Black-Scholes as a Resolution Geometry Theorem

Deriving Option Pricing from Membrane Tension Minimization

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Abstract

This companion paper demonstrates that the Black-Scholes partial differential equation emerges naturally from Resolution Geometry's framework of membrane tension minimization. The (S, t) price-time plane functions as a 2D scaffold; the option price surface V(S,t) is a fold over this scaffold; and the Black-Scholes PDE is the constrained tension-minimizing evolution (a gradient flow) for this surface under no-arbitrage conditions.

The key insight is that the option price surface behaves like a soap film (minimal surface), not a stiff plate. The system minimizes Delta-squared (gradient/tension), and Gamma (curvature) emerges as the reaction force. By transforming to logarithmic coordinates—the 'fundamental scaffold' where the geometry is flat—the Black-Scholes equation reveals itself as pure diffusion with drift, with all metric corrections vanishing.

This mapping suggests that financial derivatives pricing and gravitational physics share a common mathematical substrate: both are optimization problems on 2D manifolds with finite distinguishability capacity. Resolution Geometry is therefore not merely a framework for physics, but for any system characterized by finite capacity, conservation requirements, and cost minimization dynamics.

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1. The Structural Correspondence

Resolution Geometry (Connerty, 2026) derives physical laws from two premises: Exchange-Consistency (identity must be intrinsically encoded) and Finite Distinguishability Capacity (information density is bounded by surface area). The framework demonstrates that General Relativity emerges as the cost-minimization condition for maintaining a folded 2D scaffold.

Remarkably, the same mathematical structure appears in financial derivatives pricing. The correspondence is not metaphorical but structural:

Resolution Geometry	Black-Scholes
2D scaffold	(S, t) price-time plane
Fold function Φ	Option price surface $V(S,t)$
Fundamental frame (flat)	Log-coordinates ($x = \ln S$)
Emergent frame (curved)	Linear price coordinates (S)
Membrane tension $(\nabla\Phi)^2$	Delta squared $(\partial V/\partial S)^2$
Curvature (reaction)	Gamma ($\Gamma = \partial^2 V/\partial S^2$)
Gradient flow (evolution)	Tension-minimizing dynamics
Scaffold diffusivity	Volatility σ
Finite capacity bound	No-arbitrage constraint
Ticker rate c	Risk-free rate r
Receipt conservation	Delta hedging (Δ-neutral)

2. The Geometric Interpretation

2.1 The Price-Time Scaffold

In the Black-Scholes framework, an option's value V depends on two variables: the underlying asset price S and time t. This (S, t) plane is the financial analogue of Resolution Geometry's 2D scaffold.

The option price surface V(S, t) is a *fold* over this scaffold. Just as the physical 2D scaffold folds to create the illusion of 3D depth, the (S, t) plane 'folds' into V-space to encode the option's fair value at every point.

2.2 The Membrane Interpretation

A critical point: the option price surface behaves like a *soap film* (minimal surface), not a stiff plate. This distinction is mathematically essential.

Plate bending (minimizing curvature squared, H^2) yields fourth-order biharmonic equations. But Black-Scholes is second-order. The correct physical analogy is membrane tension minimization—the system minimizes the gradient squared (Delta^2), and curvature (Gamma) emerges as the reaction force.

This aligns with Layer 5 (Foam Optimization) of Resolution Geometry: the option price is the minimal-area surface connecting the boundary condition (payoff at expiry T) to the present. The surface resists tilting; Gamma is the inevitable byproduct of trying to keep a surface flat when the boundary conditions are convex.

2.3 Delta as Tension, Gamma as Reaction

In this corrected interpretation:

- **Delta ($\Delta = \partial V/\partial S$)** is the tension—the 'tilt' of the price surface. This is the cost being minimized.
- **Gamma ($\Gamma = \partial^2 V/\partial S^2$)** is the reaction force—the curvature required to balance tension when boundary conditions prevent flatness.
- **Volatility (σ)** is the diffusivity of the membrane—see Section 4.

The cost of maintaining a portfolio with delta exposure is real: non-zero delta means market exposure. Delta-hedging is literally trying to keep the surface flat (minimize tilt). The market *pays* for this tension, just as the scaffold *pays* for bending.

2.4 The Boundary Value Problem: Anchoring the Film

The 'soap film' analogy is not merely poetic; it describes a well-posed boundary value problem. The price surface is a minimal surface stretched between specific anchors:

Terminal Anchor (The Payoff): At expiry $t = T$, the surface is fixed to the payoff function:

$$V(x, T) = \text{Payoff}(e^x)$$

The evolution runs backward from this fixed boundary, smoothing out the 'kinks' in the payoff profile (e.g., the sharp corner of a call option at the strike) as time to expiry increases.

Asymptotic Anchors: As $S \to 0$ (or $x \to -\infty$), the option value vanishes or becomes its intrinsic value. As $S \to \infty$ (or $x \to +\infty$), the option behaves like the underlying asset (for calls) or approaches zero (for puts).

The Black-Scholes PDE is the 'equation of motion' for the film as it relaxes away from the sharp payoff boundary at T into the smooth surface observed at $t < T$.

3. No-Arbitrage as Finite Capacity

Resolution Geometry's second premise is Finite Distinguishability Capacity: information density is bounded by surface area. This manifests in physics as the holographic principle.

In finance, the operational manifestation of this capacity bound is *no-arbitrage*. Just as the holographic principle limits information density, the no-arbitrage condition limits value extraction density—there is a ceiling on how much profit can be extracted from a given position without taking on risk.

The risk-free rate r plays the role of the ticker rate c in Resolution Geometry: it is the maximum 'refresh rate' of value creation without violating the capacity bound. A delta-hedged portfolio must earn exactly r; earning more would violate the constraint, earning less would represent inefficiency.

4. Volatility as Scaffold Diffusivity

The volatility parameter σ defines the energetic cost of gradients on the scaffold.

In the energy functional $\mathcal{E}$, the term $\frac{1}{2}\sigma^2$ acts as the *stiffness modulus*: a high σ imposes a high penalty on 'tilt' (Delta). Gradients are expensive.

Dynamically, this high energetic stiffness manifests as high *diffusivity*. A stiff membrane snaps back to flatness faster than a loose one. High volatility means the scaffold aggressively smooths out local gradients, dissipating information more rapidly.

> **High σ:** High cost for Delta → Fast smoothing (High Diffusivity)
>
> **Low σ:** Low cost for Delta → Slow smoothing (Low Diffusivity)

In the log-scaffold, σ sets the diffusivity of tension relaxation—how quickly the surface smooths toward equilibrium. Just as Resolution Geometry does not derive the gravitational constant G from first principles, this framework does not derive σ. Both are empirical parameters characterizing the local properties of their respective scaffolds.

5. The Black-Scholes PDE as Tension-Minimizing Evolution

The Black-Scholes equation describes how the price surface evolves over time. Geometrically, this is not a static shape but a *gradient flow*: the surface is constantly relaxing toward the minimum-energy state (flatness) while simultaneously being driven by the scaffold's drift.

The Black-Scholes partial differential equation is:

$$\partial V/\partial t + \tfrac{1}{2}\sigma^2 S^2(\partial^2 V/\partial S^2) + rS(\partial V/\partial S) - rV = 0$$

Mathematically, this is the constrained gradient flow of the Dirichlet energy:

$$\partial V/\partial t = -\delta\mathcal{E}/\delta V - \mathcal{C}[V]$$

The surface 'slides' down the energy landscape defined by membrane tension, subject to the 'current' of the risk-neutral drift. The $\tfrac{1}{2}\sigma^2 S^2(\partial^2 V/\partial S^2)$ term is the tension relaxation—how the surface responds to accumulated gradient. The $\partial V/\partial t$ term is time evolution. The $rS(\partial V/\partial S) - rV$ terms encode drift under the risk-neutral measure.

This is precisely the structure of the Einstein field equations in Resolution Geometry: the geometry of the fold (curvature) must exactly match the stress required to maintain it (tension). In finance, the shape of the price surface must exactly match the hedging cost required to replicate it.

6. A Geometric Proof: The Log-Scaffold Derivation

To rigorously derive the Black-Scholes equation as a cost-minimization theorem, we must identify the 'fundamental scaffold' where the geometry is flat. In Resolution Geometry, observed distortions often arise from viewing the system in the wrong coordinate frame.

In finance, the 'natural' scaffold is not the linear price S, but the logarithmic price x = ln S. This is the frame where percentage changes are translation-invariant—the 'flat' geometry of the market.

6.1 The Fundamental Scaffold (x-Space)

Define the scaffold coordinate x on the 2D plane:

$$x = \ln S$$

In this coordinate system, the metric is flat (Euclidean). This choice is not arbitrary—it reflects the fundamental nature of multiplicative price dynamics. A 10% move is a 10% move regardless of whether the stock is at $10 or $100.

6.2 The Energy Functional (Dirichlet Action)

The option price V(x,t) minimizes the standard Dirichlet energy (membrane tension) on this flat scaffold. This represents the cost of carrying a 'tilted' portfolio (non-zero Delta).

The Action is:

$$\mathcal{E}[V] = \int \tfrac{1}{2}\mathcal{D}(\partial V/\partial x)^2 \, dx$$

Here, $\mathcal{D} = \tfrac{1}{2}\sigma^2$ is the scaffold diffusivity—the intrinsic 'stiffness' of the market medium, determined empirically by volatility. This is the hedging cost: the energy required to maintain a non-flat price surface.

6.3 The Variational Derivative

Minimizing this energy yields the standard tension-relaxation term:

$$\delta\mathcal{E}/\delta V = -\mathcal{D}(\partial^2 V/\partial x^2) = -\tfrac{1}{2}\sigma^2(\partial^2 V/\partial x^2)$$

This is the 'force' that flattens the surface—the system's tendency to eliminate gradients.

6.4 The Constraint (Frame Drift)

The scaffold is not static; it flows due to the risk-free rate and the geometric correction required by the log-transformation. The 'Drift Constraint' in the x-frame is the convective derivative required to maintain the risk-neutral measure:

$$\mathscr{C}[V] = (r - \tfrac{1}{2}\sigma^2)(\partial V/\partial x) - rV$$

Note: The $(r - \tfrac{1}{2}\sigma^2)$ drift term is the frame velocity required to maintain translation invariance for multiplicative dynamics. In stochastic calculus, this appears via Itô's lemma applied to geometric Brownian motion; geometrically, it is the necessary convective drift to preserve the martingale property on the logarithmic scaffold. This is not an ad hoc addition—it arises necessarily from the coordinate transformation.

6.5 The Evolution Equation (Gradient Flow)

The price surface evolves to minimize energy relative to the drifting frame:

$$\partial V/\partial t - \delta\mathscr{E}/\delta V + \mathscr{C}[V] = 0$$

Note on signs: Options are priced backward from expiry—the terminal condition is the payoff at T, and we solve backward to find present value. The 'gradient flow' therefore runs opposite to physical time.

Substituting terms:

$$\partial V/\partial t + \tfrac{1}{2}\sigma^2(\partial^2 V/\partial x^2) + (r - \tfrac{1}{2}\sigma^2)(\partial V/\partial x) - rV = 0$$

This is Black-Scholes in the fundamental (log) coordinates.

6.6 Recovery of Standard Black-Scholes

Transforming back to the emergent linear coordinates ($S = e^x$):

Using the chain rules $\partial/\partial x = S(\partial/\partial S)$ and $\partial^2/\partial x^2 = S^2(\partial^2/\partial S^2) + S(\partial/\partial S)$:

$$V\square + \tfrac{1}{2}\sigma^2(S^2V\_SS + SV\_S) + (r - \tfrac{1}{2}\sigma^2)SV\_S - rV = 0$$

Grouping the first-derivative terms:

$$\tfrac{1}{2}\sigma^2SV\_S + rSV\_S - \tfrac{1}{2}\sigma^2SV\_S = rSV\_S$$

The diffusion correction cancels the drift correction exactly. This is not coincidence—it reflects the self-consistency of the geometric framework. The 'metric corrections' from the coordinate transformation annihilate each other, leaving:

$$\boldsymbol{\partial V/\partial t + \tfrac{1}{2}\sigma^2S^2(\partial^2V/\partial S^2) + rS(\partial V/\partial S) - rV = 0}$$

This is the standard Black-Scholes equation, derived purely from membrane tension minimization on the logarithmic scaffold.

7. Implications

7.1 The Fundamental vs. Emergent Distinction

This derivation reveals a profound parallel with Resolution Geometry's core thesis. The framework argues that 'observed' 3D geometry is an emergent distortion of a flatter, fundamental 2D scaffold. The same is true in finance:

Fundamental scaffold: $x = \ln S$ — flat geometry, pure diffusion

Emergent space: S — curved, with 'mysterious' correction terms

The $\frac{1}{2}\sigma^2$ terms: Not arbitrary coefficients, but geometric artifacts of the coordinate transformation

Black-Scholes looks 'complicated' in S-coordinates because we are viewing a flat scaffold through distorted coordinates. In the fundamental frame (log-space), it is simply: tension minimization + drift = equilibrium.

7.2 Exotic Options as Non-Flat Scaffolds

Path-dependent options (Asian, barrier, lookback) correspond to pricing on *non-flat* scaffolds—where the (S, t) plane itself has intrinsic curvature induced by the path dependency. The additional complexity of pricing these options reflects the additional tension cost of maintaining folds over curved base manifolds.

7.3 Market Microstructure as Scaffold Resolution

The tick size and minimum price increment in financial markets are analogous to the Planck-scale discretization in Resolution Geometry. Below this scale, the scaffold cannot resolve distinct states. Market microstructure effects (bid-ask spread, price impact) may be interpretable as resolution-floor phenomena.

7.4 Volatility Smile as Scaffold Curvature

The famous 'volatility smile'—the empirical observation that implied volatility varies with strike price—may indicate that the real financial scaffold is not perfectly flat even in log-coordinates. Local curvature in the scaffold would manifest as strike-dependent effective volatility, just as local spacetime curvature manifests as gravitational effects in Resolution Geometry.

8. Summary of the Correspondence

The Black-Scholes equation is not merely analogous to Resolution Geometry—it is a *special case* of scaffold tension minimization:

Component	Interpretation
The (S, t) plane	A 2D scaffold
The option price V	A fold over the scaffold
Log-coordinates $x = \ln S$	The fundamental (flat) frame
Delta Δ	Membrane tension (cost being minimized)
Gamma Γ	Reaction force (curvature)
No-arbitrage	Finite capacity constraint
The Black-Scholes PDE	Constrained gradient flow for minimum tension

This suggests that Resolution Geometry is not merely a framework for physics, but a framework for *any system* characterized by finite distinguishability capacity, conservation requirements, and cost minimization dynamics.

Physics, finance, and potentially information theory, biology, and network topology may all be downstream consequences of the same constraint geometry.

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The scaffold folds to encode value.

The fold has tension.

The tension creates the price.

Black-Scholes minimizes the tension.

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9. Acknowledgments

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References

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