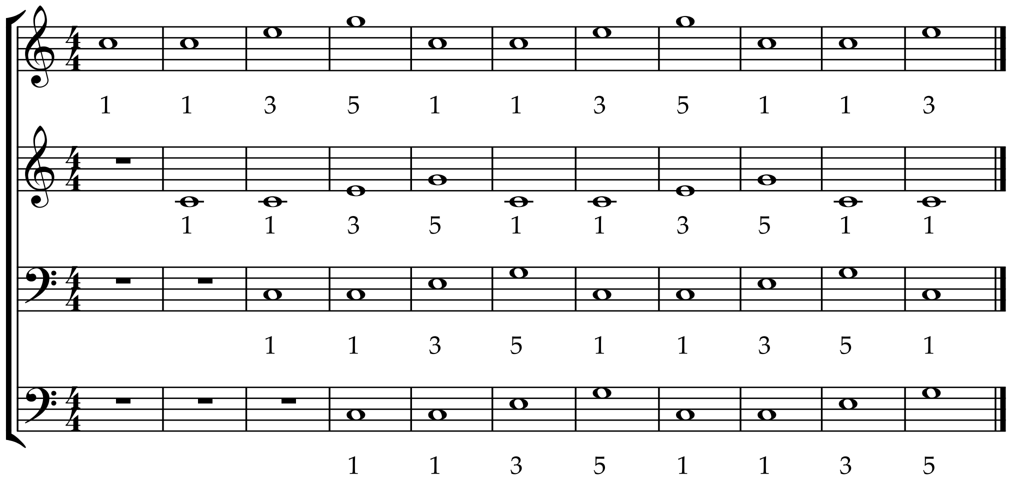

Figure 1: Canon at the octave, generated from the **Abstract Seed** 1–1–3–5.

2.5 Parallel Octaves and Fifths

Traditional treatises prohibit parallel octaves and fifths for any number of voices.[4] In the present article, this prohibition is deliberately adopted as the sole general constraint.

In order to keep the model as simple and formalizable as possible, other constraints are not taken into consideration—such as direct octaves and fifths, or intervallic harshnesses like the tritone—which depend on local factors of melodic voice-leading and do not affect the combinatorial structure of the system.

In accordance with the rules of the early contrapuntal tradition as codified in the treatises, parallel octaves and fifths are here considered errors in a technical sense, that is, violations of a system's structural constraint, independently of stylistic, aesthetic, or historically subsequent evaluations.

When, in our examples, we attempt to increase the number of voices to five or more, a critical problem arises. If we wish to avoid errors of parallel fifths and octaves, we cannot add numbers arbitrarily. Let us consider, for example, the sequence

$$1 - 1 - 3 - 1 - 1$$

for five voices. At first glance, this might appear to be an acceptable solution; however, when translated into music and analyzed—examining, for instance, what occurs between Voice 1 (V1) and Voice 2 (V2) as the canon unfolds—we observe that:

- the sequence contains the pair $1 \to 1$ at the beginning;
- the sequence again contains the pair $1 \to 1$ at the end.

When the canon overlaps, we inevitably encounter a point at which two different voices simultaneously perform the same melodic motion ($1 \to 1$), starting from the same pitch, thereby producing **consecutive unisons** or **parallel octaves**.

1	**1**	**3**	**1**	**1**	**1**	**1**	**3**	**1**	**1**	**1**	...
	1	**1**	**3**	**1**	**1**	**1**	**1**	**3**	**1**	**1**	...
		1	**1**	**3**	**1**	**1**	**1**	**1**	**3**	**1**	...
			1	**1**	**3**	**1**	**1**	**1**	**1**	**3**	...

Table 6: Development of the canon. Cells with a red background indicate scholastically forbidden parallel octaves.

The same problem arises with fifths. Let us imagine a seed containing:

$$1 - 1 - 5 - 5$$

Here, the pairs $1 \to 1$ and $5 \to 5$ are repeated, which forces two voices to move in parallel at the distance of a fifth (C–G and C–G), generating—if the fifth is above—**parallel fifths**.

1	**1**	**5**	**5**	**1**	**1**	**5**	**5**	**1**	**1**	**5**	...
	1	**1**	**5**	**5**	**1**	**1**	**5**	**5**	**1**	**1**	...
		1	**1**	**5**	**5**	**1**	**1**	**5**	**5**	**1**	...
			1	**1**	**5**	**5**	**1**	**1**	**5**	**5**	...

Table 7: Development of the canon. Cells with a red background indicate forbidden parallel fifths.

[4] Cf. L. Cherubini, *Cours de contrepoint*, p. 4.

To avoid parallel fifths, it is therefore necessary to ensure that the pairs $1 \to 1$ and $5 \to 5$ are not used within the same seed.

2.6 The Law of Unique Pairs

To avoid errors (parallelisms) of unisons and octaves, it is not sufficient to examine individual notes; one must instead consider **melodic intervals** (that is, pairs of adjacent numbers).

With the numbers $1, 3, 5$, the possible pairs are exactly nine:

- Movements from 1: $1 \to 1, 1 \to 3, 1 \to 5$
- Movements from 3: $3 \to 1, 3 \to 3, 3 \to 5$
- Movements from 5: $5 \to 1, 5 \to 3, 5 \to 5$

In order to write a correct eight-voice canon, we must construct a chain of eight numbers (thus seven transitions) using these pairs **without ever repeating any of them**. Only by guaranteeing the uniqueness of each pair (e.g., if $1 \to 3$ has been used, it cannot be used again) can we be mathematically certain that the staff will never contain two voices moving in parallel in the same way. Let us try using the seed 11315335:

1	**1**	**3**	**1**	**5**	**3**	**3**	**5**	**1**	**1**	**3**	...
	1	**1**	**3**	**1**	**5**	**3**	**3**	**5**	**1**	**1**	...
		1	**1**	**3**	**1**	**5**	**3**	**3**	**5**	**1**	...
			1	**1**	**3**	**1**	**5**	**3**	**3**	**5**	...
				1	**1**	**3**	**1**	**5**	**3**	**3**	...
					1	**1**	**3**	**1**	**5**	**3**	...
						1	**1**	**3**	**1**	**5**	...
							1	**1**	**3**	**1**	...

Table 8: Development of the infinite canon at the unison or at the octave with the seed 11315335.

The eight-voice system is rigid, because we have only nine "building blocks" (pairs) available and must use seven of them all at once, fitting them together perfectly like a puzzle.

The sequence shown above, $1 \to 1 \to 3 \to 1 \to 5 \to 3 \to 3 \to 5$, for example, is valid for an infinite eight-voice canon at the unison and at the octave, as shown in Figure 2.

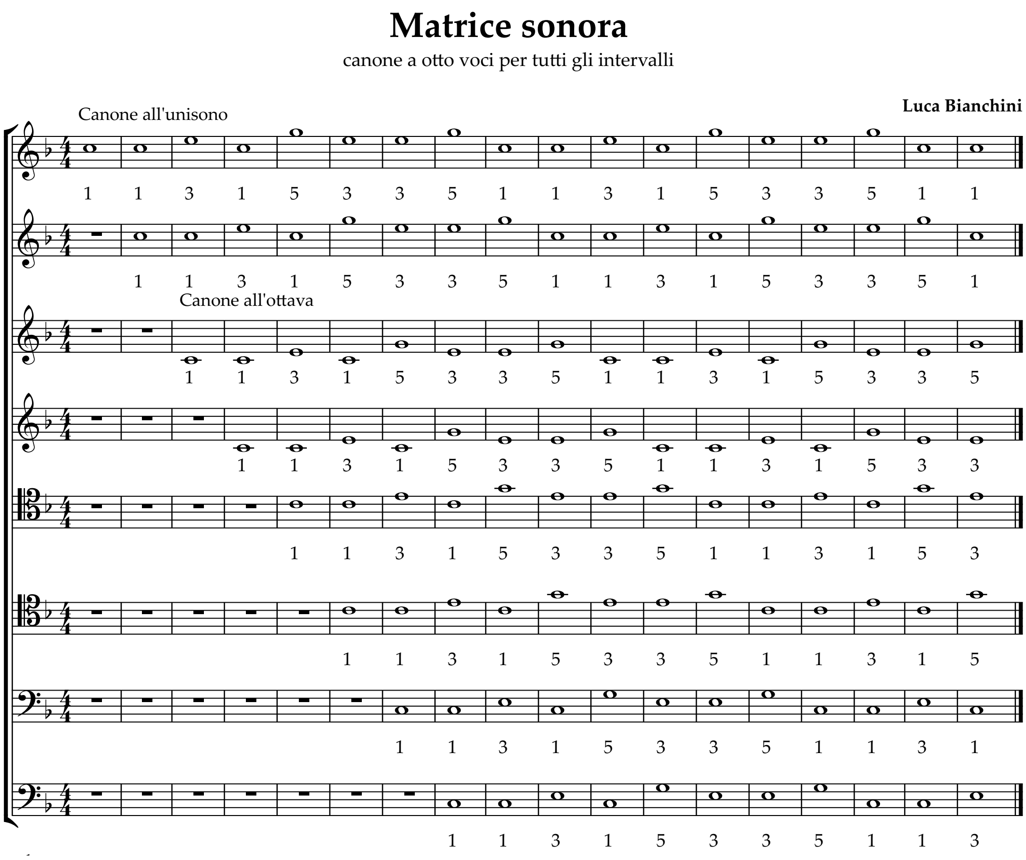

Figure 2: Canon at the octave, generated from the **Abstract Seed** 1–1–3–1–5–3–3–5.

2.7 The Calculation of Permutations

How many different columns (Seeds) can we construct using a chain of seven numbers chosen from 1, 3, or 5? This is not an infinite calculation. By using the formula for **permutations with repetition** (multiset permutations), we can define the space of possibilities exactly.[5]

Given a set of $n = 8$ elements, with groups of identical elements $n_1 = 3$ (the 1s), $n_2 = 3$ (the 3s), and $n_3 = 2$ (the 5s), the number of distinct permutations P is:

$$P = \frac{n!}{n_1! \cdot n_2! \cdot n_3!} = \frac{8!}{3! \cdot 3! \cdot 2!}$$

Expanding the factorial:

$$P = \frac{40{,}320}{6 \cdot 6 \cdot 2} = \frac{40{,}320}{72} = \mathbf{560}$$

There are therefore exactly **560** unique ways to arrange the voices vertically while respecting ideal saturation. This constitutes our "domain of existence."

2.8 The Rule of Adjacent Pairs

How does one concretely construct a valid seed? The golden rule is that each pair of consecutive numbers (n_i, n_{i+1}) generates a potential harmonic motion. If an identical pair is repeated at a short distance, errors of parallelism (parallel octaves or fifths) inevitably arise when the voices overlap.

Let us analyze the growth of complexity:

- **2 Voices:** A sequence $1 \to 1$ generates the pair $(1, 1)$. This is sufficient for a two-voice beginning.
- **3 Voices (Error):** If we extend the sequence to $1 \to 1 \to 1$, we obtain two identical consecutive pairs: $(1, 1)$ and $(1, 1)$. This immediately produces parallel octaves. The canon fails.
- **3 Voices (Correct):** If instead we write $1 \to 1 \to 3$, we obtain the pairs $(1, 1)$ and $(1, 3)$. Being different, they allow the entrance of a third voice without collisions.
- **4 Voices:** Since the sequence $1 \to 1 \to 3 \to 1$ generates the pairs $(1, 1)$, $(1, 3)$, and $(3, 1)$, all of which are unique, we can superimpose three additional voices above the base voice without errors.

The more voices we wish to add, the longer the chain of unique pairs must be. If the sequence were $1 \to 1 \to 3 \to 1 \to 1$, the final pair $(1, 1)$ would duplicate the first one, preventing the clean entrance of a fifth voice.

2.9 The Perfect Seed for Eight Voices

In order to handle a canon with eight real voices, the available margin of maneuver narrows drastically. We must find a sequence of eight numbers (chosen from $1, 3, 5$) that generates seven internal pairs, all different from one another, and that reconnects with its beginning without creating duplications, thereby allowing an infinite canon.

The sequence examined above satisfies this condition:

$$S_8 = \{1, 1, 3, 1, 5, 3, 3, 5\}$$

Let us verify the pairs generated:

1. $1 \to 1$ (Pair 1–1)

[5]A permutation is a way of arranging all the elements of a set by changing their order without excluding any of them. If there are n distinct elements, a permutation is any possible sequence of those n elements.

2. $1 \to 3$ (Pair 1–3)
3. $3 \to 1$ (Pair 3–1)
4. $1 \to 5$ (Pair 1–5)
5. $5 \to 3$ (Pair 5–3)
6. $3 \to 3$ (Pair 3–3)
7. $3 \to 5$ (Pair 3–5)

In order to make the canon **infinite** (circular), the last number (5) must reconnect with the first (1). The connecting pair is $(5, 1)$, which is unique and has never appeared before. The circle thus closes perfectly, as shown in Figure 2.

2.9.1 The Fragility of the Equilibrium

A single error is enough to cause the entire architecture to collapse. If the previous sequence were to end with 1:

$$1 - 1 - 3 - 1 - 5 - 3 - 5 - \mathbf{1}$$

then, at the moment of looping ("da capo"), we would obtain the succession $5 \to 1 \to 1$, with the pair $(1, 1)$ appearing immediately after the pair $(5, 1)$. However, the pair $(1, 1)$ was already present at the beginning, and the result would be:

$$\ldots\ 5 \to 1 \to \mathbf{1} \to 1 \to 3\ \ldots$$

Three consecutive 1s generate parallel octaves. The system would thus be mathematically blocked, preventing infinite circularity.

2.10 The Statistical Rarity of the Seed (The Needle in the Haystack)

Here the true nature of the historical difficulty becomes clear. We have seen that there are theoretically 560 possible permutations of the numbers $\{1, 1, 1, 3, 3, 3, 5, 5\}$. However, only very few of them satisfy the "Rule of Unique Pairs" (that is, without adjacent triple 1s or triple 3s, and without repeated pairs in infinite cycles).

Attempting to find such a sequence "by ear" or through trial and error, by writing notes on the staff, is equivalent to searching for a needle in a haystack of thousands of failed combinations. The matrix, by contrast, delivers the needle already formed.

2.11 Census of Valid Cycles and the Coverage Theorem

Combinatorial analysis, filtered through the system's consonance rules (in particular, the prohibition of forbidden parallelisms), reduces the space of possibilities to a finite set of "Seeds" (generating cycles). Below we list the admissible sequences for each number of voices N of the infinite canon (from 2 to 8). Note that sequences read in reverse order (e.g., 13 and 31) are equivalent and have therefore been omitted in order to avoid unnecessarily lengthening the table. The first column indicates the number of voices, the second the number of valid seeds, and the third the list of seeds.

Voices (N)	Qty.	Valid Seeds (Canonical Representatives)
2	3	`13, 15, 35`
3	8	`113, 115, 133, 135, 153, 155, 335, 355`
4	11	`1133, 1135, 1153, 1315, 1335, 1353, 1355, 1533, 1535, 1553, 3355`
5	16	`11315, 11335, 11353, 11513, 11533, 11535, 13155, 13315, 13353, 13355, 13533, 13553, 15335, 15355, 15533, 15535`
6	13	`113315, 113353, 113533, 115133, 115335, 131535, 133155, 133553, 135153, 135315, 135533, 153355, 155335`
7	18	`1131535, 1135153, 1135315, 1151353, 1153135, 1153513, 1315335, 1315355, 1315535, 1331535, 1335153, 1335315, 1351533, 1351553, 1353155, 1353315, 1355153, 1355315`
8	24	`11315335, 11331535, 11335153, 11335315, 11351533, 11353315, 11513353, 11513533, 11531335, 11533135, 11533513, 11535133, 13153355, 13155335, 13315355, 13315535, 13351553, 13353155, 13355153, 13355315, 13515533, 13533155, 13551533, 13553315`

Table 9: Catalogue of canonical Seeds by system dimension (93 valid Seeds in total).

It should also be noted that we have excluded seeds that can generate fifths. For example, in the four-voice case, the theoretically possible seed `1155` is absent. Without the specific rule preventing the coexistence of the bigrams $(1, 1)$ and $(5, 5)$, this seed would be mathematically generable; however, it has been discarded in order to avoid the two consecutive fifths that would necessarily arise when the two 1s are in the lower parts.

2.11.1 Principle of Downward Validity

The validation of the eight-voice system reveals a fundamental property of backward robustness.

Theorem: If a Seed S generates a valid eight-voice canon, then S necessarily generates valid canons for any number of voices $k < 8$.

Proof: Since an eight-voice Seed is valid, its bigrams are unique by definition. Consequently, the equality $\text{bigram}[p] = \text{bigram}[p + d]$ is impossible for any $d \neq 0$. As coincidences of motion cannot occur at any distance d, forbidden parallelisms cannot arise in any configuration with a smaller number of voices. The system is therefore *downward compatible*: what withstands maximal complexity is intrinsically safe for any lower degree of polyphonic density. By contrast, a seed that is valid for $N = 5$ does not guarantee validity for $N = 8$, since its bigrams are not constrained to total uniqueness.

2.12 The Statistical Inevitability of Error

This calculation allows us to understand scientifically why an empirical approach is either destined to fail or requires prohibitively long periods of time.[6]

A composer working without matrices does not operate within these **93** "safe" seeds, but rather navigates an open combinatorial space in which each voice can theoretically assume any of three notes ($3^8 = 6{,}561$ combinations, excluding rests); however, only these **93** balanced configurations guarantee optimal fluency.

Contrapuntal error (parallel octaves or fifths, or dead ends) arises when:

[6] Giancarlo Bizzi explains how two-voice canons, including the augmented canon, can be derived through trial and error. The resulting path, however, is long and arduous. See G. Bizzi, *Specchio invisibile. I canoni enigmatici di J. S. Bach*, Rome, Kappa, 1982.

1. an unbalanced vertical configuration is chosen (e.g., four 1s);
2. two valid columns are connected through motions not permitted by the vector matrix.

Without a pre-determination of the structure, the probability that a composer, working "by ear," will select a sequence of eight consecutive columns that corresponds exactly to one of the mere **24** valid cyclic configurations of an eight-voice canon (out of an infinity of possible combinations) is statistically negligible, within the assumed constraints. This is why our system inverts the paradigm: instead of searching for the needle in the haystack (attempting to correct errors *a posteriori*), we provide in advance the complete set of needles we require.

3 System Activation: Keys and Reductions

3.1 Beyond the Utopia of the Single Table

Historically, many attempts at compositional automation—such as the celebrated *Tabula Mirifica* or musical dice games (*Musikalisches Würfelspiel*)—have been based on the idea that a single permutation matrix could contain all possible solutions. However, considering a single table as a universal problem-solver is a combinatorial utopia, since music is a complex system that operates across multiple simultaneous dimensions (pitch, duration, intensity, timbre).[7]

Using a single table to manage this hyperspace inevitably leads to a trial-and-error approach. The composer is forced to manually discard hundreds of combinations generated by the table that turn out to be incoherent or forbidden by the rules of counterpoint. Searching for the correct solution in this way requires extremely long time spans, rendering the system inefficient precisely where it promises speed.

To bypass this "bottleneck," it is necessary to separate the parameters into distinct and superimposable matrices:

- The **Sound Matrix** (M_S) rigorously determines the intervallic texture, guaranteeing harmonic validity.
- The **Key Matrix** (M_K) transforms actual pitch, providing the melodic profile.
- The **Prolation Matrix** (M_T) establishes rhythmic duration.

There is no magic table that does everything, but rather a **system of matrices** which, through their interaction, uniquely determines the result without the need for *a posteriori* corrections. Complexity is not resolved by simplifying the tool into an all-purpose table, but by organizing the structure into multiple specialized matrices.

3.2 Keys as Transformers

The canons described above possess internal intervallic coherence, but they do not yet "sound," because they lack the **Keys** on the staff that assign a concrete name to the notes (pitch). To determine which key to apply in each case, we use the **Key Matrix** derived from the Pythagorean Table. This corresponds exactly to the elementary multiplication table, which here assumes the role of a database of transformers.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Table 10: The Key Matrix (the core of the Pythagorean Table).

[7] M. Nicolella and D. Celletti, *I segreti dell'Harmonia. Comporre canoni musicali con la Tabula mirifica*, Vatican City, Libreria Editrice Vaticana, 2021.

3.3 Reductions and Modular Arithmetic (0–6)

The Pythagorean Table generates mathematical progressions that grow rapidly toward infinity. In order to adapt these values for immediate reading on the staff, we adopt modular arithmetic modulo 7. In this context, the number 7 and its multiples (14, 21, ...) are reduced to **0**.

3.3.1 The Dual Nature of Zero

It is essential to distinguish the meaning of zero according to the context in which it operates:

- **In the Sound Matrix** (M_S): 0 indicates the absence of an event, that is, a **rest**.
- **In the Key Matrix** (M_K): 0 is an active number indicating the **Reference Key** (or identity).

If we conventionally establish that Key 0 corresponds to the **Key of C**, then:

- $0 =$ Key of C (reference);
- $1 =$ Key of D (second);
- ...
- $7 \equiv 0 =$ Key of C at the upper octave (return to identity).

The system is purely relational: if the composer were to decide that 0 corresponds to the **Key of A** (at 441 Hz), Key 1 would automatically become the **Key of B**, one step above that A. The internal relationships with all other numbers would of course remain unchanged.

3.3.2 The Reduction Formula

The formula for computing the reduced value x_{red} is simply the remainder of the division by 7:

$$x_{\text{red}} = x \pmod 7 \tag{1}$$

This operation removes any excess and always returns a number between 0 and 6.

Examples of immediate calculation:

- 7 becomes **0** (the octave acts as the unison/identity);
- 8 becomes **1** (a compound second $\rightarrow$ simple second);
- 14 becomes **0** (two octaves $\rightarrow$ identity);
- 24 becomes **3** ($24 = 21 + 3$, leaving a fourth).

Applying this logic, the infinite Pythagorean Table crystallizes into a **magic square** bordered by zeros, where the last row and the last column cancel motion, representing octave stability. As stated above, 0 indicates by convention the Key of C (the neutral element). Excluding the zeros, one can observe that the first row at the top, read from left to right, is identical to the first column read from top to bottom, and that the second top row corresponds to the second column, and so forth. Along the diagonals, there are also intriguing inverted sequences, depending on the point of observation.

1	2	3	4	5	6	**0**
2	4	6	1	3	5	**0**
3	6	2	5	1	4	**0**
4	1	5	2	6	3	**0**
5	3	1	6	4	2	**0**
6	5	4	3	2	1	**0**
0	**0**	**0**	**0**	**0**	**0**	**0**

Table 11: The Modular Pythagorean Table.

4 Matrix Algebra

4.1 The Principle of Superposition

Once the reduction rules have been defined, we can observe the system in action. The generative mechanism consists of a simple matrix addition of two components.[8]

- The **Sound Matrix** (M_S), which contains the **Abstract Seed** (1–3–5) arranged vertically, horizontally, or diagonally.
- The **Key Matrix** (M_K), derived from a row of the Pythagorean Table.

The operational rule is straightforward: if a cell of the Sound Matrix contains a number, that number represents a note and is added to the corresponding value of the Key Matrix. If the Sound Matrix contains a zero, that is, a rest, the result remains zero.

4.2 Canon at the Upper Second

To obtain, for example, a canon in which the voices imitate each other at the upper second, we add to the Sound Matrix the Pythagorean progression derived from the Key Matrix, that is, from the row of 1 in the Pythagorean multiplication table $(0, 1, 2, 3, 4, \dots)$.

$$\underbrace{\begin{bmatrix} 1 & 3 & 5 & 0 & 0 \\ 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix}}_{M_S} + \underbrace{\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}}_{M_{K1}} = \underbrace{\begin{bmatrix} 1 & 4 & 7 & 0 & 0 \\ 0 & 2 & 5 & 8 & 0 \\ 0 & 0 & 3 & 6 & 9 \end{bmatrix}}_{\text{Result (Canon)}}$$

Musical interpretation:

- Voice 1 (Row 1): $1 \to 4 \to 7$ (e.g. C–F–B), followed by two rests (0, 0).
- Voice 2 (Row 2): Enters after one bar. The original **abstract seed** 1–3–5 becomes 2–5–8 (D–G–C, octave), followed by a rest (0).
- Voice 3 (Row 3): Enters after two bars. The seed becomes 3–6–9 (E–A–D, octave).

The voices thus pursue one another regularly at the upper second. The linearity of the operation guarantees that the internal intervallic structure (the triad 1–3–5) is preserved intact, preventing harmonic errors by construction.

4.3 Canon at the Third and Beyond

By applying the row of the Pythagorean multiplication table for 2 $(0, 2, 4, 6, \dots)$, the system generates a canon at the third.

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 9 \\ 0 & 3 & 7 \\ 0 & 0 & 5 \end{bmatrix}$$

The principle is universal: with the progression of 3 one obtains a canon at the fourth, with that of 4 a canon at the fifth, and so on ad infinitum. In short, to obtain a canon at an interval n, the modular Pythagorean row to be applied always corresponds to the multiplication table of $n - 1$.

[8] From this point onward, the tables previously introduced will be formalized as **matrices**. This change in notation is not merely stylistic, but functional, as it allows us to apply the operators of linear algebra (addition, transposition, scalar multiplication) directly to musical structures, transforming composition into an exact calculation.

4.4 Canon at the Octave

By adding the progression of 7 (which is equivalent to 0, or harmonic identity), one instead obtains the canon at the octave.

$$M_S + M_{K7} \to \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{(since } 1+7 \equiv 1\text{)}$$

The voices sing the same notes (C–E–G), offset in time.

4.5 Canons at Lower Intervals

To obtain descending imitations—for example, a canon at the lower second—it is not necessary to subtract, since it suffices to add the corresponding row of the Key Matrix read in reverse order. The position occupied by 1 (canon at the upper second), when the row of the multiplication table is read backwards, is occupied by 6 (canon at the seventh). Mathematically, a canon at the lower second is therefore equivalent to a canon at the upper seventh transposed down by one octave.

1	**2**	**3**	**4**	**5**	**6**	**0**
2	**4**	**6**	**1**	**3**	**5**	**0**
3	6	2	5	1	4	**0**
4	1	5	2	6	3	**0**
5	3	1	6	4	2	**0**
6	**5**	**4**	**3**	**2**	**1**	**0**
0	**0**	**0**	**0**	**0**	**0**	**0**

Table 12: The Modular Pythagorean Table.

4.6 Infinite Canons

Within the double-matrix system, infinite canons are easier to realize than finite ones, since no special cadence is required to conclude them. To construct an infinite canon, it is sufficient to repeat cyclically the numbers of the **Sound Matrix** and to apply to each column (that is, to each temporal unit or bar) a specific key from the **Key Matrix**.

The key acts vertically on all voices sounding at a given moment: if the key changes, it changes for all voices simultaneously, preserving vertical harmonic integrity.

To obtain an infinite canon in which the voices pursue one another at the **lower second**, we combine the cyclic Sound Matrix with the **descending row** of keys $(0, 6, 5, 4, \dots)$. At the second bar, the key descends to 6: this lowers the pitch of the first voice (which continues) and at the same time determines the entry pitch of the second voice, thereby guaranteeing the correct interval of imitation.

$$\underbrace{\begin{bmatrix} 1 & 3 & 5 & 1 & 3 & 5 & \dots \\ \cdot & 1 & 3 & 5 & 1 & 3 & \dots \\ \cdot & \cdot & 1 & 3 & 5 & 1 & \dots \end{bmatrix}}_{M_S \text{ (Cyclic)}} + \underbrace{\begin{bmatrix} 0 & 6 & 5 & 4 & 3 & 2 & \dots \\ 0 & 6 & 5 & 4 & 3 & 2 & \dots \\ 0 & 6 & 5 & 4 & 3 & 2 & \dots \end{bmatrix}}_{M_K \text{ (Vertical)}}$$

$$= \underbrace{\begin{bmatrix} 1+0 & 3+6 & 5+5 & 1+4 & 3+3 & 5+2 & \dots \\ \cdot & 1+6 & 3+5 & 5+4 & 1+3 & 3+2 & \dots \\ \cdot & \cdot & 1+5 & 3+4 & 5+3 & 1+2 & \dots \end{bmatrix}}_{\text{Result (Column-by-Column Sum)}}$$

Analysis of the result: Each column represents a chord transposed by the current key.

- **Time 1 (Key 0):** Only the first voice (V1) sounds, with note 1.
- **Time 2 (Key 6):** V1 sounds $3+6=9$ ($\equiv 2$). The second voice (V2) enters with $1+6=7$ ($\equiv 0$). The interval between the entry of V1 (1) and the entry of V2 (0) is a descending second.
- **Time 3 (Key 5):** V1 sounds $5+5=10$ ($\equiv 3$). V2 sounds $3+5=8$ ($\equiv 1$). V3 enters with $1+5=6$.

The system thus automatically generates a correct infinite descending spiral, without requiring the composer to calculate individual intervals.

In Figure 3, the upper system shows the matrix of a canon at the unison and the octave for eight voices, while the lower system presents the same canon after the application of the Key Matrix (row 2 of the modular Pythagorean multiplication table). The staggered voices can be observed responding to the first at each interval of the scale.

Matrice sonora

canone a otto voci per tutti gli intervalli

Luca Bianchini

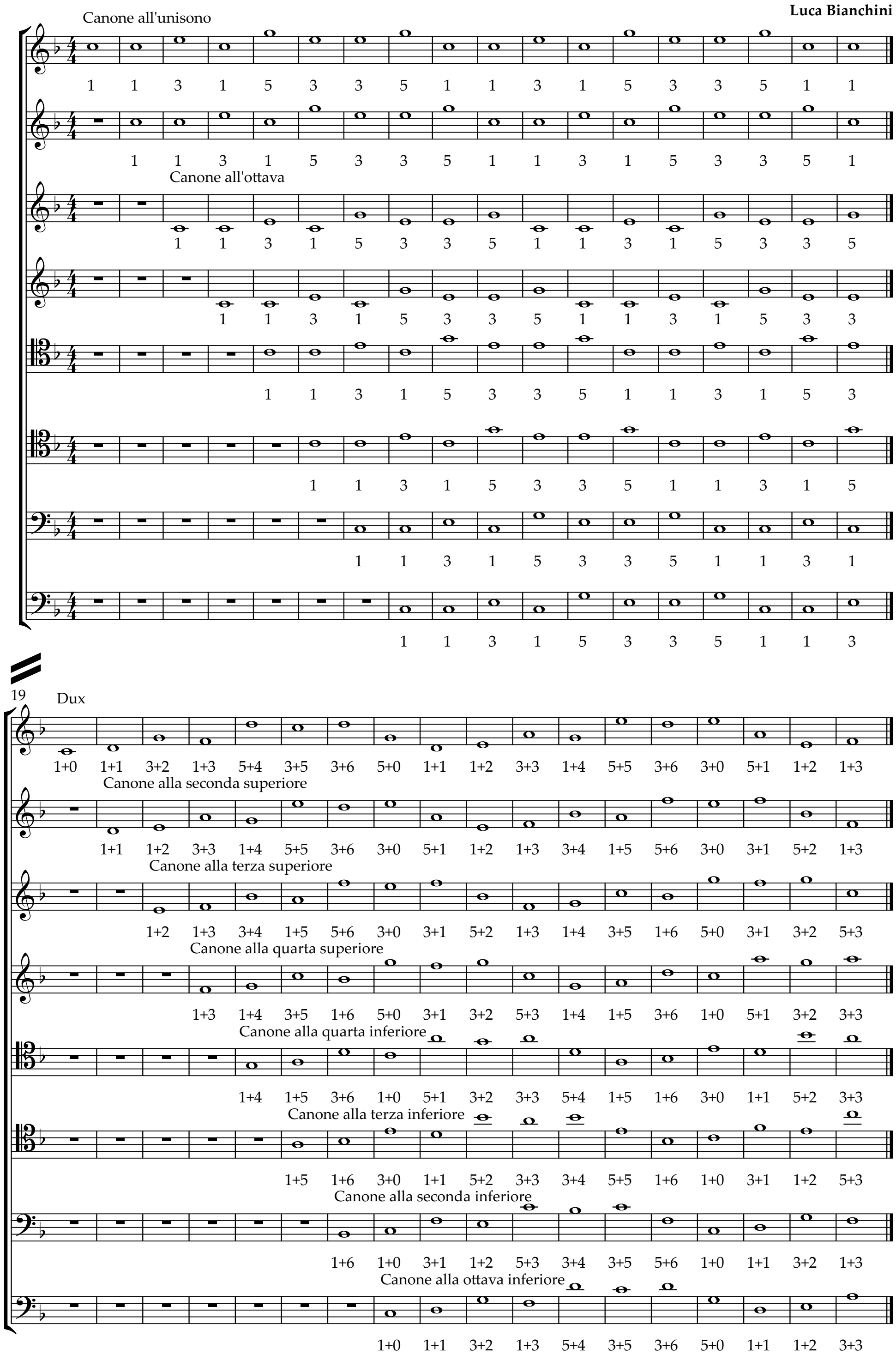

Figure 3: Starting from an **Abstract Seed** in the upper system of staves, from bar 1 to 18, the simple application of additive **Keys** (+1, +2, . . .) instantaneously generates an eight-voice canon at every interval of the scale (second, third, fourth, etc.), both ascending and descending, from bar 19 to the end. What traditional pedagogy considers "complex" is here reduced to a simple series of transformations.

5 The Structure of the Seed: Harmonic Constraints

A possible misunderstanding must be corrected at once. Although the matrix is a mathematical container, numbers cannot be inserted arbitrarily. The numbers of the seed represent the **structural pillars** (the framework) of the composition. In order for the canon to remain harmonically stable and free of errors, these pillars must obey strict rules of consonance.

5.1 The Rule of Structural Consonance

By analyzing the texture of the seed before applying the Key Matrix, we observe that there are no leaps of sevenths or fourths, nor stepwise motions (seconds) between the structural nodes. The admissible motions, replicable at the upper or lower octave, are exclusively:

- **Third motions** (ascending or descending);
- **Fifth motions** (ascending or descending);
- **Octave motions** (or unison).

5.2 Hierarchical Inversion

Contrary to intuition, after the initial entry of the **Dux** the hierarchical roles are reversed: it is the **Comes** that dictates the rules to the **Dux**.[9]

The Dux states a number, and the Comes decides how to respond. From that point onward, every decision is again determined by the Comes, and the Dux must adjust by placing its note above or below that of the Comes, in accordance with the criteria of consonance.

1. The Dux sings a note (e.g., 1).
2. At the following bar, the Comes performs that same note 1 at the pitch it chooses. Regardless of its position (whether in the bass or the upper voice), this pitch becomes an unavoidable **harmonic constraint**.
3. The Dux, which must now sing a new note, must compute it so as to remain consonant with the note just imposed by the Comes.

This is why stepwise degrees are prohibited in the framework. If the Dux were to move by a second (e.g., $1 \to 2$), at the moment it reaches 2 the Comes would be sounding 1. This would create a vertical clash of a second (1 against 2), which the system cannot resolve on its own, regardless of whether the 1 lies below or above the 2. In effect, the Dux is bound in constrained counterpoint to the Comes.

5.3 Load-Bearing Structure and Decorative Elements

If the numbers of the Sound Matrix are constrained to be consonant $(1, 3, 5)$, where does melodic complexity reside? It resides in the distinction between **load-bearing elements** and **decorative elements**.

- **The pillars (Sound Matrix):** the numbers 1–3–5 written in the matrix. They guarantee that the structure remains stable.
- **The decorative elements (Diminutions):** everything that occurs *between* one number and the next.

---

[9] According to tradition, the *dux* is the voice that begins the canon, states the theme, and leads the process. The *comes* is the following voice, which enters later.

The structural transition $1 \to 3$ in the matrix is a framework element. The composer may clothe it by inserting passing tones, suspensions, or figurations (e.g., $1 \to 2 \to 3$), or even additional bars and phrases. The golden rule of the system is that **the Key acts upon the load-bearing structure and everything built upon it**: when the Key Matrix transforms a structural number (for instance, when 1 becomes 2), it coherently transposes the entire melodic "ornament" that lies between that pillar and the next.

5.4 Example of an Extended Seed

To demonstrate the mechanics of the load-bearing structure and decorative elements—and the robustness of the system beyond the 135 triad—we take as a basis a seed that serves as the framework for a wide and jagged melody, touching distant scale degrees:

$$S_{\text{pillars}} = [1, 3, 1, 6, 4, 2, 7, 5, 3]$$

The rule of admissible motions is respected: $1 \to 3$ is an ascending third, $3 \to 1$ a descending third, $1 \to 6$ a descending third, and so forth. These numbers represent the melodic pillars. From this seed we can build a canon matrix and apply the row of the multiplication table for 1 $(0, 1, 2, \dots)$ from our Pythagorean Table. In this way we obtain a canon at the second. Arithmetic guarantees that if a pillar n becomes $n + 1$, the entire melodic "ornament" that follows it is coherently transposed as well.

$$\begin{array}{lccccccccc}
 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 & t_9 \\
\text{V1} & 1 & 3 & 1 & 6 & 4 & 2 & 7 & 5 & 3 \\
\text{V2} & \cdot & 1 & 3 & 1 & 6 & 4 & 2 & 7 & 5 \\
 & & & & + & & & & & \\
\text{Keys} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 \\
 & & & & = & & & & & \\
\text{Res. V1} & 1 & 4 & 3 & 2 & 1 & 7 & 6 & 5 & 4 \\
\text{Res. V2} & \cdot & 2 & 5 & 4 & 3 & 2 & 1 & 7 & 6
\end{array}$$

Despite the apparent "angularity" of the chosen numbers (note the presence of 2 and 7, or 6 and 4), the texture is always anchored to the vertical 135 chord. As shown in Figure 4, the additive mechanism preserves perfect imitation without requiring the composer to calculate interval species.

Figure 4: Analytical comparison for two voices. Top: canon framework at the octave generated by the matrix system, which guarantees the absence of parallelisms by construction. Middle: realization with the addition of the "skin" (ornaments). Bottom: canon at the upper second obtained from the previous one by applying the row 1 of the Pythagorean key table. Note how the ornaments follow the pillars in their displacement. Between one pillar and the next, the space either expands or contracts depending on the case; consequently, the ornaments—which previously filled a given interval—must be adjusted by the composer to accommodate the new distances.

6 Comparison with Cherubini

6.1 The Reduction of the Space of Choices

In general, even eight-voice counterpoint, when treated through the matrix system, proves paradoxically more manageable than two-part writing. This occurs because, by using exclusively the numbers of the **Abstract Seed** $(1, 3, 5)$, the set of available choices within the **Sound Matrix** (M_S) is drastically reduced and becomes almost compulsory.

The harmonic motions permitted between two contiguous columns are limited to the vectorial combinations of the triad:

$$\{11, 13, 15, 31, 33, 35, 51, 53, 55\}$$

To guarantee essential structural validity, the system imposes three constraints (two logical and one prudential):

1. **Controlled Saturation (Logical Constraint):** In each column of the matrix there may not be more than three 1s, three 3s, and three 5s. This limitation is necessary in order not to exhaust the possible motions toward the subsequent column.
2. **Prohibition of Parallels (Logical Constraint):** In two contiguous columns, no two identical pairs of numbers may occur (e.g., 3–3 in one part and 3–3 in another), in order to avoid parallel octaves, nor may 1–1 appear in a lower part simultaneously with 5–5 in an upper part, which would generate parallel fifths.
3. **Bass Stability (Prudential Constraint):** It is preferable to avoid the value 5 in the lowest part. Although the six–four chord (6/4) is technically admissible under certain conditions, at this "framework" stage it is deliberately restricted in order to ensure greater harmonic stability without requiring further calculations.

6.2 Example of a Matrix

Below we present the complete **Sound Matrix** (M_S) over nine time units, for an eight-voice counterpoint in first species (note against note).

Voice	t1	t2	t3	t4	t5	t6	t7	t8	t9
V1	3	1	5	3	1	1	1	5	3
V2	1	5	3	1	5	3	5	3	5
V3	5	3	5	1	1	5	3	3	1
V4	3	5	1	5	1	3	3	1	3
V5	5	1	3	3	5	5	5	1	5
V6	3	3	3	5	3	5	1	3	3
V7	1	3	1	1	3	3	1	1	1
V8 (B)	1	1	1	3	3	1	3	5	1

Table 13: Eight-voice Sound Matrix (validated).

This structure rigorously satisfies all imposed constraints. By analyzing the vertical distribution in this specific example, we observe a balanced configuration:

- Transitions from the number **1** occur three times per column.
- Transitions from the number **5** occur twice per column.

- Consequently, transitions from the number **3** account for the remaining three.

From one column to the next, the number 1 moves only once to 1, 3, or 5. Similarly, from 5 the motion always proceeds to 1 or 3, never to 5. This system of crossed constraints mathematically guarantees the absence of parallel octaves or fifths. Finally, the fact that the bass voice (V8) consistently carries either 1 or 3—except at the final cadence, where it crosses with the other bass (V7)—avoids the formation of a six–four chord (64), thus ensuring harmonic stability.

By applying to this matrix structure a given sequence of keys—this time independent of the modular Pythagorean table (Table 14)—we generate the actual musical realization, raising the bass and the upper parts only in selected bars. As usual, Key 0 (C) functions as the initial and final reference and closes the cycle.

By summing column by column, we obtain from the previous matrix the **Resultant Matrix** (M_R), which contains the real scale degrees (where 1 = C, 2 = D, . . . , 7 = B).

Voice (Key)	**t1** (0)	**t2** (1)	**t3** (3)	**t4** (0)	**t5** (2)	**t6** (3)	**t7** (0)	**t8** (4)	**t9** (0)
V1	3	2	1	3	3	4	1	2	3
V2	1	6	6	1	7	6	5	7	5
V3	5	4	1	1	3	1	3	7	1
V4	3	6	4	5	3	6	3	5	3
V5	5	2	6	3	7	1	5	5	5
V6	3	4	6	5	5	1	1	7	3
V7	1	4	4	1	5	6	1	5	1
V8 (B)	1	2	4	3	5	4	3	2	1

Table 14: Resultant Matrix (sum $M_S + M_K$). The numbers represent real scale degrees.

As can be observed in the first system of staves in Figure 5, the structure preserves vertical harmonic coherence while continuously varying pitch. For instance, at time t_2 (Key 1), the entire column is transposed upward by one degree with respect to the original configuration, transforming the absolute intervals while preserving the correct relative distances. If the chord was previously on C, it is now on D.

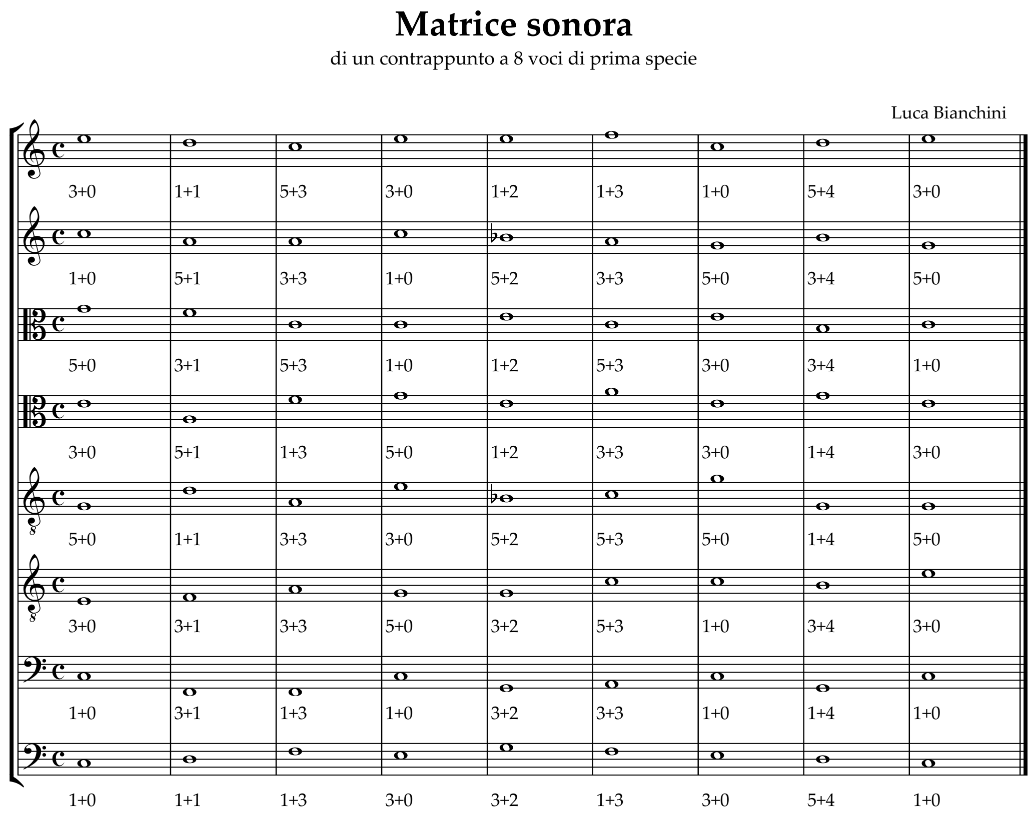

Figure 5: Realization using the Matrix System, which guarantees the absence of parallelisms by construction.

6.3 Comparative Analysis (Cherubini)

If the previous example contains no errors of octaves or fifths, an analysis of the "numerical texture" of Cherubini's exercise in Figure 6—drawn from the only example of eight-voice first-species counterpoint (note against note) in his treatise[10]—reveals how the application of scholastic rules, in the absence of a global theory, inevitably leads to a problematic domino effect.

By reproducing the exact numerical reduction of Cherubini's example, we highlight the critical points in bold:

1. **Cols. 2–3:** Rigid application of the "harmonic tie" (held notes) → saturation of 1.
2. **Cols. 3–4:** Emergency resolved through "octaves by contrary motion."
3. **Cols. 8–9:** Fatal saturation (three 5s and three 1s) → parallel-octave error (direct motion).

Voice	**t1**	**t2**	**t3**	**t4**	**t5**	**t6**	**t7**	**t8**	**t9**
V1	3	1	5	3	1	5	1	**5**	**3**
V2	1	5	3	1	5	3	5	3	5
V3	5	3	3	5	3	5	3	**1**	3
V4	3	5	**1**	**1**	1	1	5	**1**	5
V5	5	1	3	3	1	3	3	3	1
V6	3	**3**	**1**	5	5	5	5	**5**	**3**
V7	1	**3**	**1**	**1**	3	3	1	**1**	1
V8	1	1	**1**	3	3	1	3	**5**	1

Table 15: Analysis of Cherubini's counterpoint.

6.3.1 The Chain of Errors (Domino Effect)

The final error is not an isolated event, but the outcome of a defensive strategy.

Phase 1: The Trap of "Common Notes" ($t_2 \to t_3$)
Between the second and third columns, Cherubini does not commit a grammatical error, but falls victim to a pedagogical rule: the principle of the *harmonic tie*. Since the bass ascends by a third, several notes are common to both chords. Cherubini chooses to "play it safe" by holding the common notes or repeating the same intervals (Tenor II and Bass I repeat the motion 3 → 1). Although widely adopted, this strategy leads to an unforeseen accumulation of **four occurrences of** 1 in column t_3. The scholastic rule "hold common notes to avoid errors" paradoxically induces error, creating an unmanageable saturation.

Phase 2: The Patch of Contrary Motion ($t_3 \to t_4$)
Confronted with four voices on the fundamental (1) at time t_3, the composer is blocked. In order to move the parts without generating immediate errors in such a dense texture, he is forced to resort to an artifice: octaves by contrary motion between Contralto II and Bass I. This is the unmistakable symptom of "navigation by sight": lacking degrees of freedom, Cherubini forces the lock with a technical license.

Phase 3: The Final Collapse ($t_8 \to t_9$)
The latent instability explodes in the penultimate bar. At time t_8, the matrix exhibits a saturated configuration:

- **Three occurrences of** 5 (V1, V6, V8).

[10] L. Cherubini, *Cours de contrepoint et de fugue*, Paris, Maurice Schlesinger, 1835, p. 39.

- **Three occurrences of** 1 (V3, V4, V7).

The simultaneous presence of three fifths (5) and three fundamentals (1) is combinatorially critical. With three voices singing 5, all legitimate escape routes are mathematically exhausted. The First Soprano (V1) descends $5 \rightarrow 3$, and Tenor II (V6), having no alternative, is forced to double it with the identical motion $5 \rightarrow 3$. The error of parallel octaves (direct motion) thus becomes inevitable: it is the final bill paid for the "prudent" choices made in the preceding bars.

6.3.2 Analytical Conclusion

The comparison shows that the rule of the "harmonic tie," a cornerstone of traditional pedagogy, is a *false friend* in large-scale counterpoint.

Whereas the traditional system—adopted and taught not only by Cherubini, its most authoritative representative—seeks security in the preservation of individual notes, thereby producing saturation and error, the new system proposed in this article abandons local harmonic control as the primary constraint and instead relies on a **Geometry of Saturation**.

By ensuring upstream that the vertical distribution never exceeds a critical threshold of repetition (no more than three occurrences of the same note), contrapuntal motion becomes structurally immune to parallel fifths and octaves, without any need to resort to licenses, exceptions, or corrective artifices.

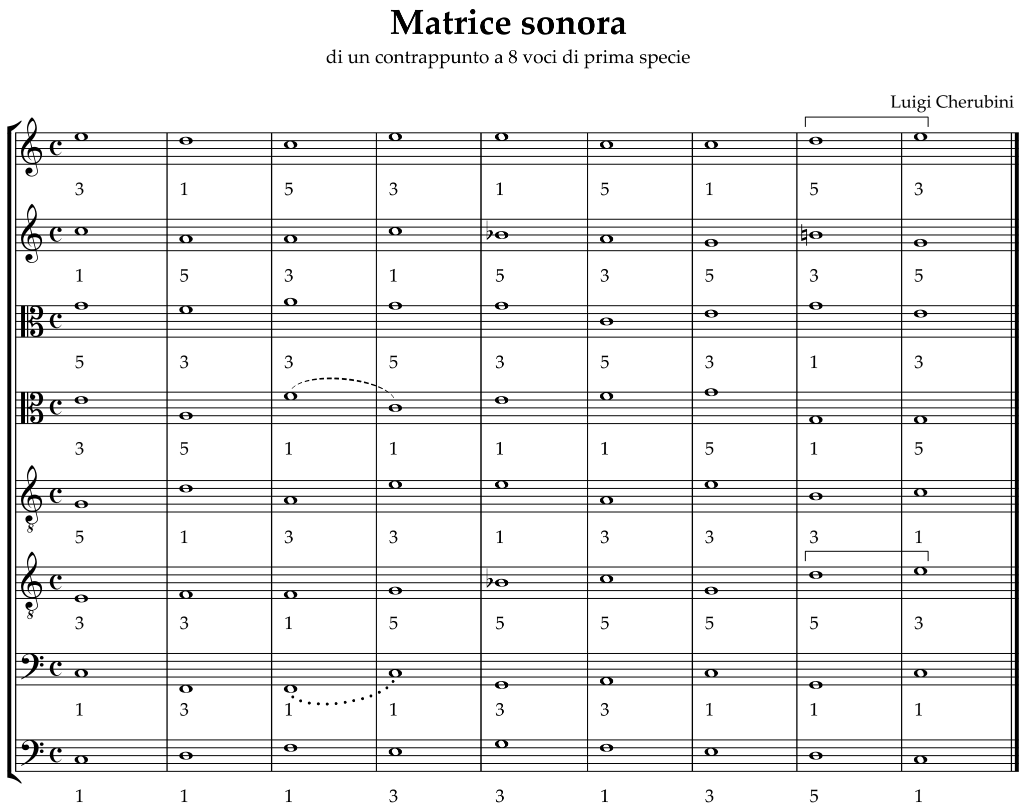

Figure 6: Analytical comparison at eight voices. Realization by Cherubini, highlighting octaves by contrary motion (dotted slurs) and the error of parallel octaves (phrase brackets).

7 The Double Canon and Modular Variation

7.1 The Abstract Framework

Returning to canons, we now extend the method to a more complex structure: a double canon within a 4×10 matrix.

In this system:

- The **Third Row** (V3, Dux 1) presents the first theme, which is imitated by the **First Row** (V1, Comes 1) with a delay of two columns.
- The **Second Row** (V2, Dux 2) presents the second theme, which is imitated by the **Fourth Row** (V4, Comes 2), again at a distance of two columns.

7.2 Algebraic Formalization

To transform this structure into music, we apply an alternating key vector ($\mathbf{K}$) to the structural matrix ($\mathbf{S}$). The operation is defined by the matrix sum $\mathbf{S} + \mathbf{K} = \mathbf{R}$.

$$
\begin{array}{rcl}
\begin{array}{r} \text{V1 (C1)} \\ \text{V2 (D2)} \\ \text{V3 (D1)} \\ \text{V4 (C2)} \end{array} &
\begin{bmatrix}
\cdot & \cdot & 1 & 3 & 3 & 1 & 1 & 1 & 3 & 1 \\
1 & 1 & 6 & 6 & 1 & 4 & 3 & 6 & 1 & 4 \\
1 & 3 & 3 & 1 & 1 & 1 & 3 & 1 & 1 & 1 \\
\cdot & \cdot & 1 & 1 & 6 & 6 & 1 & 4 & 3 & 6
\end{bmatrix} & (\text{Matrix } \mathbf{S}) \\
+ &
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix} & (\text{Matrix } \mathbf{K}) \\
= &
\begin{bmatrix}
\cdot & \cdot & 1 & 4 & 3 & 2 & 1 & 2 & 3 & 2 \\
1 & 2 & 6 & 7 & 1 & 5 & 3 & 7 & 1 & 5 \\
1 & 4 & 3 & 2 & 1 & 2 & 3 & 2 & 1 & 2 \\
\cdot & \cdot & 1 & 2 & 6 & 7 & 1 & 5 & 3 & 7
\end{bmatrix} & (\text{Matrix } \mathbf{R})
\end{array}
$$

The resulting matrix $\mathbf{R}$ shows how the voices interlock perfectly, generating a complex polyphonic texture from two simple linear themes (Figure 7). It is noteworthy that this specific configuration displays striking structural analogies with an example reported by Father Giambattista Martini in the first volume of his *Esemplare*, bearing witness to the fact that the combinatorial logic formalized here was already intuitively present in the pedagogy of the great historical masters.[11]

8 Expansion to Four Voices: Intervallic Variations

8.1 Horizontal Freedom and Vertical Constraint

Nothing prevents us from varying the horizontal intervals in order to create more articulated melodic profiles. For example, by setting a seed with the sequence 1–3–1–6, we obtain a melody that descends to the lower sixth, thereby enriching the motion of the parts.

In this new example, we construct a four-voice canon. Unlike the previous cases, in which the *comes* responded to the *dux* at a distance of one note, here the voices respond at a distance of **two notes**.

[11] G. B. Martini, *Esemplare o sia saggio fondamentale pratico di contrappunto sopra il canto fermo*, Vol. 1, Bologna, Lelio dalla Volpe, 1774, p. 209.

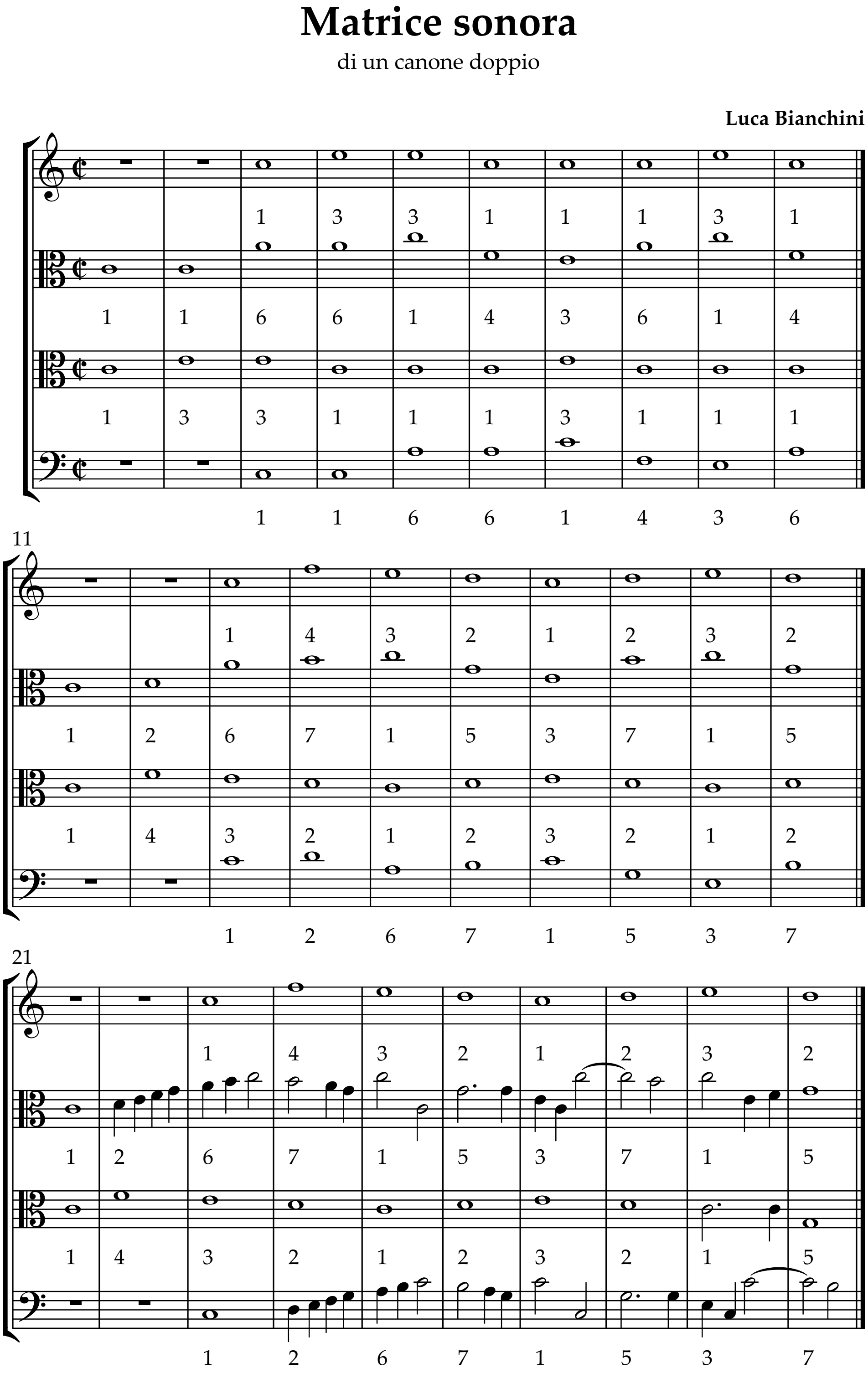

Figure 7: Sound matrix and realization of a four-voice double canon. The themes (Dux 1 and Dux 2) and their responses (Comes 1 and Comes 2) are managed simultaneously through matrix addition. From bar 21 onward, the same canon is embellished with figurations that fill the spaces between the melodic pillars.

8.2 Structural Analysis (Matrix S)

In this new example of a four-voice canon, the *comes* answers the *dux* at a distance of **two notes**. Within the structural matrix, the entries are staggered: the second voice begins on 1, and the third responds with 1 at a distance of two notes; the remaining two voices begin on 5 and enter into canon after four and six notes, respectively.

Below is the formal representation of the structural matrix **S**. The use of placeholders (empty spaces) highlights the staggered entries of the voices:

$$
\begin{array}{r}
\mathrm{V1} \\ \mathrm{V2} \\ \mathrm{V3} \\ \mathrm{V4}
\end{array}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & 5 & 5 & 7 & 7 & 2 & 7 & 6 & \cdot & 6 & 6 & 1 & 1 & 3 & 1 & 7 & \cdot & \cdot \\
1 & 1 & 3 & 3 & 5 & 3 & 2 & \cdot & 2 & 2 & 4 & 4 & 6 & 4 & 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 & 5 & 7 & 7 & 2 & 7 & 6 & \cdot & 6 & 6 & 1 & 1 & 3 & 1 & 7 \\
\cdot & \cdot & 1 & 1 & 3 & 3 & 5 & 3 & 2 & \cdot & 2 & 2 & 4 & 4 & 5 & 4 & 3 & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
$$

8.3 Application of Keys and Result (Matrix R)

To vary the canon and render it less static, we combine the table with a sequence of alternating keys. The key vector **K** is defined by the alternation between 0 (identity) and 6 (diatonic transposition):

$$\mathbf{K} = [0, 6, 0, 6, 0, 6, \dots]$$

By applying this transformation (where the subscript +6 indicates the alteration applied on even bars), we obtain the resultant matrix **R**:

$$
\begin{array}{r}
\mathrm{R1} \\ \mathrm{R2} \\ \mathrm{R3} \\ \mathrm{R4}
\end{array}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & 5 & 5_{+6} & 7 & 7_{+6} & 2 & 7_{+6} & 6 & \cdot & 6 & 6_{+6} & 1 & 1_{+6} & 3 & 1_{+6} & 7 & \cdot & \cdot \\
1 & 1_{+6} & 3 & 3_{+6} & 5 & 3_{+6} & 2 & \cdot & 2 & 2_{+6} & 4 & 4_{+6} & 6 & 4_{+6} & 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 & 5_{+6} & 7 & 7_{+6} & 2 & 7_{+6} & \cdot & \cdot & 6 & 6_{+6} & 1 & 1_{+6} & 3 & 1_{+6} & 7 \\
\cdot & \cdot & 1 & 1_{+6} & 3 & 3_{+6} & 5 & 3_{+6} & 2 & \cdot & 2 & 2_{+6} & 4 & 4_{+6} & 5 & 4_{+6} & 3 & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
$$

If we transcribe this matrix onto the staff, adding the ornamental notes, we obtain the complete canon (Fig. 8).

Matrice sonora

di un canone a 4 voci

Figure 8: Musical realization of the four-voice canon with intervallic variations (Matrix **R**).

8.3.1 Historical Comparison: Nicola Sala

A significant historical antecedent was composed by **Nicola Sala**, who included a structurally similar canon among the final exercises of his counterpoint course.[12]

The difference lies in the method of variation. Instead of applying our 0–6 alternation (which generates a descending motion), Sala raises the second half of each measure by a second. In mathematical terms, within our modular system, this operation corresponds to the alternating application of Key 1 (which denotes the interval of a second) and Key 0 (which denotes the identity):

$$K_{\text{Sala}} = [0, 1, 0, 1, 0, 1, \dots]$$

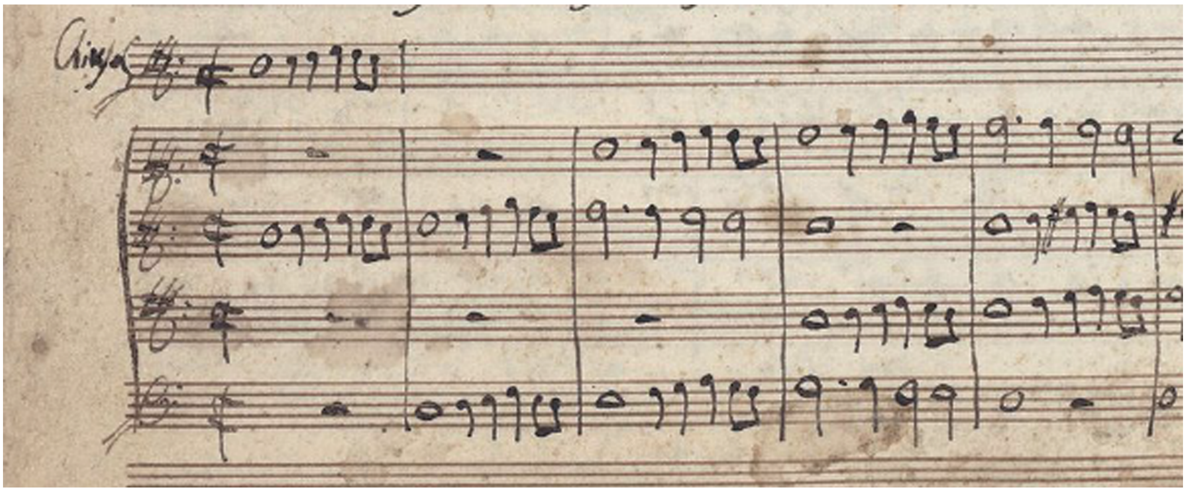

Figure 9: **Four-voice canon by N. Sala**.

[12] N. Sala, *Regole del Contrappunto Pratico*, 1787, autograph manuscript, p. 96 (pencil numbering). Naples, Biblioteca del Conservatorio di Musica San Pietro a Majella. Also available on IMSLP.

9 Advanced Symmetries: The Retrograde Canon

9.1 Geometric Construction: Mirror and Chiasmus

Let us begin with a matrix containing an elementary two-voice piece of three measures. The first row contains the numbers of the first voice, and the second row those of the second voice.

$$M_{start} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 1 & 5 \end{bmatrix}$$

This polyphony can be transformed into a retrograde (cancrizans) canon by exploiting the symmetry of the modular Pythagorean table. The procedure is purely geometric and unfolds in two stages.

Stage 1: Mirror Expansion
We copy the numbers symmetrically around a **central pivot** (axis of symmetry), which we fix here on the number 5 (the last column of the original matrix).

$$M_{mirror} = \begin{bmatrix} 1 & 3 & \mathbf{5} & 3 & 1 \\ 3 & 1 & \mathbf{5} & 1 & 3 \end{bmatrix}$$

At this stage, we do not yet have a canon, but two independent "palindromic" voices: the first half is identical to the second half read backwards.

Stage 2: The Chiasmus (Crosswise Exchange)
To activate the canon, we apply a crosswise exchange (chiasmus): the halves to the right of the pivot are inverted, assigning to the upper voice what was previously in the lower one, and vice versa.

$$M_{final} = \begin{bmatrix} 1 & 3 & \mathbf{5} & 1 & 3 \\ 3 & 1 & \mathbf{5} & 3 & 1 \end{bmatrix}$$

This yields a fully resolved enigmatic canon. It suffices to notate only the first voice (1–3–5–1–3) with a motto (e.g. *"Seek my end in my beginning"*). The performer will discover that the second voice is nothing other than the first read in reverse (3–1–5–3–1).

9.2 Mathematical Formalization

We can divide the structure into four sectors plus a central pivot P:

- A: Voice 1 (left side)
- B: Voice 2 (left side)
- a: Retrograde of A (located on the right, lower voice)
- b: Retrograde of B (located on the right, upper voice)

The final structure of the upper line becomes $A - P - b$. Since b is the retrograde of B, the lower line $(B - P - a)$ is the exact retrogradation of the upper one.

9.3 Key Invariance

The structural matrix defined above represents intervallic relationships, but it is not yet tonally determined. In the case of the retrograde canon, the application of keys requires additional mathematical care: in order not to destroy the constructed mirror symmetry, the key vector added must itself be **palindromic**.

9.3.1 Case A: Canon at the Unison (Constant Vector)

It is sufficient to add a constant vector, for example the series of 0 (or equivalently of 7). Since the number added is identical for every column, the symmetry remains unaltered.

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 3 \\ 3 & 1 & 5 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 & 1 & 3 \\ 3 & 1 & 5 & 3 & 1 \end{bmatrix}$$

9.3.2 Case B: Expansion of the Ambitus (Symmetric Key Vector)

If we wish to make the melody more dynamic, we must apply a **Palindromic Key Vector** (e.g. 0–1–2–1–0). Note how the key value increases up to the central pivot and then decreases symmetrically.

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 3 \\ 3 & 1 & 5 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 & 2 & 3 \\ 3 & 2 & 7 & 4 & 1 \end{bmatrix}$$

(Note: $3+1=4$, $5+2=7$, $1+1=2$*)*

The sum produces a new melody $(1,4,7,2,3)$ that fully preserves the properties of the canon: the second voice $(3,2,7,4,1)$ is the exact retrogradation of the first, while the melodic contour now spans a wider and more articulated range.

9.4 Case C: Retrograde Canon with Staggered Entries (Bilateral Symmetry)

There exists a third type of retrograde canon, simpler to construct yet highly effective, which exploits temporal displacement $(t+1)$. Instead of building complex specular voices, this method relies on a fundamental property: if the underlying melodic matrix is palindromic and the applied keys are themselves palindromic, the result is necessarily a retrograde canon.

9.4.1 Generative Procedure

Phase 1: The Palindromic Matrix
We define a numerical sequence that functions as its own mirror. In this example, we use the theme $1-3-5-5-3-1$. The second voice enters one time unit later, repeating the same sequence.

$$M_{pal} = \begin{bmatrix} 1 & 3 & 5 & 5 & 3 & 1 & \cdot \\ \cdot & 1 & 3 & 5 & 5 & 3 & 1 \end{bmatrix}$$

Phase 2: Activation via Symmetric Keys
Pitch determination is obtained by applying an "arch-shaped" key vector $(0-1-2-3-2-1-0)$, which ascends and descends symmetrically. The sum of these two symmetric elements (Melody + Keys) generates a structure in which the second voice is the exact retrogradation of the first, shifted in time.

$$\begin{bmatrix} 1 & 3 & 5 & 5 & 3 & 1 & \cdot \\ \cdot & 1 & 3 & 5 & 5 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 7 & 8 & 5 & 2 & \cdot \\ \cdot & 2 & 5 & 8 & 7 & 4 & 1 \end{bmatrix}$$

Result Analysis:

- Voice 1 performs: $1,4,7,8,5,2$.
- Voice 2 performs: $2,5,8,7,4,1$.

As can be observed, the second voice is the exact "crab" of the first (read backwards).

9.5 The Family of Derived Compositions

A final consideration opens up new perspectives on algorithmic composition. So far, we have applied specific vectors, but the system is intrinsically open. The sound matrix S (the seed) is a neutral topological structure: it defines **when** the voices move, but not **where** they go until a key intervenes.

If we consider the Pythagorean Table as a database of transformers, by applying to the same matrix S any other row of the table (the row of 3, 4, 6, etc.), we obtain a different canon each time. This demonstrates that, once a single valid "seed" (free of parallel errors) has been defined, the composer instantaneously possesses not a single work, but an entire **family of compositions** (F), all of them mathematically correct.

$$F = \{S + R_i \mid R_i \in \text{Pythagorean Table}\}$$

10 The Inverse Retrograde Canon and the Reopening of the System

The inverse retrograde canon (or inverse cancrizans canon) represents, for some authors, the mathematical and artistic apex of polyphonic construction. In this type of canon, the second voice imitates the first by reading the melody from the last note to the first (retrograde motion) while simultaneously inverting the intervals (inversional motion).

10.1 The Reopening of Compatibility

The most promising feature of this form is its extraordinary generative flexibility. Whereas the simple retrograde canon imposes the strict constraint of palindromic key vectors, the inverse retrograde canon, thanks to its double transformation (temporal and spatial), reopens compatibility with standard progressive sequences.

This means that we can apply to the retrograde-inverse structure the normal progressive rows of the modular Pythagorean table ($1, 2, 3 \ldots$ or $2, 4, 6 \ldots$). Each of these rows generates a valid canon. The system is no longer confined to a small set of symmetric combinations, but once again draws upon the full potential of the Pythagorean table.

10.2 Practical Example

Let us construct the basic mirror structure (chiasmus) using an extended seed $(1-3-5-7-2)$. After applying the "X" exchange, we add a standard progressive key $(0, 1, 2, 3, 4)$, using the row of 1.

$$\underbrace{\begin{bmatrix} 1 & 3 & 5 & 2 & 7 \\ 3 & 1 & 5 & 7 & 2 \end{bmatrix}}_{\text{Chiasmus Matrix}} + \underbrace{\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}}_{\text{Progressive Key}} = \begin{bmatrix} 1 & 4 & 7 & 5 & 4 \\ 3 & 2 & 7 & 3 & 6 \end{bmatrix}$$

(Note: $2 + 3 = 5$*;* $7 + 4 = 11 \to 4$*)*

The first voice $(1, 4, 7, 5, 4)$ rises and oscillates. The second voice $(3, 2, 7, 3, 6)$ responds with an inverse logic that is complex yet coherent. The first voice begins with an ascending fourth $(1 \to 4)$; the second, when read backwards $(6 \to 3)$, descends by a fourth.

10.3 Offset Inverse Retrograde

Even in the inverse retrograde canon, it is possible to introduce the voices diagonally, with staggered entries. By applying the row of 1 of the Pythagorean multiplication table $(0, 1, 2 \dots)$ to an offset matrix, the voices proceed in inverse retrograde motion and imitate each other at the upper second.

$$\underbrace{\begin{bmatrix} 1 & 3 & 5 & 5 & 7 & 2 & \cdot \\ \cdot & 1 & 3 & 5 & 5 & 7 & 2 \end{bmatrix}}_{S_{\text{offset}}} + \underbrace{\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}}_{K_{\text{progressive}}} = \begin{bmatrix} 1 & 4 & 7 & 1 & 4 & 7 & \cdot \\ \cdot & 2 & 5 & 1 & 2 & 5 & 1 \end{bmatrix}$$

With this method, inverse retrograde canons, ascending canons, and descending canons can be superimposed, resulting in a total polyphonic architecture limited only by the composer's imagination.

11 Extension of the Principle: Diagonalization of Historical Materials

The principle of diagonalization presented so far does not apply exclusively to abstract numerical structures (such as the Pythagorean triad), but can be fruitfully extended to pre-existing concrete musical materials. Instead of three discrete numerical values, one may consider three **distinct musical segments** (S_A, S_B, S_C), each consisting of a portion of a voice of defined duration, vertically coherent but not necessarily in a canonical relationship with the original source.

11.1 Linearization Algorithm

Consider a vertical section extracted from a three-voice composition, for instance a work by Corelli. The segments are defined according to register:

- S_A: Segment of the upper voice.
- S_B: Segment of the middle voice.
- S_C: Segment of the lower voice.

The generative operation consists in transforming vertical simultaneity into a linear concatenation. The sequence $S_A \to S_B \to S_C$ is assigned in its entirety to the upper voice (Dux). The lower voices (Comes) reproduce the same concatenation with staggered entries, respectively delayed by the duration of one segment (Δt_1) and by two segments (Δt_2).

The resulting structure is a canon based not on point-by-point imitation (note against note), but on a **block offset**:

$$\begin{array}{lllll} \text{V1:} & [S_A] & [S_B] & [S_C] & \dots \\ \text{V2:} & \text{rest} & [S'_A] & [S'_B] & \dots \\ \text{V3:} & \text{rest} & \text{rest} & [S''_A] & \dots \end{array} \tag{2}$$

In this system, the minimal imitative unit is not the individual note, but the entire musical segment. See an example in Figure 10, in which a fragment taken from a *Concerto grosso* by Corelli (Op. 6, No. 4) is transformed into an infinite canon at the unison and the octave. The constituent parts A, B, C, D may be freely permuted, depending on compositional requirements. In this case, the first voice sings A, C, B, and D.

Diagonalizzazione di materiali storici

Canone a 4 voci

Figure 10: Musical realization of a four-voice infinite canon derived from a section of a *Concerto grosso* by A. Corelli.

11.2 Variation through Intervallic Translation

Once the basic canonical structure has been obtained through diagonalization, it becomes possible to apply the **Key Matrices** in order to generate novel variants in which the Comes responds at different pitch levels.

Such operations preserve the internal structural coherence of the whole, guaranteed by the validity of the original fragment, while at the same time producing a new sonic texture that is difficult to reduce to a simple literal quotation of the source model. The historical piece thus becomes, in turn, a “complex seed” for new generations of infinite canons.

12 Temporal Expansion: Augmented and Mensural Canons

In augmented canons and canons by prolations[13], the factor of **time** becomes operative, whereas in other types of canons (isochronous canons) it remains constant. To construct an augmented canon, we employ three coordinated matrices:

1. the **sound matrix** (defining the intervallic texture);
2. the **key matrix** (determining the real pitch level);
3. the **time matrix** (assigning durations to the measures).

In Figure 11, the first system of staves presents a single bar for three voices in 4/2. The lower part is written in long note values (part A), the middle part in medium values (part B), and the upper part in shorter values (part C), as schematically represented below.

C
B
A

Table 16: Vertical representation (of the parts).

In the second system of the same Figure 11, we derive an octave canon by simply copying, in each voice and in succession, part A, then part B, and finally part C.

A	**B**	**C**		
	A	**B**	**C**	
		A	**B**	**C**

Table 17: Structure of the finite octave canon.

Since the canon terminates too early, we composed an additional new part D, placed after C, corresponding to the middle voice of the third system in Figure 11. In this way, the canon is extended by continuing the imitative process.

A	**B**	**C**	**D**		
	A	**B**	**C**	**D**	
		A	**B**	**C**	**D**

Table 18: Structure of the extended octave canon.

[13]A type of canon in which the same musical line is performed simultaneously with different rhythmic proportions.

Figure 11: **Preparatory canon**. The first system at the top shows the *vertical nucleus*, with parts A, B, and C in vertical simultaneity (synchronous state). In the subsequent systems, a process of *temporal displacement* occurs: the parts are shifted along the temporal axis to generate a unison canon, preserving the sonic material while projecting it diagonally. This latter canon serves as the basis for the augmented canon.

As can be observed in Figure 11, part A proceeds with longer note values, part B with faster notes, part C faster still, and part D extremely rapid, with progressively smaller values. This effect is intentional, since the canon just constructed serves as the scaffolding for the more complex structure of an augmented canon, in which one part sings a given set of values, another doubles them, and a third multiplies them by four. At each column, durations are multiplied by powers of 2, as shown in Figure 12.

A×1	**B×2**	**C×4**	**D×8**		
	A×2	**B×4**	**C×8**	**D×16**	
		A×4	**B×8**	**C×16**	**D×32**

Table 19: Structure of the augmented canon.

In this type of canon, the only points at which it is necessary to verify the absence of parallel octaves or fifths are those in which note changes occur simultaneously. Such points are easily predictable, since they fall on arithmetic progressions determined by the augmentation factor (for instance, when values are doubled or quadrupled).

Figure 12: **Augmented canon.** To the durations of the preparatory canon shown in the previous figure, the following multiplication coefficients have been applied: $\times 2$ (*proportio dupla*), $\times 4$ (*proportio quadrupla*), and $\times 8$ (*proportio octupla*). Note the strategic entrance of voice D (measure 14), which fulfills the principle of *conservation of density.* It introduces new kinetic material precisely when the energy of the preceding voices weakens as a consequence of temporal dilation.

12.1 The Compromise of Nicola Sala

A comparison with historical practice reveals a fascinating aesthetic divergence. By examining, in Figure 13, an augmented canon by Nicola Sala, we observe that the Neapolitan master introduces the third voice (in quadrupled values) with a delay of one measure with respect to the theoretical point of mathematical interlocking. This “license” has a precise structural consequence: the canon functions perfectly for the first few measures, but becomes mathematically unsustainable from the midpoint onward. As a result, Sala abandons strict imitation in the upper voices, allowing only the lowest part to maintain thematic rigor, while the other two parts proceed as free voices.

This choice reveals the spirit of the Italian school, oriented toward maximum freedom: the listener is offered the suggestive idea of an augmented canon, while the architectural framework is bent in favor of immediate musicality and cadential closure. We bow to these aesthetic choices, which privilege artistic effect over strict numerical coherence.

The matrix-based system proposed in this article is not intended to impose rigidity, but rather to simplify the task of those who wish to follow in Sala’s footsteps, by managing free sections with ease, while at the same time satisfying the expectations of those who pursue an ideal of absolute mathematical rigor. The flexibility of the model demonstrates that, under the guidance of the matrix, canonical counterpoint and free counterpoint are not antithetical, but can coexist productively, always leaving the final decision to the composer.

Figure 13: **Enigmatic (augmented) canon by N. Sala.** The voices proceed according to *proportio dupla*, *proportio quadrupla*, and *proportio octupla*. Note in particular the entrance of the third voice, which occurs in the fifth measure rather than the fourth. From that point onward, geometric proportionality necessarily breaks down. When the three voices reach the point marked in the score by the sign $-SS-$, they cease to sing in canon.

12.2 Synchronization by Means of a “Pivot” (Prefix)

To transform a canon with staggered entrances into a canon in which all voices begin simultaneously, it is sufficient to add an initial note that functions as a **pivot** (for example, the value **3**). In the resulting matrix, each number in the second row has a value twice that of the corresponding number in the first row, and each number in the third row has a value four times that of the first row (the dash indicates a sustained note):

$$
\begin{array}{c}
V_1(\times 1) \\
V_2(\times 2) \\
V_3(\times 4)
\end{array}
\begin{bmatrix}
3 & 1 & 1 & 3 & 3 & 1 & 6 & 6 & 1 & \dots \\
3 & - & 1 & - & 1 & - & 3 & - & 3 & \dots \\
3 & - & - & - & 1 & - & - & - & 1 & \dots
\end{bmatrix}
$$

12.3 Mensural Canons

The system makes it possible to generate infinitely many species of canon by assigning different meters to each voice. As in fourteenth-century Italian practice, we combine the meters of a canon with successive entrances according to the following relations:

- the first meter is half the second;
- the third is equal to the second;
- the fourth is double the third.

The cycle then resumes by multiplication by two. The voices enter in a staggered fashion, but according to these complex metric relations:

- **V1:** 2/4 → 4/4 → 4/4 → 8/4 . . .
- **V2:** 6/4 → 12/4 . . . (sesquialtera / triple proportion)
- **V3:** 18/4 → 36/4 . . . (nonuple proportion)

<table>
<tr><th>Meter V1</th><th>2/4</th><th>4/4</th><th>4/4</th><th>8/4</th><th>4/4</th><th>8/4</th><th>8/4</th><th>16/4</th><th>. . .</th></tr>
<tr><td>V1 (Seed)</td><td>1</td><td>1</td><td>1</td><td>3</td><td>5</td><td>3</td><td>3</td><td>. . .</td><td></td></tr>
<tr><td>Meter V2</td><td colspan="2">6/4</td><td colspan="2">12/4</td><td colspan="2">12/4</td><td colspan="2">24/4</td><td>. . .</td></tr>
<tr><td>V2 (Seed)</td><td colspan="2">1</td><td colspan="2">1</td><td colspan="2">1</td><td colspan="2">3</td><td>. . .</td></tr>
<tr><td>Meter V3</td><td colspan="4">18/4</td><td colspan="4">36/4</td><td>. . .</td></tr>
<tr><td>V3 (Seed)</td><td colspan="4">1</td><td colspan="4">1</td><td>. . .</td></tr>
</table>

Table 20: Synchronized alignment (by pivot) of the mensural canon.

In the musical notation it is not necessary to indicate every change of meter explicitly. By writing only the initial meter, it is possible to adjust the number of beats so as to align them with the theoretical measure. To compute this canon—of apparently “improbable construction,” though only in appearance—it is sufficient to construct a sonic matrix and apply the matrix of durations (2/4, 6/4, 18/4, . . .), as shown in Figure 14.

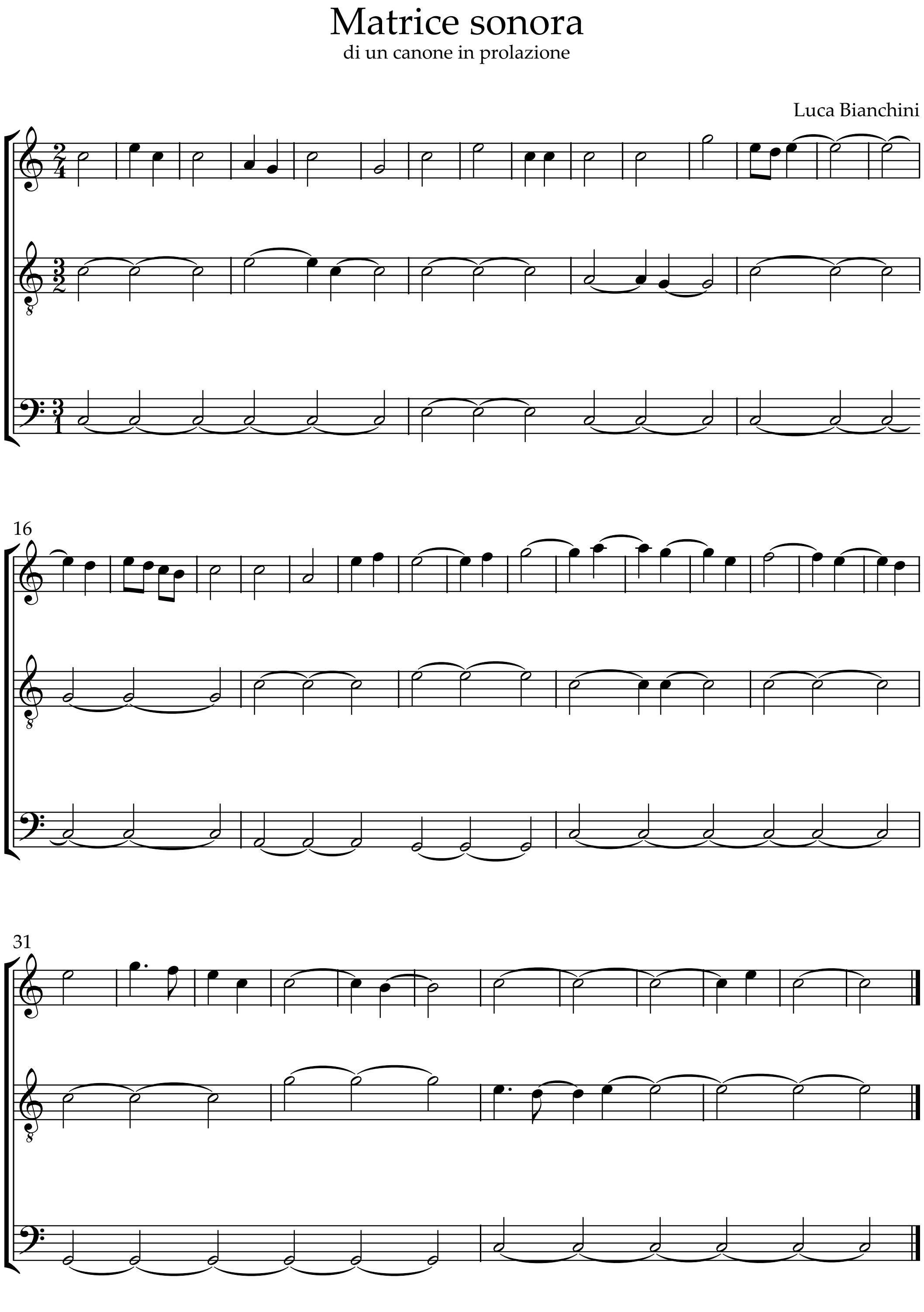

Figure 14: **Realization of a canon in prolations** by means of the matrix system. In each part the pitches are identical, but their durations change according to a geometric proportion. In the first staff, for example, the initial C has a value of 2/4; in the second staff 3/2; and in the third 3/1.

13 The Connection with the Neapolitan School: Partimenti as Implicit Matrices

The very term *partimento*, referring to a musical system in use in the Neapolitan Conservatories from the early eighteenth century onward, suggests a dynamic nature that goes beyond the static realization of figured-bass chords. As the name implies, partimenti are "moving parts," in which numerical indications do not merely serve to fill vertical harmony, but rather to suggest a precise contrapuntal texture. They are typically written on a single staff and, in their horizontal unfolding, display an ingenious system for indicating not only harmony, but also counterpoint and the entrances of voices, through the combined use of notes, clefs, and metrical signs.

In close analogy with our matrix-based system, the historical partimento is neither pure counterpoint (in the sense of our *Stilnovo* in music) nor pure harmony (in the manner of Rameau), but rather a **third genre**, in which numbers function as positioning vectors, clefs convey information about entrances and pitch displacement, and time signatures provide temporal organization.

A distinctive feature that clearly separates the partimento from figured bass is the frequent and alternating use of clefs within the same piece.

In our model, this way of conceiving musical structures corresponds exactly to the action of the **key matrix**, alongside the sonic matrix and the time matrix. When a Neapolitan master changed clef within a partimento or indicated complex numerical figures above the staff, he was not merely requesting a transposition, but signaling the entrance of a new "logical voice" or a specific imitative configuration.

In the eighteenth century, the study of partimenti was carried out alongside that of counterpoint and solfeggio. Partimenti explained the structural frameworks; solfeggi addressed ornamentation—that is, how to diminish notes within a voice; and counterpoint provided the rules governing the interaction of voices with one another. The student who realized partimenti thus put into practice everything that was taught at the Conservatory in the field of composition.

Even today, partimenti can be reread not only as exercises in free improvisation, but as **complex algorithms**, the legacy of numerical formulas that can be perfectly described through sonic matrices, in which the number designates the part and the clef defines the transformer required to activate it within tonal space.

Double counterpoint as taught in the Neapolitan Conservatories—for example in the time of Tritto—can likewise be greatly simplified by combining matrices with one another.

A final reflection brings us back to the Renaissance schools and their notational systems. In choral books, polyphonic music was written not in score, but horizontally—a practice that has long fascinated music theorists, both at the upper and lower levels. Within melodic horizontality, the entirety of polyphony and counterpoint is already contained. Indeed, those who first devised musical systems based on mathematical ratios—one may think of the Greek musicographers—had already conceived polyphony in principle, anticipating by millennia the emergence of medieval multi-voice compositions.

14 Conclusions

The pedagogical tradition of the great theorists and composers of the past—especially in the late nineteenth century—gradually abandoned the art of partimenti. Practical explanations of how to write canons increasingly relied on methods of "local adjustment" based on *trial and error* (write, check, correct). Canons involving more than two voices came to be regarded as exceptional cases and were presented as curiosities from a past age in which composers had reached the highest peaks of counterpoint, never to be equaled again.

Within the system proposed here, it becomes evident that the greater the number of voices, the easier the canon is to construct, since the combinatorial possibilities are correspondingly reduced. In such cases, the canon is, so to speak, almost written by itself. The same holds true for the enigmatic canon: what may appear impossible to replicate becomes straightforward once one starts from the sonic matrix and subsequently applies the key matrix and the time matrix.

The system of the *abstract seed* overturns a long-established paradigm. It is more advantageous to begin study with first-species counterpoint in eight voices—which is elementary within this framework—and only later move to two-voice counterpoint; likewise, one may start with an eight-voice real canon before tackling the two-voice canon, which is considerably more demanding, since many more variables are involved.

1. **Upstream validation:** if the vertical nucleus (the 1–3–5 triad) is correct at the outset, its temporal projection *must* necessarily be correct. Harmonic validity is an axiom of the system, not the result of empirical verification.
2. **An aesthetics of necessity:** ornamentation and melodic embellishment are not stylistic ornaments, but structural necessities. They serve to compensate for the decay of rhythmic energy caused by temporal dilation in the lower voices.
3. **Towards automation (C++):** the logical–mathematical approach described here opens the way to computational musicology. A system based on vectors, time matrices (2^n), and transposition operators is natively translatable into algorithms.

In fact, once its underlying principles have been revealed, knowing how to write an enigmatic canon is comparable to being able to perform arithmetic using the Pythagorean multiplication tables taught in elementary school—tables no one would have expected to "sound," like the monochord of the Pythagorean school, which was itself also called a *canon*.

While the classical contrapuntist manipulates notes on a staff, the modern theorist manipulates variables within a data flow. The "Pythagorean table," taught in elementary education, thus becomes the *reference table* of a generative process capable of managing not merely two, but N voices within a potentially infinite sonic space. The canon therefore reveals itself not as an artistic puzzle, but as a self-generating crystalline structure: the triumph of mathematical order over the chaos of trial and error.

To sketch the canon in prolations presented in Section 12—considered unattainable by many treatise writers, who in fact never explained how such constructions were historically achieved—required approximately half an hour of work. Even without taking into account the more than thirty years of study leading to this point (from 1985 to the present), this result appears, at least from our perspective, entirely appreciable.

14.1 Delimitation of the Field of Inquiry

In this study we have deliberately limited ourselves to the analysis of the fundamental triad $(1, 3, 5)$, which constitutes the primary harmonic framework. This methodological choice was necessary in order to isolate the generative principles of the matrix-based system without introducing an excessive number of variables during the phase of theoretical definition. We acknowledge that the exclusive application to the triad represents, in this context, a formal delimitation with respect to the vastness of the historical musical literature.

14.2 System Scalability and Future Developments

Nevertheless, it is important to emphasize that the theoretical architecture presented here is not confined to the triad. The system is intrinsically scalable and is fully capable of handling:

- **Heptatonic extension:** The matrix can accommodate complex chords based on the complete stacking of thirds $(1, 3, 5, 7, 2, 4, 6)$.
- **Historical partimenti:** It is possible to translate the celebrated "Rules of the Octave" of the Neapolitan School into matrices, thereby automating the standard motions of the bass.
- **Cadences and dissonances:** The system can encode compound cadences (including the management of the six-four chord, 6/4, often problematic in early treatises) and the resolution of suspensions.

Even in these cases of greater complexity, such as a **Fugue**, the ontological principle remains unchanged: there always exists a **Seed Matrix** (the vital harmonic nucleus) that is placed in space and time through the action of the **Key Matrices** and the **Prolation Matrices**. While the Seed contains harmonic necessity (the chord), the matrices of keys and of time act as transformation operators, defining respectively the real pitch and the duration.

In this article we have set out the foundational theoretical principles and provided the essential applied examples; the demonstration that matrix theory can encompass the totality of contrapuntal language will be the subject of future publications.[14]

[14]The generative model described here has also been applied retrospectively to well-known repertoires, producing infinite canons derived from works by D. Scarlatti (e.g., the so-called "Cat Fugue") and G. P. da Palestrina. Selected examples have been published online for demonstrative purposes and can be heard at `https://www.youtube.com/playlist?list=PLNuUwTDYnDaba9Qg5P2YM983DuXI7_V5R`.

15 Towards a New Digital Philology

By demonstrating how a matrix-based system makes it possible to control the combinatorial space of counterpoint, we have drastically reduced the incidence of undesirable outcomes typical of empirical procedures lacking global constraints. The scope of the *abstract seed*, however, is not limited to compositional pedagogy: it can also function as a diagnostic tool in the field of musical philology.

An analytical application of a structural approach consistent with the present model is documented in Bianchini–Trombetta (2016), where the corpus of canons attributed to Mozart is examined, including a critical reassessment of several editorial realizations.[15]

In particular, the analysis shows that the solution proposed in the *Neue Mozart-Ausgabe* for an enigmatic canon (K.^{2}. 89a II; K.^{6}. 73r) produces, in vertical superposition, dissonant combinations and patterns of voice-leading that call for a reassessment of the realization. By contrast, the application of an explicit generative model suggests an alternative reading based on coherent polyphonic linearity and controlled constraints, avoiding the systematic dissonances produced by certain modern transcriptions.

The system proposed here, therefore, is not intended solely for the generation of new music, but also provides an operational framework for the verification and decoding of historical repertoires, in cases where heuristic reading alone fails to make explicit the structural constraints at play.

16 Acknowledgements

The author wishes to thank Dr. Anna Trombetta for her careful readings of the manuscript and for her valuable suggestions, aimed at improving its clarity of presentation.

References

[1] L. Bianchini and A. Trombetta, *Mozart. La caduta degli dei. Parte prima*, Tricase, Youcanprint, 2016; English ed.: *Mozart: The Fall of the Gods.*

[2] G. Bizzi, *Specchio invisibile. I canoni enigmatici di J. S. Bach*, Roma, Kappa, 1982.

[3] L. Cherubini, *Cours de contrepoint et de fugue*, Parigi, Maurice Schlesinger, 1835.

[4] G. B. Martini, *Esemplare o sia saggio fondamentale pratico di contrappunto sopra il canto fermo*, Vol. 1, Bologna, Lelio dalla Volpe, 1774.

[5] C. E. R. Morsink, *The Composition of New Music Inspired by Music Philosophy and Musical Theoretical Writings from Ancient Greece*, PhD Diss., Goldsmiths, University of London, 2013.

[6] M. Nicolella e D. Celletti, *I segreti dell'Harmonia. Comporre canoni musicali con la Tabula mirifica*, Città del Vaticano, Libreria Editrice Vaticana, 2021.

[7] N. Sala, *Regole del Contrappunto Pratico*, 1787, Manoscritto autografo, Napoli, Biblioteca del Conservatorio di Musica San Pietro a Majella.

[8] G. Zarlino, *Le istitutioni harmoniche*, Venezia, 1558.

[15] L. Bianchini and A. Trombetta, *Mozart: The Fall of the Gods. Part One*, Tricase, Youcanprint, 2016.