(*Paper No. 2969.*)

"Equiangular Prisms."

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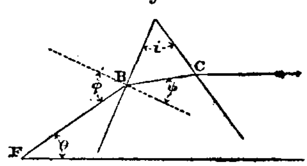
THE primary object in lighthouse optics is to parallelize as nearly as possible the rays emitted by the burner; but with such a luminary as a 6-wick burner in the focus, the area of flame is such as to produce a cone of divergent rays, incident upon every point of the interior of the apparatus, of an angle varying between 2° and 3° for small flames and between 5° and 6° for large burners. As it is impossible to reduce this cone of rays internally, except by increasing the diameter of the apparatus, the prisms must be constructed to reduce, as far as possible, the angle of this cone outside the apparatus, or at least to preserve it constant both inside and outside. The latter condition is satisfied by the equiangular prism, or prism of minimum deviation, a form of lens which has been introduced by Mr. Charles A. Stevenson, M. Inst. C.E. The difference in the angles of the external cone only becomes marked as the distance from the focal plane becomes great; for at the focal plane itself the difference between the conical angle of spherical and equiangular sections amounts to only 8″ in an initial angle of cone of 4°. At 15° the angle of the external cone of the spherical refractor exceeds that of the plano refractor by 31′ 14″ and exceeds the equiangular by 43′ 24″. At 30° above the focal plane the spherical exceeds the plano by 3° 28′ 34″ and the spherical exceeds the equiangular refractor by 4° 17′ 3″. The advantages derived from this property are at once apparent, for the intensity of the light varies inversely as the angle of this cone. On account of this property it is possible to carry the refracting portion of the apparatus to a height impossible in the spherical section and certainly highly disadvantageous in the Fresnel or plano section. For, at a height of 40° from the focal plane, the angle subtended by the burner inside the apparatus being

Fig. 1.



4°, the cone of divergent rays in the equiangular section measures only 4° 0′ 10″, and in the plano section amounts to 5° 25′ 50″; while it is impossible to construct a spherical section at this angle, as the outside face would become so obtuse to the ray in glass that total reflection would ensue. In the equiangular profile also coloured dispersion is reduced to a minimum. For if the ray F B, *Fig. 1*, is refracted to C, where it is finally refracted, then, D being the total deviation, θ the vectorial angle, ϕ and ϕ^1 the angles between the rays and normals at B, ψ and ψ^1 the angles between the rays and normals at C and i the refracting angle of the prism, $D = \psi - \psi^1 + \phi - \phi^1$ and $i = \psi^1 + \phi^1$ so $D = \psi + \phi - i$, whence—

$$\sin(D + i - \phi) = \sin i \sqrt{\mu^2 - \sin^2\phi} - \sin\phi\cos i;$$

for $\sin(D + i - \phi) = \sin\psi$,

and $\sin i \sqrt{\mu^2 - \sin^2\phi} - \sin\phi\cos i = \mu\sin\psi^1$,

or $\sin\psi = \mu\sin\psi^1$,

which is true from the law of refraction. Now if the difference between the refractive-indices for the mean and the extreme rays of the spectrum of the light considered amounts to 0·01, this equation can take the form

$$\sin D^1 = \sin i \sqrt{\mu^2 - \sin^2\phi} - \sin\phi\cos i,$$

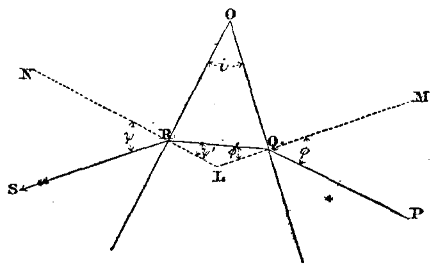
where i and ϕ are regarded as constants and D^1 and μ variables; hence the coloured dispersion will now be obtained on differentiating this equation, $dD^1 = \dfrac{\sin i}{\cos D^1 \cos\phi^1}\, d\mu$. In the Fresnel section: $D^1 = D + i - \phi = i$; hence $dD^1 = \dfrac{\tan i \times 0{\cdot}01}{\cos\phi^1}$. In the spherical lens: $D = \theta$; $\phi = 0$ and $\phi^1 = 0$, $dD^1 = \dfrac{\sin i \times 0{\cdot}01}{\cos(i + \theta)}$, and in the equiangular: $\phi = \dfrac{D + i}{2}$; $D = \theta$, $dD^1 = \dfrac{2\tan\phi \times 0{\cdot}01}{\mu}$.

The following Table, derived from the above equations, shows the equiangular section is in this respect superior to either the Fresnel or the spherical. Spherical aberration is also more nearly eliminated in it than in any of the usual forms of lighthouse lenses:—

Angle above Focal Plane.	Fresnel.	Spherical.	Equiangular.
1°	1′ 8″	1′ 8″	1′ 8″
15°	17′ 46″	18′ 35″	17′ 22″
30°	40′ 45″	52′ 50″	37′ 15″

With regard to the essential optical property of such prisms, let P Q R S, *Fig. 2*, be the path of a ray through a prism the apex of which is at O, and the refracting angle i. The normals at R and Q are L N and L M. If the total deviation,

Fig. 2.



D, be a minimum, $D = \psi + \phi - i$, $i = \phi^1 + \psi^1$; hence also $\frac{dD}{d\phi} = 0 = \frac{d\psi}{d\phi} + 1$, $\therefore \frac{d\psi}{d\phi} = -1$ and $\frac{d\phi^1}{d\phi} + \frac{d\psi^1}{d\psi} = 0.$

Also, since $\sin\phi = \mu \sin\phi^1$; $\therefore \cos\phi = \mu\cos\phi \frac{d\phi^1}{d\phi}$,

and $$\sin\psi = \mu\sin\psi^1;\ \therefore \cos\psi\frac{d\psi}{d\phi} = \mu\cos\psi^1\frac{d\psi^1}{d\phi};$$

$$\therefore \quad \frac{d\phi^1}{d\phi} = \frac{\cos\phi}{\mu\cos\phi^1},\ \frac{d\psi^1}{d\phi} = -\frac{\cos\psi}{\mu\cos\psi^1},$$

whence $$\frac{\cos\phi}{\mu\cos\phi^1} - \frac{\cos\psi}{\mu\cos\psi^1} = 0;$$

$$\therefore \quad (1-\sin^2\psi^1)(1-\mu^2\sin^2\phi^1) = (1-\sin^2\phi^1)(1-\mu^2\sin^2\psi^1),$$

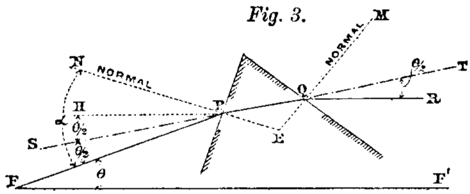
whence $$\sin^2\phi^1 = \sin^2\psi^1,$$

or $$\phi^1 = \psi^1 \text{ and } \therefore \phi = \psi.$$

Hence when the angle of incidence is equal to the angle of emergence, the deviation is a unique minimum.

It is possible to find an expression which will give the tangents to the outer face in terms of the vectorial angle θ. Let F F^1 be the optic axis, *Fig. 3*, F the focus, P a point on the inner face, Q a point on the outer face, and R Q the horizontal ray projected from the apparatus. Now the condition that the surfaces of the prism at P and Q, to which P N and Q M are normals, be such as to produce equal deviation of the focal ray F P, is, that the angle F P N be equal to the angle R Q M; or, considering the ray in glass Q P to occupy the mean position, this ray produced through Q and P will be seen to bisect the angle H P F, because the ray F P is altered from its original position to the horizontal P H in its final position, and from the definition of equiangular surfaces, where each face refracts equally, it follows that the ray in glass when produced is such that P S bisects the angle F P H,

Fig. 3.



and that Q T makes an angle $\frac{\theta}{2}$ with R Q. Hence, from the law of refraction, it follows that $\frac{\sin F P N}{\sin Q P E} = \mu$; so $\frac{\sin R Q M}{\sin P Q E} = \mu$, but $P Q E = Q P E = T Q M = R Q M - \frac{\theta}{2}$.

$$\therefore \qquad \frac{\sin R Q M}{\sin\left(R Q M - \frac{\theta}{2}\right)} = \mu,$$

whence, expanding, $\mu \cot R Q M \sin\frac{\theta}{2} = \mu \cos\frac{\theta}{2} - 1$,

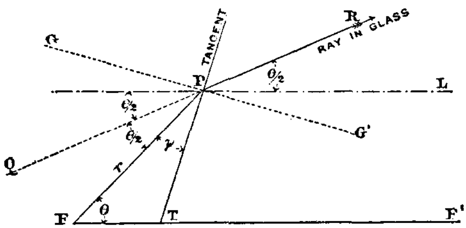
then
$$\cot R Q M = \frac{\cos\frac{\theta}{2} - \frac{1}{\mu}}{\sin\frac{\theta}{2}},$$

which gives the direction of normals and tangents.

With regard to the curves of the inner face, let F, *Fig. 4*, be the focus, and let the course of the ray be F P R. Let P T be the tangent to the surface of the lens at P. Then, as has been shown above, if $P F T = \theta$, $F P Q = \frac{\theta}{2} = Q P M = R P L$; by the law of refraction $\sin F P G = \mu \sin R P G^1$, where μ = index of refraction; therefore, if ψ is the angle between the ray and the tangent,

$$\cos \psi = \mu \cos \left(\psi + \frac{\theta}{2}\right) = \mu \left(\cos \psi \cos \frac{\theta}{2} - \sin \psi \sin \frac{\theta}{2}\right).$$

Fig. 4.



But $\cos \psi = \frac{d\,r}{d\,s}$, $\sin \psi = \frac{r\,d\,\theta}{d\,s}$, where $d\,s$ is an element of the arc of the curve in question.

$$\therefore \quad \frac{d\,r}{d\,s} = \mu \left(\frac{d\,r}{d\,s} \cos \frac{\theta}{2} - \frac{r\,d\,\theta}{d\,s} \sin \frac{\theta}{2}\right),$$

$$\frac{d\,r}{r} = - \frac{\mu \sin \frac{\theta}{2} d\,\theta}{1 - \mu \cos \frac{\theta}{2}},$$

hence, integrating, $r = \frac{C}{\left(\mu \cos \frac{\theta}{2} - 1\right)^2}$, or, if a be taken as the focal distance from F, the above can be written

$$r = \frac{a\,(\mu - 1)^2}{\left(\mu \cos \frac{\theta}{2} - 1\right)^2} \quad . \quad . \quad . \quad . \quad (1),$$

which is the equation to the first surface.

To derive the equation to the second or outside surface, *Fig. 5*, let F be the focus, and let the two rays F P and F P¹ be very close, then $K P F = S Q R = \frac{\theta}{2}$, where $\theta = P F X$. Now draw P¹ K parallel to P Q, and P¹m and Q¹n perpendicular to P Q. Then let $F P = r$ and $P Q = U$, and let ϕ be the angle which F P makes with the tangent to the first surface, χ the angle which P Q produced makes with the tangent to the second surface.

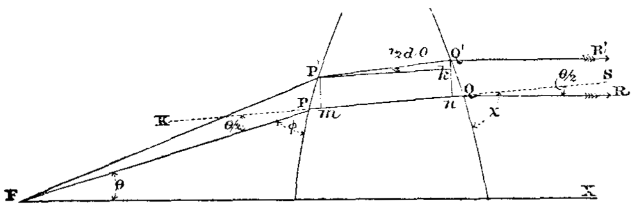
Then $\quad -d\,\mathrm{U} = \mathrm{P}\,m + \mathrm{Q}\,n;$

hence $\quad \mathrm{Q}\,n = -d\,\mathrm{U} - \mathrm{P}\,m.$

$$= -d\,\mathrm{U} - d\,s\cos\left(\phi + \frac{\theta}{2}\right)$$

$$= -d\,\mathrm{U} - d\,r\cos\frac{\theta}{2} + r\,d\,\theta\sin\frac{\theta}{2}.$$

Fig. 5.



Also $\quad \mathrm{Q}^1\,n = \mathrm{Q}^1\,\mathrm{K} + \mathrm{K}\,n = \mathrm{Q}^1\,\mathrm{K} + \mathrm{P}^1\,m,$

$$= \frac{1}{2}d\,\theta\,\mathrm{U} + d\,s\sin\left(\phi + \frac{\theta}{2}\right).$$

$$= \left(r\cos\frac{\theta}{2} + \frac{\mathrm{U}}{2}\right)d\,\theta + d\,r\sin\frac{\theta}{2}.$$

Hence, since

$$\tan\chi = \frac{\mathrm{Q}^1\,n}{\mathrm{Q}\,n},$$

$$\tan\chi = \frac{\left(r\cos\frac{\theta}{2} + \frac{\mathrm{U}}{2}\right)d\,\theta + \sin\frac{\theta}{2}\,d\,r}{r\sin\frac{\theta}{2}\,d\,\theta - \cos\frac{\theta}{2}\,d\,r - d\,\mathrm{U}}.$$

Again, by the law of refraction at the second surface, $\mu\cos\chi = \cos\left(\chi - \frac{\theta}{2}\right)$,

whence $\quad \tan\chi = \dfrac{\cos\frac{\theta}{2} - \mu}{\sin\frac{\theta}{2}};$

$$\therefore \qquad \sin\frac{\theta}{2}\left\{\left(r\cos\frac{\theta}{2}+\frac{U}{2}\right)d\,\theta+\sin\frac{\theta}{2}\,d\,r\right\}$$

$$+\left(\cos\frac{\theta}{2}-\mu\right)\left\{r\sin\frac{\theta}{2}\,d\,\theta-\cos\frac{\theta}{2}\,d\,r-d\,U\right\}=0;$$

$$\therefore \qquad \left(\mu-\cos\frac{\theta}{2}\right)\frac{d\,u}{d\,\theta}+\frac{1}{2}\sin\frac{\theta}{2}U+r\sin\theta-\frac{d\,r}{d\,\theta}\cos\theta$$

$$+\mu\cos\frac{\theta}{2}\frac{d\,r}{d\,\theta}-\mu\,r\sin\frac{\theta}{2}=0.$$

$$\therefore \qquad \frac{d}{d\,\theta}\left\{U\left(\mu-\cos\frac{\theta}{2}\right)\right\}-\frac{d}{d\,\theta}(r\cos\theta)+\mu\frac{d}{d\,\theta}\left(r\cos\frac{\theta}{2}\right)$$

$$-\frac{1}{2}\mu\,r\sin\frac{\theta}{2}=0;$$

but $\qquad r=\dfrac{C}{\left(\mu\cos\frac{\theta}{2}-1\right)^2}$ which leads to

$$U\left(\mu-\cos\frac{\theta}{2}\right)-r\cos\theta+\mu\,r\cos\frac{\theta}{2}-\frac{C}{\left(\mu\cos\frac{\theta}{2}-1\right)^2}=K,$$

where K is an arbitrary constant. Substituting, it now appears that

$$U\left(\mu-\cos\frac{\theta}{2}\right)=\frac{C\left(2\cos^2\frac{\theta}{2}-1\right)}{\left(\mu\cos\frac{\theta}{2}-1\right)^2}-\frac{\mu\,C\cos\frac{\theta}{2}}{\left(\mu\cos\frac{\theta}{2}-1\right)^2}+\frac{C}{\mu\cos\frac{\theta}{2}-1}+K$$

$$=\frac{C}{\left(\mu\cos\frac{\theta}{2}-1\right)^2}\left\{2\cos^2\frac{\theta}{2}-1-\mu\cos\frac{\theta}{2}+\mu\cos\frac{\theta}{2}-1\right\}+K;$$

whence $\qquad U=\dfrac{K}{\mu-\cos\frac{\theta}{2}}-\dfrac{2\,C\sin^2\frac{\theta}{2}}{\left(\mu-\cos\frac{\theta}{2}\right)\left(\mu\cos\frac{\theta}{2}-1\right)^2}.$

By means of this last equation it would be easy to construct the second curve point by point from the first curve. It must be noticed, however, that the second curve depends on the first, since its curvature involves the constant C. It has been shown how this constant is determined for any given problem. By properly determining the constant K of the second curve, which is possible when the angle which the lens is to subtend at the focus is given, the second curve can be drawn.

The Cartesian equations to the second surface are—

$$x = r\cos\theta + \mathrm{U}\cos\frac{\theta}{2};\ y = r\sin\theta + \mathrm{U}\sin\frac{\theta}{2};$$

that is—

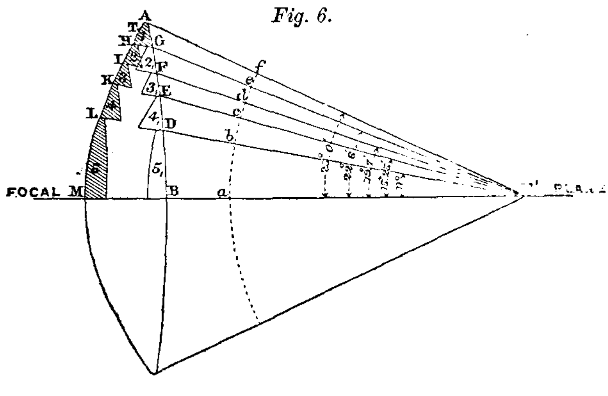
$$x = \frac{\mathrm{K}}{\mu - \cos\frac{\theta}{2}} + \frac{\mathrm{C}\left(\mu\cos\theta - \cos\frac{\theta}{2}\right)}{\left(\mu - \cos\frac{\theta}{2}\right)\left(\mu\cos\frac{\theta}{2} - 1\right)^2} \quad . \quad . \quad (2)$$

and

$$y = \frac{\mathrm{K}\sin\frac{\theta}{2}}{\mu - \cos\frac{\theta}{2}} + \frac{2\,\mathrm{C}\sin\frac{\theta}{2}}{\left(\mu - \cos\frac{\theta}{2}\right)\left(\mu\cos\frac{\theta}{2} - 1\right)^2} \quad . \quad . \quad (3)$$

With regard to the practical application of these equations, supposing the lens to be constructed is of the new form shown

Fig. 6.



hatched in *Fig. 6*, the equiangular curve of the outer face A M can be plotted from equations (2) and (3).

In the construction of the refractor two courses can be adopted; the first is to grind the outer face of each annulus, as A H, to the nearest circular arc corresponding to this part of the true equiangular curve. To do this it would be necessary to determine from equations (2) and (3) the Cartesian co-ordinates of the points A and H and of the point T midway between A and

H. Now it is desired to find the circle which passes through these three points. This is found by taking the general equation of a circle, $x^2 + y^2 + 2gx + 2fy + C = 0$, and determining the constants g, f and C, for the co-ordinates of the centre are $(-g, -f)$, while the radius of the required circular arc is $\sqrt{g^2 + f^2 - C}$. The second method is to consider the whole length A H I K L as a circular arc which approximates most nearly to this part of the true equiangular curve. This would give a common centre for each of the prisms 1, 2, 3 and 4. This circular arc could be arrived at in a manner precisely analogous to that described, and though this method is not so accurate, it possesses the advantage of greater simplicity and ease of construction. The inside faces of the prisms in this case would not be parts of equiangular curves, but would depend upon the outer circular arc. In this case the bull's-eye would be formed of separate circular arcs approximating to the equiangular curves of the outside and inside faces.

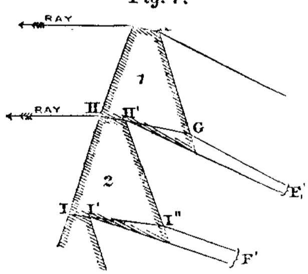
The inside curves of the prisms A′G, H′I″, I′K″, &c., have now to be considered. The point A′ being given, the equiangular curve A′G can be drawn from equation (1), and, by taking a point intermediate between A′ and G and finding the Cartesian co-ordinates of the three, it is possible, as in the case of the outer face, to determine the co-ordinates of the circular arc for the portion A′G of the inner face. The point H′ is fixed when the depth of arris H H′ has been determined. Reconstructing equation (1) to suit the new conditions the face H′I″ can be drawn; and again by finding co-ordinates as above the centre for prism No. 2 is determined, and so on with Nos. 3, 4 and 5.

The essential feature of the inverse equiangular form of refractor consists in the inversion of the facets of the lens elements, so that they project inwards instead of outwards, and thus leave the outer face smooth. This arrangement with any other form of prism but the equiangular would entail a loss of light at each arris, which, above 15° from the focal plane, would be very considerable. With the equiangular form of prism this loss is inappreciable up to 15°, and, as will be shown, is far more than compensated for by the great gain obtained by the increase of focal length, even when the inverse refractor is carried up to a total height above and below the focal plane of 30°, thus subtending a total vertical angle of 60°.

The new form of refractor being approximately spherical possesses all the advantages of the truly spherical form, while at the same time it has none of the disadvantages arising from abnormal divergences above 20°. By using this form the same power of light can be obtained from a smaller apparatus, thus reducing the actual cost, enabling smaller lanterns to be used, and reducing the weight of the revolving apparatus—a great consideration when quick-flashing lights are so much employed. The inverse lens has also the merit of utilizing to the fullest advantage the best part of the light. Being practically spherical in form on the outside it is impossible for the rings of this apparatus to fall inwards, hence the setting can be greatly simplified, eliminating loss of light, and reducing the weight of the apparatus.

The actual gain in power of this inverse lens remains to be shown. The profile with which it is compared is that in which the true equiangular curve is on the inside, from which the facets project outwards from the focus. This, as has been shown by Mr. Alan Brebner,[1] is 1·8 per cent. more powerful than the Fresnel section, and 15·9 per cent. than the spherical, the refractors under comparison being carried up to an angle of 31° above the focal plane.

Fig. 7.



Now in the comparison of the profile A B with A M it is seen that as the prisms of each are truly equiangular, their divergences will be altered only in respect of the greater focal distance of the prisms on A M as compared with those on A B. But in the case of those on A M there are losses of light at each arris. This loss is most easily computed when the area of the illuminated portion of a sphere subtended by the cone, the angle of which in the case before considered is 50°, is compared with the same surface of sphere minus the zones of partial darkness produced by the loss at the arrises, represented in *Fig. 7* by G F″ H′, I″ F″ I′, &c. Summing all these zones it will be found that the total area : reduced area :: 1·06 : 1; or, in this respect, the inverse refractor is 6 per cent. less powerful than that with which it is compared. To simplify the calculations, the areas of all the zones, *e d*, *d c* and *c d*, *Fig. 6*, have each been made equal to the area of the bull's-eye, and therefore equal to one another. The powers of the rings of prisms and bulls'-eyes can now be taken simply as the squares of their focal distances. Hence—

[1] Minutes of Proceedings Inst. C.E., vol. cxxii. p. 300.

$$\frac{\text{Power of } 5}{\text{Power of } 5_1} = \frac{1{,}134^2}{980^2} = 33 \cdot 9 \text{ per cent. more powerful.}$$

$$\frac{\text{Power of } 4}{\text{Power of } 4_1} = \frac{1{,}126^2}{1{,}009^2} = 24 \cdot 5 \quad ,, \quad ,, \quad ,,$$

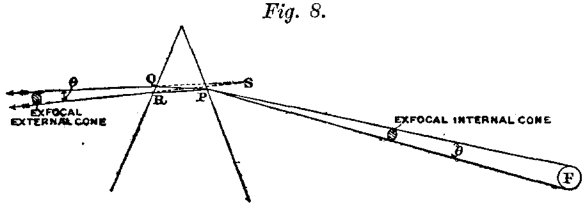
$$\frac{\text{Power of } 3}{\text{Power of } 3_1} = \frac{1{,}116^2}{1{,}038^2} = 15 \cdot 5 \quad ,, \quad ,, \quad ,,$$

$$\frac{\text{Power of } 2}{\text{Power of } 2_1} = \frac{1{,}112^2}{1{,}069^2} = 8 \cdot 2 \quad ,, \quad ,, \quad ,,$$

$$\frac{\text{Power of } 1}{\text{Power of } 1_1} = \frac{1}{1} = 0 \cdot 0 \quad ,, \quad ,, \quad ,,$$

Whence, taking the average, it is found that the whole inverse apparatus is 16·4 per cent. more powerful, but with the loss at the arrises, which amounts to 6 per cent., the total gain of the inverse equiangular refractor amounts to 10·4 per cent.

Fig. 8.



It may be remarked, in conclusion, that the abnormal divergences in the spherical refractor above 20° arises from the fact that all the work of refraction is performed by the outside face. In the Fresnel it is divided between the inside and outside faces, the latter, however, refracting more than the former. The work of each face in the case of the equiangular section has been shown to be equal, and thus a divergence smaller than either the Fresnel or the spherical is produced, *Fig. 8.* Now if the inside face be made to do more than the outside, the divergent cone will be reduced, thus producing a much more intense light. In the extreme case, when the inside face does all the work, and when the outside becomes perpendicular to the optic axis, it can be shown that at 30° the angle of the divergent cone will only be 2° 26′ 30″, the angle of cone inside being 5°. At 10° the angle of the external cone will have increased to 4° 40′ 0″, while at the optic axis itself it amounts to 7° 38′ 0″. It will be seen, therefore, that no advantage is derived by throwing all the work as far as 10°

upon the inside face, as at this point the external and internal angles of the divergent cones are practically the same; while below this angle, if the inside face performs all the work, the external cone actually exceeds in size the internal. Above 10° the angle is seen to diminish until at 30° it is practically halved. The disadvantage of such prisms would be that a considerable loss of light is sustained at each arris, for it can be seen at once that the under face of each prism must subtend a considerable angle at the focus, and so cause much loss of light.

The Paper is accompanied by four tracings, from which the *Figs.* in the text have been prepared.