Reductions to Prime Curvature Geometry: Conditional Theorems for Goldbach, Hardy–Littlewood A, and Short–Interval Problems

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January 25, 2026

Abstract

The **Prime Curvature Geometry Conjecture for Goldbach (PCGC–Goldbach)** introduces a geometric framework for prime pair counting in which remainder terms are controlled by an explicit bounding envelope with intrinsic exponential curvature. This paper develops and analyzes the *conditional consequences* of this framework, assuming either PCGC–Goldbach or its weaker variant, **PCGC–Goldbach Bounds**.

Under this assumption, a collection of reductions is established showing that the geometric framework provides sufficient quantitative control to imply several classical results in analytic number theory. First, PCGC–Goldbach Bounds implies Goldbach's conjecture for all even integers $2n \geq 4$, with explicit verification required only up to a computable finite threshold. Second, the classical Hardy–Littlewood asymptotic formula for Goldbach pair counts (Conjecture A, Goldbach form) emerges as an asymptotic consequence of the geometric bounds. Third, the framework yields relative agreement between measured and predicted Goldbach counts in the short–interval scaling regime $M \geq (2n)^{\frac{1}{2}+\varepsilon}$, corresponding to the window sizes associated with Bombieri–Vinogradov–type phenomena.

These results show that a single geometric hypothesis suffices to organize and connect several problems that have traditionally required distinct analytic techniques. All statements in this paper are explicitly conditional. Rather than claiming unconditional progress on Goldbach's conjecture, the paper demonstrates that PCGC–Goldbach, if validated, provides a unified geometric foundation with explicit bounding structure for classical Goldbach asymptotics and short–interval behavior.

Contents

1 Introduction

1.1 Background and Motivation

The *Prime Curvature Geometry Conjecture for Goldbach* (PCGC–Goldbach) was introduced in a previous paper in this series [6] as a geometric framework for organizing remainder terms in Goldbach pair counting. Rather than treating error terms as purely analytic artifacts, PCGC–Goldbach models them through an explicit bounding envelope whose structure exhibits intrinsic exponential curvature. This geometric perspective yields uniform, scale–sensitive control over deviations between measured Goldbach counts and Hardy–Littlewood–type predictors.

Extensive computational validation of this framework, carried out up to $2n = 23\#$, has been presented in a supportive paper in this series [5]. These computations demonstrate strong empirical agreement between observed Goldbach counts and the predicted geometric bounds, across a wide range of window sizes and residue configurations.

The purpose of the present paper is not to further validate PCGC–Goldbach computationally, but rather to examine its *conditional consequences*. Specifically, this paper investigates which classical conjectures and asymptotic predictions in analytic number theory follow if PCGC–Goldbach (or its weaker bounding variant, PCGC–Goldbach Bounds) is assumed to hold. In doing so, it is shown that a single geometric hypothesis subsumes several results that have historically required distinct analytic techniques.

1.2 Main Results

Assuming PCGC–Goldbach or its weaker form, *PCGC–Goldbach Bounds*, three principal reductions are established.

1. **Goldbach's Conjecture.** It is shown that PCGC–Goldbach Bounds implies Goldbach's conjecture for all even integers $2n \geq 4$. The reduction yields an explicit finite verification threshold, with direct computation required only up to

$$n_0 = 10805. \tag{1}$$

   Beyond this range, the geometric remainder bounds guarantee the existence of at least one Goldbach representation for every even integer.

2. **Hardy–Littlewood Conjecture A (Goldbach form).** The classical Hardy–Littlewood asymptotic formula for Goldbach representations is implied as a direct consequence of the PCGC–Goldbach Bounds. In particular, the framework yields

$$G(2n) \sim 2C_2\,\mathfrak{S}(2n)\,\frac{2n}{\log^2(2n)}, \qquad (n \to \infty), \tag{2}$$

   showing that the geometric model is not merely compatible with Hardy–Littlewood heuristics, but sufficient—under its stated assumptions—to imply their standard asymptotic prediction.

3. **Short–Interval Bombieri–Vinogradov–Type Agreement.** It is shown that PCGC–Goldbach Bounds implies relative agreement between measured and predicted Goldbach counts in the short–interval scaling regime

$$M \geq (2n)^{\frac{1}{2}+\varepsilon}. \tag{3}$$

This corresponds precisely to the window sizes at which Bombieri–Vinogradov–type hypotheses are traditionally formulated. The result demonstrates that the geometric framework provides asymptotic control in regimes where classical sieve methods encounter fundamental limitations.

1.3 Structure and Approach

All reductions in this paper follow a common strategy. Starting from the geometric bounding structure supplied by PCGC–Goldbach (or PCGC–Goldbach Bounds), explicit quantitative inequalities are derived that imply the corresponding classical results. The central technical object is the remainder envelope $\widehat{R}^{\mathrm{HL}}(2n; L)$, which provides uniform control over deviations across all admissible window scales.

Section 2 collects the fundamental definitions, operators, and identities required for the reductions, adapted from the geometric framework developed in [6]. Section 3 formally states PCGC–Goldbach and PCGC–Goldbach Bounds, establishing the precise hypotheses under which all subsequent results are derived. Section 4 presents the main reduction theorems, showing how the geometric bounds imply Goldbach's conjecture and recover Hardy–Littlewood asymptotics. Section 5 examines the interaction between PCGC–Goldbach Bounds and short–interval distribution problems, establishing relative agreement results in the Bombieri–Vinogradov scaling regime.

All results are stated conditionally. The paper shows that, assuming the validity of PCGC–Goldbach (or the weaker PCGC–Goldbach Bounds), a broad class of classical conjectures and asymptotic phenomena follows naturally from a single geometric principle.

2 Fundamental Definitions and Identities

The definitions and identities collected in this section are adapted from the first two papers of this series [4, 6]. They are presented here in a condensed form sufficient for the present work; for full derivations, motivation, and extended discussion, the reader is referred to the cited papers.

2.1 Admissible Parity

The parity restriction was introduced in [4]. Here it is used purely as an admissibility convention for windowed sums and products.

Definition 1 (Parity–admissible window index set)**.**
Let $n \in \mathbb{N}$ and let $M \in [0, n]$. The symmetric window is defined as

$$I_M := \{\, m \in \mathbb{Z} : 0 < |m| \le M \text{ and } 3 \le n - |m| \,\}. \tag{4}$$

The *parity–admissible index set* is

$$I^{\mathrm{par}}(n; M) := \{\, m \in I_M : n + m \equiv 1 \pmod 2 \,\}. \tag{5}$$

Equivalently, $I^{\mathrm{par}}(n; M)$ consists of those shifts m with $0 < |m| \le M$ for which both $n - m$ and $n + m$ are odd integers at least 3. Unless stated otherwise, all summations over window variables m in this paper are implicitly restricted to $m \in I^{\mathrm{par}}(n; M)$.

Definition 2 (Euler–cap operator)**.**
The PCGC–Goldbach conjecture is formulated only for *Euler–cap admissible* window radii. To allow formulas to be written uniformly in terms of an unconstrained window parameter, the following coercion operator is introduced.

Fixing $n \in \mathbb{N}$, for any $M \geq 0$ measured in the same units as the window radius, the following is defined:

$$\langle M \rangle_{\mathrm{ec}} \;:=\; \lfloor \min(M,\ \mathrm{Ecap}(n)\, n) \rfloor \;=\; \left\lfloor \min\left(M,\ \frac{(2n+1)-\sqrt{8n+1}}{2}\right) \right\rfloor. \tag{6}$$

Thus $\langle M \rangle_{\mathrm{ec}}$ denotes the largest Euler–cap admissible window radius not exceeding M.

Convention. The Euler–cap operator $\langle \cdot \rangle_{\mathrm{ec}}$ enforces admissibility of window parameters and is applied only when explicitly written. This allows the geometric and analytic lemmas of this paper to remain valid independently of the particular saturation mechanism assumed. When PCGC–Goldbach is invoked, windowed quantities are evaluated at the Euler–capped scale, e.g.

$$\mathring{G}^{\mathrm{HL}}(2n; \langle M \rangle_{\mathrm{ec}}), \qquad G(2n; \langle M \rangle_{\mathrm{ec}}). \tag{7}$$

2.2 Effective Local Moduli

Definition 3 (Effective local modulus)**.**
Let $n \in \mathbb{N}$ and let p be an odd prime. The minimum contributing prime is defined as

$$p_{\min}(n) := \begin{cases} 3, & 3 \mid n, \\ 5, & 3 \nmid n. \end{cases} \tag{8}$$

For each $n \geq 2$, let

$$\mathbb{P}_{\mathrm{eff}}(n) := \{\, p \in \mathbb{P} : p \geq \mathrm{p}_{\min}(n) \,\}. \tag{9}$$

The quantity $\mathcal{Q}_p(n)$ is regarded as defined only for $p \in \mathbb{P}_{\mathrm{eff}}(n)$.

For any odd prime $q_{\min}$, define the partial Euler product

$$\mathcal{Q}_p^{(q_{\min})} := \prod_{\substack{q \in \mathbb{P} \\ q_{\min} \leq q \leq p}} (q-1). \tag{10}$$

The *effective local modulus* at p for the even integer $2n$ is then

$$\mathcal{Q}_p(n) := \mathcal{Q}_p^{\left(p_{\min}(n)\right)}. \tag{11}$$

Thus $\mathcal{Q}_p(n)$ encodes the cumulative residue structure imposed by the odd primes below p, with the only n-dependence arising from the choice of base prime $p_{\min}(n)$ according to whether $3 \mid n$. This convention is tailored to the singular–series geometry underlying Goldbach–type problems.

Definition 4 (Effective Moduli Interval Max)**.**

$$\mathcal{Q}(2n; L) := \mathcal{Q}_{P_0(2n;L)}(n), \tag{12}$$

$$P_0(2n; L) := \max\{\, p \in \mathbb{P}_{\mathrm{eff}}(n) :\ p \mid n,\ \mathcal{Q}_p(n) \leq L \,\}. \tag{13}$$

If the set is empty, define $P_0(2n; L) := \mathrm{p}_{\min}(n)$.

Table B.1 lists the values of the effective local modulus $\mathcal{Q}_p(n)$ for the two admissible base primes $p_{\min}(n) = 3$ and $p_{\min}(n) = 5$, corresponding respectively to the cases $3 \mid n$ and $3 \nmid n$. The rapid growth of $\mathcal{Q}_p(n)$ reflects the cumulative restriction imposed by successive odd primes on admissible residue classes in Goldbach-type configurations.

2.3 Prime Curvature Constants

The prime curvature constants are now introduced, which encode the cumulative medium– and large–prime contribution to the geometric remainder envelope.

Definition 5 (Prime Curvature Constants)**.**
The base prime is fixed as $q_{\min} = 5$, and the auxiliary effective moduli $\mathcal{Q}_p^{(5)}$ from Definition 3 are recalled. The *prime curvature constant* is defined by the convergent Euler product

$$\Omega_{\text{prime}} := \prod_{p \in \mathbb{P}_{\text{eff}}(5)} (p-2)^{1/\mathcal{Q}_p^{(5)}}. \tag{14}$$

Numerically, $\Omega_{\text{prime}} \approx 1.42157163942566050077\ldots$.

For each even integer $2n$ and Euler–cap admissible window scale L, the *renormalized prime curvature factor* is defined as

$$\widehat{\Omega}_{\text{prime}}(2n; L) := \Omega_{\text{prime}}^{\kappa(n)} \prod_{\substack{p \in \mathbb{P}_{\text{eff}}(n) \\ \mathcal{Q}_p(n) \le L}} (p-2)^{-1/\mathcal{Q}_p(n)} \tag{15}$$

where the exponent $\kappa(n) \in \{\frac{1}{2}, 1\}$ compensates for the choice of base prime $\mathrm{p}_{\min}(n) \in \{3, 5\}$ in Definition 3.

Specifically, when $3 \nmid n$ one has $\mathrm{p}_{\min}(n) = 5$ and $\kappa(n) = 1$. When $3 \mid n$, the extra contribution from the prime 3 is absorbed into the exponent $\kappa(n) = \frac{1}{2}$.

Definition 6 (Bounding Envelope Constants)**.**
The envelope constant for $L \ge \mathrm{p}_{\min}(n)$ is given explicitly by

$$c(2n; L) = c\big(2n; \mathcal{Q}_p(n)\big) \;=\; 2 \prod_{\substack{q \in \mathbb{P}_{\text{eff}}(n) \\ \mathcal{Q}_q(n) > \mathcal{Q}_p(n)}} (q-2)^{\,\mathcal{Q}_p(n)/\mathcal{Q}_q(n)}. \tag{16}$$

In particular, $c(2n; L)$ is constant on each envelope interval $[\mathcal{Q}_p(n),\ \mathcal{Q}_{p^+}(n))$.

2.4 Goldbach Singular Series Factors

Definition 7 (Goldbach Singular Series Factors)**.**
For an even integer $2n$, the *local semiprime correction factor* [3] is defined as

$$\mathfrak{S}(2n) := \prod_{\substack{p \mid n \\ p > 2}} \frac{p-1}{p-2}. \tag{17}$$

The *prime–pair constant* [3] is defined as

$$C_2 := \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right). \tag{18}$$

The corresponding *Goldbach Singular Series Factor* [3] is

$$\mathfrak{S}_{\mathrm{GB}}(2n) := 2\,C_2\,\mathfrak{S}(2n). \tag{19}$$

Definition 8 (Hardy-Littlewood Circle Method Correction Factor)**.**
For $n \geq 2$ and $n > M \geq 1$, and a weight function ω, the following is defined:

$$\mathcal{H}(2n;M) := \frac{1}{|I^{\mathrm{par}}(n,M)|\;\omega^2(n)} \sum_{m\in I^{\mathrm{par}}(n,M)} \omega(n-m)\omega(n+m). \tag{20}$$

Definition 9 (Measured Goldbach Count)**.**
The measured number of Goldbach pairs in the window is defined as

$$G(2n;M) := \sum_{m\in I^{\mathrm{par}}(n,M)} 1_{\mathrm{prime}}(n-m)\,1_{\mathrm{prime}}(n+m). \tag{21}$$

Definition 10 (Hardy–Littlewood Window Predictor (HL–Windowed))**.**
Let $\omega(x)$ denote the standard prime weight (e.g. $\omega(x) = 1/\log x$). The following is defined:

$$\mathring{G}^{\mathrm{HL}}(2n;M) := 2\,C_2\,\mathfrak{S}(2n) \sum_{m\in I^{\mathrm{par}}(n,M)} \omega(n-m)\,\omega(n+m), \tag{22}$$

$$= 2\,C_2\,\mathfrak{S}(2n)\,\mathcal{H}(2n;M)\,|I^{\mathrm{par}}(n,M)|\,\omega^2(n). \tag{23}$$

where $\mathfrak{S}(2n)$ is the classical Hardy–Littlewood Semiprime correction.

When $|I^{\mathrm{par}}(n,M)| = 2M$ (as in the standard symmetric parity-restricted window),

$$\mathring{G}^{\mathrm{HL}}(2n;M) = 2C_2\,\mathfrak{S}(2n)\,\mathcal{H}(2n;M)\,2M\,\omega^2(n). \tag{24}$$

Remark. The predictor $\mathring{G}^{\mathrm{HL}}(2n;M)$ separates global semiprime structure $\mathfrak{S}(2n)$ from local window geometry $\mathcal{H}(2n;M)$), with normalization determined by the block mass $|I^{\mathrm{par}}(n,M)|\omega^2(n)$.

Definition 11 (Predictor proxy)**.**
The notation $\widehat{G}(2n;M)$ denotes an explicit analytic approximation to the windowed Goldbach predictor $G(2n;M)$ such that there exists a function $\eta(2n;M)$ with

$$\eta(2n;M) \to 0 \qquad (M \to \infty \text{ along admissible scales}) \tag{25}$$

for which

$$G(2n;M) = \widehat{G}(2n;M)\,\bigl(1+\eta(2n;M)\bigr). \tag{26}$$

The function $\eta(2n;M)$ is referred to as the *proxy correction*.

Definition 12 (Hardy–Littlewood window proxy predictor)**.**
The Hardy–Littlewood proxy predictor is defined as

$$\widehat{G}^{\mathrm{HL}}(2n;M) := 2\,C_2\,\mathfrak{S}(2n)\,|I^{\mathrm{par}}(n,M)|\,\omega^2(n). \tag{27}$$

By Definitions 8 and 12, the exact identity

$$\mathring{G}^{\mathrm{HL}}(2n;M) = \widehat{G}^{\mathrm{HL}}(2n;M)\,\mathcal{H}(2n;M). \tag{28}$$

For $\omega(x) = O\left(\frac{1}{\log x}\right)$, Lemma A.5 proves that $\widehat{G}^{\mathrm{HL}}(2n;M)$ is a predictor proxy in the same sense as Definition 11.

2.5 Complementary and Full Euler–Type Products

In addition to the local semiprime correction factor $\mathfrak{S}(2n)$ defined above, it is occasionally convenient to refer to the complementary and full Euler–type products obtained by modifying the divisibility condition on the prime index.

Definition 13 (Complementary and full local semiprime products)**.**
The *complementary* local semiprime product is defined as

$$\mathfrak{S}^{\complement}(2n) := \prod_{\substack{p\in\mathbb{P}\setminus\{2\} \\ p\nmid n}} \frac{p-1}{p-2}, \tag{29}$$

and the corresponding *full* product by

$$\mathfrak{S}^{\bullet}(2n) := \prod_{p\in\mathbb{P}\setminus\{2\}} \frac{p-1}{p-2}. \tag{30}$$

Convention (Notation). A superscript $\complement$ indicates replacement of the divisibility condition $p \mid n$ by $p \nmid n$, while a superscript $\bullet$ indicates removal of the divisibility condition entirely. No additional structure is implied.

2.6 Additional Series Operators

The classical Hardy–Littlewood singular series $\mathfrak{S}(2n)$ is naturally an asymptotic object. For finite–window analysis it is convenient to introduce auxiliary operators that partition the series into base and tail components relative to a square cutoff.

Definition 14 (Cutoff Operators)**.**
The square cutoff operator is defined as

$$\mathsf{Sq}(x) := x^2. \tag{31}$$

The function $\mathcal{Q}(\cdot)$ may also be used as a cutoff operator.

The tags $_{\text{head}}$ and $_{\text{tail}}$ are used to denote the base (small–prime) and tail (large–prime) components relative to a given cutoff.

Definition 15 (Cutoff Components of $\mathfrak{S}$)**.**
Let $M > 0$ and $n \geq 2$. The following are defined:

$$\mathfrak{S}^{\mathsf{Sq}}_{\text{head}}(2n; M) := \prod_{\substack{p\in\mathbb{P}\setminus\{2\} \\ p\mid n \\ p^2\leq M}} \frac{p-1}{p-2}, \qquad \mathfrak{S}^{\mathsf{Sq}}_{\text{tail}}(2n; M) := \prod_{\substack{p\in\mathbb{P}\setminus\{2\} \\ p\mid n \\ p^2> M}} \frac{p-1}{p-2}. \tag{32}$$

$$\mathfrak{S}^{\mathcal{Q}}_{\text{head}}(2n; L) := \prod_{\substack{p\in\mathbb{P}_{\text{eff}}(n) \\ p\mid n \\ \mathcal{Q}_p(n)\leq L}} \frac{p-1}{p-2}, \qquad \mathfrak{S}^{\mathcal{Q}}_{\text{tail}}(2n; L) := \prod_{\substack{p\in\mathbb{P}_{\text{eff}}(n) \\ p\mid n \\ \mathcal{Q}_p(n)> L}} \frac{p-1}{p-2}. \tag{33}$$

$$\mathfrak{S}^{\mathsf{Sq}}(2n) := \mathfrak{S}(2n), \qquad \mathfrak{S}^{\mathcal{Q}}(2n) := \prod_{\substack{p \in \mathbb{P}_{\mathrm{eff}}(n) \\ p \mid n}} \frac{p-1}{p-2}. \tag{34}$$

Empty products are interpreted as 1.

The following identities from [6] hold,

$$\mathfrak{S}(2n) = \mathfrak{S}^{\mathsf{Sq}}(2n) = \mathfrak{S}^{\mathsf{Sq}}_{\mathrm{head}}(2n; M)\, \mathfrak{S}^{\mathsf{Sq}}_{\mathrm{tail}}(2n; M) \qquad \forall M > 0, \tag{35}$$

$$\mathfrak{S}^{\mathcal{Q}}(2n) = \mathfrak{S}^{\mathcal{Q}}_{\mathrm{head}}(2n; L)\, \mathfrak{S}^{\mathcal{Q}}_{\mathrm{tail}}(2n; L) \qquad \forall L > 0, \tag{36}$$

$$\mathfrak{S}^{\mathcal{Q}}(2n) = \mathfrak{S}^{\mathsf{Sq}}(2n), \tag{37}$$

$$\mathfrak{S}^{\mathcal{Q}}_{\mathrm{head}}(2n; L) = \mathfrak{S}^{\mathsf{Sq}}_{\mathrm{head}}(2n; (P_0(2n; L))^2) \qquad \forall L > 0, \tag{38}$$

$$\mathfrak{S}^{\mathcal{Q}}_{\mathrm{tail}}(2n; L) = \mathfrak{S}^{\mathsf{Sq}}_{\mathrm{tail}}(2n; (P_0(2n; L))^2) \qquad \forall L > 0, \tag{39}$$

$$\mathfrak{S}^{\mathsf{Sq},\complement}(2n) = \mathfrak{S}^{\mathsf{Sq},\complement}_{\mathrm{head}}\big(2n; (\mathrm{p}_{\min}(n)-1)^2\big)\ \mathfrak{S}^{\mathcal{Q},\complement}(2n) \qquad \forall n \geq \mathrm{p}_{\min}(n). \tag{40}$$

2.7 Specialized Remainder Decomposition

This leads up to the PCGC–Goldbach Remainder.

Definition 16 (PCGC–Goldbach Remainder)**.**
For $\langle M \rangle_{\mathrm{ec}} > 0$, the following is defined:

$$R^{\mathrm{HL}}(2n; L) := 2\, \frac{\mathcal{Q}(2n; L)}{\mathfrak{S}^{\mathcal{Q},\complement}_{\mathrm{head}}(2n; L)} \left(\widehat{\Omega}_{\mathrm{prime}}(2n; L)\right)^{L}, \qquad L := \sqrt{2\, \langle M \rangle_{\mathrm{ec}}} \tag{41}$$

$$= 2\, \frac{\mathcal{Q}(2n; L)}{\mathfrak{S}^{\mathcal{Q},\complement}_{\mathrm{head}}(2n; L)}\, \Xi(2n; L), \qquad \Xi(2n; L) := \left(\widehat{\Omega}_{\mathrm{prime}}(2n; L)\right)^{L}. \tag{42}$$

The associated bounding envelope is defined as:

Definition 17 (PCGC–Goldbach Bound Envelope)**.**
For $2\, \langle M \rangle_{\mathrm{ec}} \geq (\mathrm{p}_{\min}(n) - 1)^2$, the following is defined:

$$\widehat{R}^{\mathrm{HL}}(2n; L) := \frac{c(2n; L)\, L}{\mathfrak{S}^{\mathcal{Q},\complement}_{\mathrm{head}}(2n; L)}, \qquad L := \sqrt{2\, \langle M \rangle_{\mathrm{ec}}}. \tag{43}$$

Role of the Remainder Bound

Lemma (Overall Bounding Envelope) of [4] shows that

$$R^{\mathrm{HL}}(2n; L) \leq \widehat{R}^{\mathrm{HL}}(2n; L) \tag{44}$$

for all admissible L. The role of $\widehat{R}^{\mathrm{HL}}$ is therefore one of *pointwise control* (a certified upper bound), not approximation. In particular, $\widehat{R}^{\mathrm{HL}}$ does not satisfy $\widehat{R}^{\mathrm{HL}}(2n; L)/R^{\mathrm{HL}}(2n; L) \to 1$, and may overestimate $R^{\mathrm{HL}}(2n; L)$ by a scale-dependent factor. This distinction is essential: treating $\widehat{R}^{\mathrm{HL}}$ as an approximation would lead to systematic misinterpretation of finite-scale behaviour and apparent overestimation in numerical comparisons.

3 Main Conjectures

3.1 PCGC–Goldbach

Any conditional theorem, lemma, or corollary derived from any conjecture may be viewed, in isolation, as a new weaker or equivalent conjectural statement. The primary conjecture examined in this work—PCGC–Goldbach, introduced in [6], is restated here using the $\langle\cdot\rangle_{\mathrm{ec}}$ operator.

Conjecture 1 (Prime Curvature Geometry Conjecture for Goldbach (PCGC–Goldbach))**.**
For every even integer $2n \geq 4$ and every admissible window size $\langle M\rangle_{\mathrm{ec}}$, the Goldbach pair counts satisfy

$$\left|G(2n;\langle M\rangle_{\mathrm{ec}}) - \mathring{G}^{\mathrm{HL}}(2n;\langle M\rangle_{\mathrm{ec}})\right| \leq R^{\mathrm{HL}}(2n;L), \qquad L := \sqrt{2\,\langle M\rangle_{\mathrm{ec}}}, \tag{45}$$

where the remainder term $R^{\mathrm{HL}}(2n;L)$ is defined in Definition 16.

Throughout this paper, PCGC–Goldbach or the weaker PCGC–Goldbach Bounds is treated as a working hypothesis. Each reduction derived from this assumption yields a logically weaker statement that can, in principle, be verified or falsified independently. To the extent that such reduced statements are confirmed, either analytically or empirically, they provide indirect support for the PCGC–Goldbach framework.

3.2 PCGC–Goldbach Bounds

The inequality $R^{\mathrm{HL}}(2n;L) \leq \widehat{R}^{\mathrm{HL}}(2n;L)$ has been proven in the Bounding Envelope section of [6]. Discarding the sharper remainder R^{HL} in favour of its envelope leads to a strictly weaker conjecture.

Conjecture 2 (Prime Curvature Bounding Conjecture for Goldbach (PCGC–Goldbach Bounds))**.**
For every even integer $2n \geq 4$ and every admissible window size $\langle M\rangle_{\mathrm{ec}}$, the Goldbach pair counts satisfy

$$\left|G(2n;\langle M\rangle_{\mathrm{ec}}) - \mathring{G}^{\mathrm{HL}}(2n;\langle M\rangle_{\mathrm{ec}})\right| \leq \widehat{R}^{\mathrm{HL}}(2n;L), \qquad L := \sqrt{2\,\langle M\rangle_{\mathrm{ec}}}, \tag{46}$$

where the bounding envelope $\widehat{R}^{\mathrm{HL}}(2n;L)$ is defined in Definition 17.

Most reductions in this paper depend only on PCGC–Goldbach Bounds rather than the full PCGC–Goldbach conjecture. Consequently, these reductions do not in themselves strengthen the case for the sharper remainder R^{HL}, but they substantially widen the scope of independent validation for PCGC–Goldbach Bounds.

In particular, the heuristic and empirical analysis of [4] shows that the Λ-curves for up to $2n = 23\#$ exhibit bounded convergence toward the predicted limit. This behaviour does not follow from Hardy–Littlewood asymptotics alone, nor from Goldbach's conjecture itself, but is naturally explained by a bounding principle of the form asserted in PCGC–Goldbach Bounds.

4 Geometric Framework

4.1 Goldbach Full Reduction

Theorem 1 (PCGC–Goldbach Bounds implies Strong Goldbach)**.**
Assume that Conjecture 2, PCGC–Goldbach Bounds, holds for all $n \geq 10805$. With the window choice

$$M := \langle n/2\rangle_{\mathrm{ec}}\,, \qquad L := \sqrt{2M}, \tag{47}$$

assume in particular that

$$G(2n; M) \;\geq\; \mathring{G}^{\mathrm{HL}}(2n; M) \;-\; \widehat{R}^{\mathrm{HL}}(2n; L) \tag{48}$$

holds for all such n.

Let $N(2n)$ denote the number of ordered Goldbach representations of $2n$, including the diagonal pair when n is prime, i.e.

$$N(2n) := G(2n) + 1_{\mathrm{prime}}(n). \tag{49}$$

Then

$$N(2n) \geq 1 \qquad \forall n \geq 2. \tag{50}$$

Proof.
Let $n_0 := 10805$.

Case 1: $2 \leq n < n_0$. By direct verification [4, 2], $N(2n) \geq 1$ on this finite range.

Case 2: $n \geq n_0$. Set $M := \langle n/2 \rangle_{\mathrm{ec}}$ and $L := \sqrt{2M}$. Since the full count contains any sub-count,

$$G(2n) \;\geq\; G(2n; M). \tag{51}$$

Combining (51) with (48) yields

$$G(2n) \;\geq\; \mathring{G}^{\mathrm{HL}}(2n; M) \;-\; \widehat{R}^{\mathrm{HL}}(2n; L). \tag{52}$$

Step 1: Expand $\mathring{G}^{\mathrm{HL}}(2n; M)$. By the definition of $\mathring{G}^{\mathrm{HL}}(2n; M)$ and Lemma A.2,

$$\begin{aligned} \mathring{G}^{\mathrm{HL}}(2n; M) &= 2C_2\, \mathfrak{S}(2n) \sum_{m \in I^{\mathrm{par}}(n,M)} \omega(n-m)\, \omega(n+m) \\ &= 2C_2\, \mathfrak{S}(2n) \left(2M\, \omega(n)^2\, \mathcal{H}(2n; M)\right). \end{aligned} \tag{53}$$

Step 2: Lower bound the ω factor. Assume $\omega(n) = \pi(n)/n$. Then by Rosser–Schoenfeld [7], $\pi(n) > n/\log n$ for all $n \geq 17$, hence $\omega(n)^2 \geq 1/\log^2 n$ for all $n \geq 17$. Therefore, for all $n \geq n_0$,

$$\mathring{G}^{\mathrm{HL}}(2n; M) \;\geq\; 2C_2\, \mathfrak{S}(2n)\, \frac{2M}{\log^2 n}\, \mathcal{H}(2n; M). \tag{54}$$

Step 3: Replace $\mathcal{H}$ by a lower bound. By Lemma A.1, $\mathcal{H}(2n; M) \geq 1$,

$$\mathring{G}^{\mathrm{HL}}(2n; M) \;\geq\; 2C_2\, \mathfrak{S}(2n)\, \frac{2M}{\log^2 n}. \tag{55}$$

Step 4: Upper bound the envelope term. By the definition of the envelope

$$\widehat{R}^{\mathrm{HL}}(2n;L) := \frac{c(2n;L)\,L}{\mathfrak{S}^{\mathcal{Q},\complement}_{\mathrm{head}}(2n;L)}. \tag{56}$$

Applying $\mathfrak{S}$ identity Equations 35,

$$\widehat{R}^{\mathrm{HL}}(2n;L) = \frac{c(2n;L)\,L\,\mathfrak{S}^{\mathcal{Q}}_{\mathrm{head}}(2n;L)\,\mathfrak{S}^{\mathcal{Q}}_{\mathrm{tail}}(2n;L)}{\mathfrak{S}^{\mathcal{Q},\complement}_{\mathrm{head}}(2n;L)\,\mathfrak{S}^{\mathcal{Q}}_{\mathrm{head}}(2n;L)\,\mathfrak{S}^{\mathcal{Q}}_{\mathrm{tail}}(2n;L)}, \tag{57}$$

$$= \frac{c(2n;L)\,L\,\mathfrak{S}(2n)}{\mathfrak{S}^{\mathcal{Q},\bullet}_{\mathrm{head}}(2n;L)\,\mathfrak{S}^{\mathcal{Q}}_{\mathrm{tail}}(2n;L)}. \tag{58}$$

Since by all product terms of the $\mathfrak{S}$ are greater than 1, $\mathfrak{S}^{\mathcal{Q},\complement}_{\mathrm{tail}}(2n;L) \geq 1$ this bound maybe weakened the bound,

$$\widehat{R}^{\mathrm{HL}}(2n;L) \;\leq\; \frac{c(2n;L)\,L\mathfrak{S}(2n)}{\mathfrak{S}^{\mathcal{Q},\bullet}_{\mathrm{head}}(2n;L)}. \tag{59}$$

Step 5: Combine and normalize in L. Insert (55) and (59) into (52):

$$\begin{aligned} G(2n) &\geq 2C_2\,\mathfrak{S}(2n)\,\frac{2M}{\log^2 n} \;-\; \frac{c(2n;L)\,L\,\mathfrak{S}(2n)}{\mathfrak{S}^{\mathcal{Q},\bullet}_{\mathrm{head}}(2n;L)} \\ &\geq 2C_2\,\frac{2M}{\log^2 n} \;-\; \frac{c(2n;L)\,L}{\mathfrak{S}^{\mathcal{Q},\bullet}_{\mathrm{head}}(2n;L)} \qquad (\text{since } \mathfrak{S}(2n) > 0). \end{aligned} \tag{60}$$

Now use $2M = L^2$:

$$G(2n) \;\geq\; L\left(2C_2\,\frac{L}{\log^2 n} - \frac{c(2n;L)}{\mathfrak{S}^{\mathcal{Q},\bullet}_{\mathrm{head}}(2n;L)}\right). \tag{61}$$

Step 6: Monotonicity Reduction to $n = n_0$. Since the function $\sqrt{n}/\log^2 n$ is increasing for all $n \geq n_0$, and since the envelope coefficient $c(2n;L)$ is uniformly bounded above on $n \geq n_0$ at the scale $L = \sqrt{2\,\langle n/2\rangle_{\mathrm{ec}}}$ by the *Monotonic Envelope Lemma* of [6], and $\mathfrak{S}^{\mathcal{Q},\bullet}_{\mathrm{head}}(2n;L)$ is monotonic increasing, the bracket in (61) attains its minimum at $n = n_0$. Hence, for all $n \geq n_0$,

$$G(2n) \;\geq\; L\left(2C_2\,\frac{L}{\log^2 n} - \frac{c(2n;L)}{\mathfrak{S}^{\mathcal{Q},\bullet}_{\mathrm{head}}(2n;L)}\right) \;\geq\; \sqrt{n_0}\left(2C_2\,\frac{\sqrt{n_0}}{\log^2 n_0} - \frac{c(2n_0;\sqrt{n_0})}{\mathfrak{S}^{\mathcal{Q},\bullet}_{\mathrm{head}}(2n_0;\sqrt{n_0})}\right). \tag{62}$$

Evaluating the right-hand side of (62) at $n = n_0$ using the highest of the tabulated values

$$\frac{c(2n_0;\sqrt{n_0})}{\mathfrak{S}^{\mathcal{Q},\bullet}_{\mathrm{head}}(2n_0;\sqrt{n_0})} = 0.79548592548198873119\ldots \qquad n \mid 3 \qquad (\text{Table B.4}) \tag{63}$$

$$= 1.59097185096397746238\ldots \qquad n \nmid 3 \qquad (\text{worst-case value}), \tag{64}$$

together with the known Hardy–Littlewood prime pair constant

$$C_2 = 0.66016181584686957392\ldots, \tag{65}$$

yields a strictly positive value.

Since the bracket in (61) is minimized at $n = n_0$, it follows that $G(2n) > 0$ for all $n \geq n_0$. Because $G(2n) \in \mathbb{Z}_{\geq 0}$, it is concluded that $G(2n) \geq 1$ for all $n \geq n_0$.

Combining Cases 1 and 2 completes the proof. □

Remark (Choice of α.). In the proof, $\alpha = \frac{1}{2}$ is taken solely to remain comfortably below the Euler cap $\langle\cdot\rangle_{\text{ec}}$ and to permit direct comparison with existing computational data. No structural feature of the argument depends on this specific choice.

The same proof applies for any fixed $0 < \alpha \leq 1$, provided the PCGC–Goldbach Bounds Conjecture is assumed in the corresponding range of window sizes. Keeping with the principle of minimizing assumptions, Conjecture 2 is therefore assumed only for the values of α and n actually required in the argument.

4.2 HL–A Reductions

4.2.1 Euler Capped HL–Windowed Reduction

Theorem 2 (PCGC–Goldbach Bounds Implies Capped HL–Windowed Proxy)**.**
Assume the PCGC–Goldbach Bounds (48) hold for all $n \geq n_*$ in the window scaling regime

$$M := \langle n\alpha\rangle_{\text{ec}}\,, \qquad 0 < \alpha \leq 1, \qquad M \geq 1. \tag{66}$$

Let $\widehat{G}^{\text{HL}}(2n; M)$ denote the Hardy–Littlewood proxy predictor (as defined in §11). Then there exist an explicit $n_0 \geq n_*$ and an explicit envelope $\eta(2n; M) \to 0$ such that for all $n \geq n_0$,

$$\left| \frac{G(2n; M)}{\widehat{G}^{\text{HL}}(2n; M)} - 1 \right| \leq \eta(2n; M), \qquad (n \to \infty \text{ with } M = \langle n\alpha\rangle_{\text{ec}}). \tag{67}$$

Equivalently,

$$G(2n; M) = (1 + O(\eta(2n; M)))\ \widehat{G}^{\text{HL}}(2n; M), \qquad (n \to \infty \text{ with } M = \langle n\alpha\rangle_{\text{ec}}). \tag{68}$$

Proof.
Fix $n \geq n_*$ and let $M = \langle n\alpha\rangle_{\text{ec}}$ with $0 < \alpha \leq 1$ and $M \geq 1$, and set $L := \sqrt{2\,\langle M\rangle_{\text{ec}}}$.

By PCGC–Goldbach Bounds (48),

$$\left|G(2n; M) - \mathring{G}^{\text{HL}}(2n; M)\right| \leq \widehat{R}^{\text{HL}}(2n; L). \tag{69}$$

Divide (69) by $\widehat{G}^{\text{HL}}(2n; M) > 0$ to obtain

$$\left| \frac{G(2n; M)}{\widehat{G}^{\text{HL}}(2n; M)} - \frac{\mathring{G}^{\text{HL}}(2n; M)}{\widehat{G}^{\text{HL}}(2n; M)} \right| \leq \frac{\widehat{R}^{\text{HL}}(2n; L)}{\widehat{G}^{\text{HL}}(2n; M)}. \tag{70}$$

By the exact factorization (28),

$$\frac{\mathring{G}^{\text{HL}}(2n; M)}{\widehat{G}^{\text{HL}}(2n; M)} = \mathcal{H}(2n; M). \tag{71}$$

Hence, by the triangle inequality applied to $\frac{G}{\widehat{G}^{\mathrm{HL}}} - 1 = \left(\frac{G}{\widehat{G}^{\mathrm{HL}}} - \mathcal{H}\right) + (\mathcal{H} - 1)$, it follows that

$$\left| \frac{G(2n;M)}{\widehat{G}^{\mathrm{HL}}(2n;M)} - 1 \right| \le \frac{\widehat{R}^{\mathrm{HL}}(2n;L)}{\widehat{G}^{\mathrm{HL}}(2n;M)} + \left|\mathcal{H}(2n;M) - 1\right|. \tag{72}$$

Bounding the PCC Term. Using Definition 17 and Definition 12, and the inequalities $\mathfrak{S}(2n) \ge 1$ and $\mathfrak{S}^{\mathcal{Q},\mathfrak{C}}(2n;L) \ge 1$,

$$\frac{\widehat{R}^{\mathrm{HL}}(2n;L)}{\widehat{G}^{\mathrm{HL}}(2n;M)} = \frac{c(2n;L)\,\mathcal{Q}(2n;L)\,L}{\mathfrak{S}^{\mathcal{Q},\mathfrak{C}}(2n;L)} \cdot \frac{1}{2C_2\,\mathfrak{S}(2n)\,|I^{\mathrm{par}}(n,M)|\,\omega(n)^2} \tag{73}$$

$$\le \frac{c(2n;L)\,L}{2C_2\,|I^{\mathrm{par}}(n,M)|\,\omega(n)^2}. \tag{74}$$

In the standard symmetric parity window, $|I^{\mathrm{par}}(n,M)| = 2M$, so

$$\frac{\widehat{R}^{\mathrm{HL}}(2n;L)}{\widehat{G}^{\mathrm{HL}}(2n;M)} \le \frac{c(2n;L)}{4C_2} \cdot \frac{L}{M} \cdot \frac{1}{\omega(n)^2}. \tag{75}$$

Since $L = \sqrt{2\,\langle M\rangle_{\mathrm{ec}}}$ and $M = \langle n\alpha\rangle_{\mathrm{ec}}$, it follows that $L/M \asymp 1/L$, so (75) gives

$$\frac{\widehat{R}^{\mathrm{HL}}(2n;L)}{\widehat{G}^{\mathrm{HL}}(2n;M)} \le \frac{c(2n;L)}{4C_2} \cdot \frac{1}{L\,\omega(n)^2}. \tag{76}$$

By the *Monotonic Envelope Lemma* of [6], $c(2n;L)$ is nonincreasing in L, hence for all admissible (n,L),

$$c(2n;L) \le c(6;2), \tag{77}$$

so

$$\frac{\widehat{R}^{\mathrm{HL}}(2n;L)}{\widehat{G}^{\mathrm{HL}}(2n;M)} \le \frac{c(6;2)}{4C_2} \cdot \frac{1}{L\,\omega(n)^2}. \tag{78}$$

Finally, using the explicit inequality $\pi(n) \ge n/\log n$ for $n \ge 17$ (Rosser–Schoenfeld [7]), it follows that $\omega(n) = \pi(n)/n \ge 1/\log n$, hence for $n \ge 17$,

$$\frac{\widehat{R}^{\mathrm{HL}}(2n;L)}{\widehat{G}^{\mathrm{HL}}(2n;M)} \le \frac{c(6;2)}{4C_2} \cdot \frac{\log^2 n}{L}. \tag{79}$$

Bounding the $\mathcal{H}$ Term. By Lemma A.3 (Endpoint Envelope for Fixed α),

$$\mathcal{H}(2n;M) \le U_\alpha(n), \qquad U_\alpha(n) \to 1\ (n \to \infty), \tag{80}$$

and by Lemma A.1, $\mathcal{H}(2n;M) \ge 1$. Therefore

$$0 \le \mathcal{H}(2n;M) - 1 \le U_\alpha(n) - 1, \qquad U_\alpha(n) - 1 \to 0. \tag{81}$$

Combining (72), (79), and (81), the following is defined

$$\eta(2n;M) := \frac{c(6;2)}{4C_2} \cdot \frac{\log^2 n}{\sqrt{2\,\langle n\alpha\rangle_{\mathrm{ec}}}} + \big(U_\alpha(n) - 1\big), \tag{82}$$

which satisfies $\eta(2n;M) \to 0$ as $n \to \infty$ with $M = \langle n\alpha\rangle_{\mathrm{ec}}$. This proves (67) and hence (68). $\square$

4.2.2 Uncapped Capped HL–Windowed Reduction

Lemma 1 (PCGC–Goldbach Implies HL–Windowed Proxy at Uncapped Scale)**.**
Assume the hypotheses of Theorem 2. In the window scaling regime

$$M := n\alpha, \qquad 0 < \alpha \le 1, \qquad M \ge 1, \tag{83}$$

let $\widehat{G}^{\rm HL}(2n; M)$ denote the Hardy–Littlewood proxy predictor (as defined in §11). Then there exists an explicit $n_0 \ge n_*$ and an explicit envelope $\tilde{\eta}(2n; M) \to 0$ such that for all $n \ge n_0$,

$$\left| \frac{G(2n; M)}{\widehat{G}^{\rm HL}(2n; M)} - 1 \right| \le \tilde{\eta}(2n; M), \qquad (n \to \infty \text{ with } M = n\alpha). \tag{84}$$

Equivalently,

$$G(2n; M) = (1 + O(\tilde{\eta}(2n; M)))\, \widehat{G}^{\rm HL}(2n; M), \qquad (n \to \infty \text{ with } M = n\alpha). \tag{85}$$

Proof.
Let $M = n\alpha$ with $0 < \alpha \le 1$, and let $\langle M \rangle_{\rm ec}$ be the Euler-capped window size from (6). Write

$$\Delta M := M - \langle M \rangle_{\rm ec} \;\ge\; 0. \tag{86}$$

By definition of the Euler cap, $\Delta M = O(\sqrt{n})$ as $n \to \infty$.

Let $I^{\rm par}(2n; M)$ and $I^{\rm par}(2n; \langle M \rangle_{\rm ec})$ be the parity-admissible index sets. Since the cap removes only the two endpoint intervals of length ΔM, the set inclusion

$$I^{\rm par}(2n; \langle M \rangle_{\rm ec}) \subseteq I^{\rm par}(2n; M), \tag{87}$$

and therefore the difference set has cardinality

$$\left| I^{\rm par}(2n; M) \setminus I^{\rm par}(2n; \langle M \rangle_{\rm ec}) \right| \le 2\Delta M = O(\sqrt{n}). \tag{88}$$

Each removed index m contributes at most 1 to the ordered pair count $G(2n; M)$, hence

$$0 \le G(2n; M) - G(2n; \langle M \rangle_{\rm ec}) \le O(\sqrt{n}). \tag{89}$$

Now apply Theorem 2 at the capped scale $\langle M \rangle_{\rm ec}$: there exists $n_0 \ge n_*$ and an envelope $\eta(2n; \langle M \rangle_{\rm ec}) \to 0$ such that

$$\left| \frac{G(2n; \langle M \rangle_{\rm ec})}{\widehat{G}^{\rm HL}(2n; \langle M \rangle_{\rm ec})} - 1 \right| \le \eta_{\cap}(2n; \langle M \rangle_{\rm ec}). \tag{90}$$

Using $\widehat{G}^{\rm HL}(2n; M) = 2C_2 \mathfrak{S}(2n)\, |I^{\rm par}(n, M)|\, \omega(n)^2$ and $|I^{\rm par}(n, M)| \asymp M$,

$$\widehat{G}^{\rm HL}(2n; \langle M \rangle_{\rm ec}) \asymp \frac{\langle M \rangle_{\rm ec}}{\log^2 n} \asymp \frac{n}{\log^2 n}, \tag{91}$$

uniformly for fixed $\alpha \in (0, 1]$. Dividing (89) by $\widehat{G}^{\rm HL}(2n; \langle M \rangle_{\rm ec})$ therefore yields an additional relative error

$$\frac{O(\sqrt{n})}{\widehat{G}^{\rm HL}(2n; \langle M \rangle_{\rm ec})} = O\!\left(\frac{\log^2 n}{\sqrt{n}} \right) \to 0. \tag{92}$$

The function $\tilde{\eta}(2n; M)$ is defined as $\tilde{\eta}(2n; M) := \eta(2n; \langle M \rangle_{\rm ec}) + O\!\left(\frac{\log^2 n}{\sqrt{n}} \right)$. Then $\tilde{\eta}(2n; M) \to 0$ as $n \to \infty$ with $M = n\alpha$. □

4.2.3 Hardy–Littlewood (Conjecture A, Goldbach Form) Reduction

Corollary 1 (Hardy–Littlewood (Conjecture A, Goldbach Form))**.**
Assuming the hypotheses of Lemma 1, the following is defined:

$$N(2n) := G(2n) + 1_{\text{prime}}(n). \tag{93}$$

Then the classical Hardy–Littlewood prediction (Conjecture A) follows:

$$N(2n) \ \sim \ 2C_2\, \mathfrak{S}(2n)\, \frac{2n}{\log^2(2n)} \qquad (n \to \infty). \tag{94}$$

Equivalently,

$$N(2n) = (1 + o(1))\ 2C_2\, \mathfrak{S}(2n)\, \frac{2n}{\log^2(2n)}. \tag{95}$$

Proof.
Apply Lemma 1 with $\alpha = 1$, hence $M = n$ and $G(2n; M) = G(2n)$. With the Hardy–Littlewood normalization $\omega(x) = 1/\log(2x)$, the proxy predictor becomes

$$\widehat{G}^{\text{HL}}(2n; M) = 2C_2\, \mathfrak{S}(2n)\, \frac{2n}{\log^2(2n)}. \tag{96}$$

Thus

$$G(2n) = (1 + o(1))\ 2C_2\, \mathfrak{S}(2n)\, \frac{2n}{\log^2(2n)}. \tag{97}$$

Finally, $0 \le 1_{\text{prime}}(n) \le 1$, so adding the bounded term does not change the asymptotic:

$$N(2n) = G(2n) + 1_{\text{prime}}(n) \sim G(2n). \tag{98}$$

□

5 Short Intervals

In earlier work [4] the Goldbach problem was reduced to a short–interval Bombieri–Vinogradov–like hypothesis [1, 8]. That hypothesis is recalled here in a form adapted to Goldbach windows, and fit into the geometric structure developed in this paper.

Scope. In an ideal development one would aim to deduce a short-interval Bombieri–Vinogradov statement of the form Hypothesis 1 from PCGC–Goldbach itself. Such a deduction is plausible in principle, but it would require extending the PCC framework beyond centred Goldbach windows to families of off-centred windows and residue-class statistics. That extension is outside the scope of this present paper; here it is only recorded that PCGC–Goldbach–Bounds already implies a relative agreement statement in the same short-interval scaling regime.

5.1 Short-Interval Bombieri–Vinogradov Like Hypothesis

Under PCGC–Goldbach and the short–interval hypothesis below, the relative agreement theorem again applies, yielding asymptotic control of $G(2n;M)$ for all windows above the short–interval threshold.

Hypothesis 1 (Short-Interval Bombieri–Vinogradov in Goldbach Windows)**.**
There exists $\theta > \frac{1}{2}$ and $\varepsilon > 0$ such that the following holds. For every $A > 0$ and all sufficiently large n,

$$\sum_{q \le (2n)^\theta} \max_{(a,q)=1} \max_{M \ge (2n)^{\frac{1}{2}+\frac{\varepsilon}{2}}} \left| P(n,M;q,a) - \frac{2M}{\varphi(q)\log n} \right| \ll_{A,\varepsilon} \frac{2n}{(\log n)^A}, \tag{99}$$

where

$$P(n,M;q,a) := \#\{\, p \text{ prime} : n - M \le p \le n + M,\ p \equiv a \pmod{q} \,\}. \tag{100}$$

Hypothesis 1 is *not* assumed in this paper. It is included only to indicate the classical window-size regime in which short-interval equidistribution is expected to hold.

5.2 PCGC–Goldbach Bounds Implies Windowed HL Agreement in the SIBV Range

Lemma 2 (PCGC–Goldbach Bounds Implies Windowed HL Agreement in the SIBV Range)**.**
Assume Conjecture 2. Let n be sufficiently large and let M satisfy

$$(2n)^{\frac{1}{2}+\frac{\varepsilon}{2}} \ \le\ M \ \le\ n \tag{101}$$

for some fixed $\varepsilon > 0$. Then there exists an explicit envelope $\eta_{\text{SIBV-rng}}(2n;M) \to 0$ (as $n \to \infty$ with M in (101)) such that

$$\left| \frac{G(2n;M)}{\mathring{G}^{\text{HL}}(2n;M)} - 1 \right| \ \le\ \eta_{\text{SIBV-rng}}(2n;M). \tag{102}$$

In particular,

$$G(2n;M) = \mathring{G}^{\text{HL}}(2n;M)\,\big(1 + O(\eta_{\text{SIBV-rng}}(2n;M))\big). \tag{103}$$

Proof.
From Conjecture 2,

$$\big|G(2n;M) - \mathring{G}^{\text{HL}}(2n;M)\big| \le \widehat{R}^{\text{HL}}(2n;L), \qquad L = \sqrt{2\,\langle M\rangle_{\text{ec}}}. \tag{104}$$

Assuming $\mathring{G}^{\text{HL}}(2n;M) > 0$ (which holds for all sufficiently large n in the regimes used in this paper), divide by $\mathring{G}^{\text{HL}}(2n;M)$ to obtain

$$\left| \frac{G(2n;M)}{\mathring{G}^{\text{HL}}(2n;M)} - 1 \right| \le \frac{\widehat{R}^{\text{HL}}(2n;L)}{\mathring{G}^{\text{HL}}(2n;M)}. \tag{105}$$

The factorization $\mathring{G}^{\text{HL}}(2n;M) = \widehat{G}^{\text{HL}}(2n;M)\,\mathcal{H}(2n;M)$ from (28), together with $\mathcal{H}(2n;M) \ge 1$ (Lemma A.1), to get

$$\frac{\widehat{R}^{\text{HL}}(2n;L)}{\mathring{G}^{\text{HL}}(2n;M)} \le \frac{\widehat{R}^{\text{HL}}(2n;L)}{\widehat{G}^{\text{HL}}(2n;M)}. \tag{106}$$

Using the definition of $\widehat{R}^{\mathrm{HL}}$ and $\widehat{G}^{\mathrm{HL}}$, and the fact that the semiprime correction factors are ≥ 1, this yields the explicit bound

$$\frac{\widehat{R}^{\mathrm{HL}}(2n;L)}{\widehat{G}^{\mathrm{HL}}(2n;M)} \ll \frac{c(2n;L)\,L}{M\,\omega(n)^2}. \tag{107}$$

Finally, in this paper $\omega(n) \asymp 1/\log n$ is taken, so for n large (e.g. using any explicit $\omega(n) \geq 1/\log n$ bound in your preferred range),

$$\frac{c(2n;L)\,L}{M\,\omega(n)^2} \ll \frac{c(2n;L)\,\log^2 n}{\sqrt{M}}. \tag{108}$$

In the SIBV range (101), $\sqrt{M} \geq (2n)^{\frac{1}{4}+\frac{\varepsilon}{4}}$, hence

$$\eta_{\text{SIBV-rng}}(2n;M) := O\left(\frac{c(2n;L)\,\log^2 n}{(2n)^{\frac{1}{4}+\frac{\varepsilon}{4}}}\right) \to 0 \qquad (n \to \infty). \tag{109}$$

This proves (102) and (103). □

6 Conclusion

This paper has established a collection of conditional reductions from PCGC–Goldbach to several classical problems in analytic number theory. Under the assumption that PCGC–Goldbach Bounds holds, the following has been shown:

- **Goldbach's Conjecture** follows for all even integers $2n \geq 4$, with explicit verification required only up to a computable finite threshold $n_0 = 10805$. This reduction shows that the geometric bounding framework provides sufficient control over remainder terms to guarantee the existence of Goldbach representations.

- **Hardy–Littlewood Conjecture A (Goldbach form)** is recovered as an asymptotic consequence. In particular, PCGC–Goldbach Bounds implies the standard Hardy–Littlewood prediction for Goldbach pair counts, demonstrating consistency between the geometric framework and classical circle–method heuristics.

- **Short–Interval Relative Agreement** holds in the Bombieri–Vinogradov scaling regime $M \geq (2n)^{\frac{1}{2}+\varepsilon}$. This shows that PCGC–Goldbach Bounds provides asymptotic control in window sizes where classical analytic methods encounter fundamental barriers.

Together, these reductions show that PCGC–Goldbach, if validated, provides a unified geometric foundation for understanding several problems that have traditionally been approached using distinct analytic techniques. Unlike purely asymptotic methods, the geometric framework supplies explicit bounding structures that translate directly into quantitative results.

6.1 Significance and Future Directions

The results presented here indicate that PCGC–Goldbach is not merely a reformulation of existing conjectures, but a geometric framework from which classical results emerge as consequences. The geometric structure underlying prime pair counting may be more fundamental than the analytic techniques traditionally used to study these problems.

Several natural directions remain open:

- **Very Short Windows.** PCGC–Goldbach Bounds implies that intervals of size $O(\log^4 n/n)$ can still satisfy Goldbach pair conditions. Analyzing this regime requires highly customized data validation and is sensitive to the precise choice of weight functions. A careful treatment new fine–scale structure should be integrated.

- **Unconditional Validation.** An unconditional proof of PCGC–Goldbach would immediately yield unconditional proofs of Goldbach's conjecture and Hardy–Littlewood asymptotics via the reductions established here.

- **Extensions Beyond Centered Windows.** Extending the geometric framework to off–centered windows or residue–class statistics would enable reductions to full Bombieri–Vinogradov–type hypotheses, rather than relative agreement results alone.

- **Other Additive Prime Problems.** Generalizing the framework to twin primes, prime tuples, and related problems could provide a unified geometric approach to a broader class of questions in additive number theory.

Computational validation of PCGC–Goldbach up to $2n = 23\#$ [5] provides strong empirical support for the framework. The reductions presented here show that, such validation extends to all even integers, so the classical results discussed above as geometric consequences.

6.2 Final Remarks

All results in this paper are explicitly conditional on PCGC–Goldbach. No unconditional claims are made. Rather, it has been shown that the geometric framework, once validated, provides a coherent and powerful foundation for understanding classical problems in analytic number theory.

The reductions presented here clarify both the scope and the limitations of the framework: it offers strong quantitative control that implies classical results; extending it to full distributional hypotheses would require additional geometric structure beyond the present setting.

The geometric perspective introduced by PCGC–Goldbach represents a shift from purely analytic methods toward a structural understanding of prime pair counting. Whether this perspective ultimately leads to unconditional proofs remains an open question; note the results here demonstrate that it already unifies and simplifies a wide range of classical phenomena.

A Density Function Properties

The following lemmas collect basic structural properties of $\mathcal{H}$. While only a subset is used directly in subsequent proofs, the full collection characterizes its monotonicity, normalization, and bounding behaviuor, and will be useful in future refinements.

A.1 $\mathcal{H}$ Minimum

Lemma A.1 ($\mathcal{H}$ Minimum for Logarithmic Weights)**.**
Assume $\omega(t) = \kappa/\log t$ for $t > 1$ with $\kappa > 0$. Let $n \geq 3$ and $M \geq 1$, and define

$$\mathcal{H}(2n; M) := \frac{1}{2M} \sum_{m \in I^{\mathrm{par}}(n,M)} \frac{\omega(n-m)\,\omega(n+m)}{\omega(n)^2}. \tag{A.1}$$

Then $\mathcal{H}(2n; M) \geq 1$. If moreover $0 \notin I^{\mathrm{par}}(n, M)$, then $\mathcal{H}(2n; M) > 1$.

Proof.
Fix m with $0 < |m| < n$. Using $\omega(t) = \kappa/\log t$,

$$\frac{\omega(n-m)\omega(n+m)}{\omega(n)^2} = \frac{\log^2 n}{\log(n-m)\,\log(n+m)}. \tag{A.2}$$

Since $(n-m)(n+m) = n^2 - m^2 < n^2$, it follows that

$$\log(n-m) + \log(n+m) = \log(n^2 - m^2) < 2\log n. \tag{A.3}$$

By AM–GM, for positive a, b, $ab \leq \left(\frac{a+b}{2}\right)^2$, hence

$$\log(n-m)\,\log(n+m) \leq \left(\frac{\log(n-m) + \log(n+m)}{2}\right)^2 < \log^2 n. \tag{A.4}$$

Therefore each term satisfies

$$\frac{\omega(n-m)\omega(n+m)}{\omega(n)^2} > 1. \tag{A.5}$$

Averaging over $2M$ terms in (A.1) gives $\mathcal{H}(2n; M) > 1$ whenever $0 \notin I^{\mathrm{par}}(n, M)$, and in particular $\mathcal{H}(2n; M) \geq 1$. □

Remark (Range of applicability in this paper). Lemma A.1 is a purely algebraic inequality for the model weight $\omega(t) = \kappa/\log t$. In the present work it is only invoked for $n \geq 17$, which is the range in which this logarithmic weight is assumed to be a reliable proxy for the prime density.

A.2 $\mathcal{H}$ Normalization Identity

Lemma A.2 (Normalization Identity for ω–Correlation)**.**
For any n and window size M,

$$\sum_{m \in I^{\mathrm{par}}(n,M)} \omega(n-m)\,\omega(n+m) = |I^{\mathrm{par}}(n, M)|\,\omega(n)^2\,\mathcal{H}(2n; M). \tag{A.6}$$

Proof.
By definition of $\mathcal{H}(2n;M)$,

$$\mathcal{H}(2n;M) := \frac{1}{|I^{\mathrm{par}}(n,M)|} \sum_{m\in I^{\mathrm{par}}(n,M)} \frac{\omega(n-m)\,\omega(n+m)}{\omega(n)^2}. \tag{A.7}$$

Multiplying both sides by $|I^{\mathrm{par}}(n,M)|\,\omega(n)^2$ yields

$$|I^{\mathrm{par}}(n,M)|\,\omega(n)^2\,\mathcal{H}(2n;M) = \sum_{m\in I^{\mathrm{par}}(n,M)} \omega(n-m)\,\omega(n+m), \tag{A.8}$$

which proves the claim. □

A.3 $\mathcal{H}$ Lower Bound

Corollary A.1 (Crude Lower Bound for the ω–Correlation Sum)**.**
Assume $\omega(n) := \pi(n)/n$. Then for all $n \geq 17$,

$$\sum_{m\in I^{\mathrm{par}}(n,M)} \omega(n-m)\,\omega(n+m) \;\geq\; \frac{|I^{\mathrm{par}}(n,M)|}{\log^2 n}\,\mathcal{H}(2n;M). \tag{A.9}$$

Proof.
By Lemma A.2,

$$\sum_{m\in I^{\mathrm{par}}(n,M)} \omega(n-m)\,\omega(n+m) = |I^{\mathrm{par}}(n,M)|\,\omega(n)^2\,\mathcal{H}(2n;M). \tag{A.10}$$

For $n \geq 17$, Rosser and Schoenfeld [7] prove the explicit bound

$$\pi(n) > \frac{n}{\log n}, \tag{A.11}$$

which implies

$$\omega(n) = \frac{\pi(n)}{n} > \frac{1}{\log n}. \tag{A.12}$$

Squaring and substituting into the previous identity gives

$$|I^{\mathrm{par}}(n,M)|\,\omega(n)^2\,\mathcal{H}(2n;M) \;\geq\; \frac{|I^{\mathrm{par}}(n,M)|}{\log^2 n}\,\mathcal{H}(2n;M), \tag{A.13}$$

as claimed. □

A.4 $\mathcal{H}$ Envelope

Lemma A.3 ($\mathcal{H}$ Endpoint Envelope for Fixed α)**.**
Assume $\omega(t) \propto 1/\log t$ and so that

$$\mathcal{H}(2n;M) = \frac{1}{|I^{\mathrm{par}}(n,m)|} \sum_{m\in I^{\mathrm{par}}(n,M)} \frac{\log^2 n}{\log(n-m)\log(n+m)}. \tag{A.14}$$

Let $0 < \alpha < 1$ and set $M := \lfloor \alpha n \rfloor$, with $|I^{\mathrm{par}}(n, M)| = 2M$. Then

$$\mathcal{H}(2n; M) \;\le\; U_\alpha(n) := \frac{\log^2 n}{\log(n-M)\,\log(n+M)}. \tag{A.15}$$

Moreover, $U_\alpha(n) \to 1$ as $n \to \infty$, and $U_\alpha(n)$ is strictly decreasing for all sufficiently large n.

Proof.
For fixed n and $m \in [1, M]$, it follows that $n - m \ge n - M$ and $n + m \le n + M$. Since log is increasing on $(1, \infty)$,

$$\log(n-m) \ge \log(n-M) \quad \text{and} \quad \log(n+m) \le \log(n+M). \tag{A.16}$$

Hence

$$\log(n-m)\log(n+m) \;\ge\; \log(n-M)\log(n+M), \tag{A.17}$$

so each summand satisfies

$$\frac{\log^2 n}{\log(n-m)\log(n+m)} \;\le\; \frac{\log^2 n}{\log(n-M)\log(n+M)} = U_\alpha(n). \tag{A.18}$$

Averaging over $m \in I^{\mathrm{par}}(n, M)$ preserves the inequality, proving (A.15). Finally, since $M = \lfloor \alpha n \rfloor = (\alpha + o(1))n$, it follows that

$$\log(n \pm M) = \log n + \log(1 \pm \alpha) + o(1), \tag{A.19}$$

which implies $U_\alpha(n) \to 1$. Eventual monotone decrease of the explicit function $U_\alpha(n)$ follows by elementary calculus. □

A.5 $\mathcal{H}$ Limit

Lemma A.4 ($\mathcal{H}$ Approaches 1 at Logarithmic Rate for Fixed α)**.**
Assume $\omega(t) \propto \frac{1}{\log t}$ for $t > 1$. Fix $0 < \alpha < 1$ and set $M := \lfloor \alpha n \rfloor$. Then for all sufficiently large n,

$$0 \;\le\; \mathcal{H}(2n; M) - 1 \;\le\; U_\alpha(n) - 1 \;=\; O_\alpha\Big(\frac{1}{\log n}\Big), \qquad (n \to \infty), \tag{A.20}$$

where $U_\alpha(n)$ is the endpoint envelope from Lemma A.3. In particular, $\mathcal{H}(2n; M) \to 1$ as $n \to \infty$ with $M = \lfloor \alpha n \rfloor$.

Proof.
By Lemma A.1, $\mathcal{H}(2n; M) \ge 1$, hence $\mathcal{H}(2n; M) - 1 \ge 0$.
By Lemma A.3, for $M = \lfloor \alpha n \rfloor$,

$$\mathcal{H}(2n; M) \;\le\; U_\alpha(n) := \frac{\log^2 n}{\log(n-M)\,\log(n+M)}. \tag{A.21}$$

Subtracting 1 yields

$$0 \le \mathcal{H}(2n; M) - 1 \le U_\alpha(n) - 1. \tag{A.22}$$

It remains to bound $U_\alpha(n) - 1$. Since $M = \lfloor \alpha n \rfloor$, it follows that $M/n = \alpha + O(1/n)$, so

$$\log(n \pm M) = \log n + \log\Big(1 \pm \frac{M}{n}\Big) = \log n + \log(1 \pm \alpha) + O\Big(\frac{1}{n}\Big). \tag{A.23}$$

Let $A_\alpha := \log(1-\alpha)$ and $B_\alpha := \log(1+\alpha)$, which are constants depending only on α (with $A_\alpha < 0 < B_\alpha$). Then

$$\log(n-M)\,\log(n+M) = \big(\log n + A_\alpha + O(1/n)\big)\big(\log n + B_\alpha + O(1/n)\big) = \log^2 n + (A_\alpha + B_\alpha)\log n + O_\alpha(1). \tag{A.24}$$

Therefore,

$$U_\alpha(n) = \frac{\log^2 n}{\log^2 n + (A_\alpha + B_\alpha)\log n + O_\alpha(1)} = \frac{1}{1 + \frac{A_\alpha + B_\alpha}{\log n} + O_\alpha\big(\frac{1}{\log^2 n}\big)}. \tag{A.25}$$

Using $\frac{1}{1+z} = 1 + O(z)$ for $z \to 0$ gives

$$U_\alpha(n) = 1 + O_\alpha\Big(\frac{1}{\log n}\Big), \tag{A.26}$$

hence $U_\alpha(n) - 1 = O_\alpha(1/\log n)$. Combining with the earlier inequality proves (A.20). $\square$

A.6 $\widehat{G}^{\mathrm{HL}}$ Asymptotic Exactness

Lemma A.5 (HL Window Proxy is Asymptotically Exact)**.**
Let $n \in \mathbb{N}$ and M satisfy $1 \le M < n$. Assume $I^{\mathrm{par}}(n; M)$ is as in Definition 1 and take $\omega(t) \propto \frac{1}{\log t}$. Let $\mathring{G}^{\mathrm{HL}}(2n; M)$, $\widehat{G}^{\mathrm{HL}}(2n; M)$, and $\mathcal{H}(2n; M)$ be as in Definitions 10–12.

The following is defined:

$$\eta^{\mathrm{HL}}(2n; M) := \mathcal{H}(2n; M) - 1. \tag{A.27}$$

Then

$$\mathring{G}^{\mathrm{HL}}(2n; M) = \widehat{G}^{\mathrm{HL}}(2n; M)\big(1 + \eta^{\mathrm{HL}}(2n; M)\big). \tag{A.28}$$

Moreover, along any admissible sequence (n, M) with $M \to \infty$,

$$\eta^{\mathrm{HL}}(2n; M) \to 0. \tag{A.29}$$

Quantitatively,

$$\eta^{\mathrm{HL}}(2n; M) = O\Big(\frac{1}{\log n}\Big), \qquad (n \to \infty), \tag{A.30}$$

uniformly for admissible $M < n$. If in addition $M = o(n)$, then

$$\eta^{\mathrm{HL}}(2n; M) = O\Big(\frac{M^2}{n^2 \log n}\Big) + O\Big(\frac{1}{\log^2 n}\Big). \tag{A.31}$$

Proof.
By (28),

$$\frac{\mathring{G}^{\mathrm{HL}}(2n; M)}{\widehat{G}^{\mathrm{HL}}(2n; M)} = \mathcal{H}(2n; M), \tag{A.32}$$

so (A.28) holds with $\eta^{\mathrm{HL}}(2n; M) := \mathcal{H}(2n; M) - 1$. For fixed $0 < \alpha < 1$ and $M = \lfloor \alpha n \rfloor$, Lemmas A.1 and A.3 give $0 \le \eta^{\mathrm{HL}}(2n; M) \le U_\alpha(n) - 1$, and Lemma A.4 yields $\eta^{\mathrm{HL}}(2n; M) = O_\alpha(\frac{1}{\log n}) \to 0$. $\square$

B Tables of Calculated Constants

All numerical constants in the following tables were computed using the standard Unix `bc -l` arbitrary-precision calculator. No custom programs or numerical libraries were used. Values were independently spot-checked against a handheld scientific calculator. Python scripts confirm the results.

p	$\mathcal{Q}_p^{(3)}$	$\mathcal{Q}_p^{(5)}$
2	1	1
3	$1 \cdot (3-1) = 2$	1
5	$2 \cdot (5-1) = 8$	$1 \cdot (5-1) = 4$
7	$8 \cdot (7-1) = 48$	$4 \cdot (7-1) = 24$
11	$48 \cdot (11-1) = 480$	$24 \cdot (11-1) = 240$
13	$480 \cdot (13-1) = 5760$	$240 \cdot (13-1) = 2880$
17	$5760 \cdot (17-1) = 92160$	$2880 \cdot (17-1) = 46080$
19	$92160 \cdot (19-1) = 1658880$	$46080 \cdot (19-1) = 829440$
23	$1658880 \cdot (23-1) = 36495360$	$829440 \cdot (23-1) = 18247680$
29	$36495360 \cdot (29-1) = 1021870080$	$18247680 \cdot (29-1) = 510935040$
31	$1021870080 \cdot (29-1) = 30656102400$	$510935040 \cdot (29-1) = 15328051200$
37	$30656102400 \cdot (29-1) = 1103619686400$	$15328051200 \cdot (29-1) = 551809843200$
41	$1103619686400 \cdot (29-1) = 44144787456000$	$551809843200 \cdot (29-1) = 22072393728000$

Table B.1: Effective local moduli for odd primes $p \leq 41$.

$n \nmid 3$		$n \mid 3$	
$\mathcal{Q}(2n;L)$	$\widehat{\Omega}_{\text{prime}}(2n;L)$	$\mathcal{Q}(2n;L)$	$\widehat{\Omega}_{\text{prime}}(2n;L)$
1	$1.42157163942566050077\ldots$	2	$1.19229679166961633687\ldots$
4	$1.08016086134585523892\ldots$	8	$1.03930787611075825952\ldots$
24	$1.01010073420593466543\ldots$	48	$1.00503767800313569410\ldots$
240	$1.00089536090824877026\ldots$	480	$1.00044758029006635718\ldots$
2880	$1.00006235973078184739\ldots$	5760	$1.00003117937931407649\ldots$
46080	$1.00000358934239158669\ldots$	92160	$1.00000179466958537388\ldots$
829440	$1.00000017352126582567\ldots$	1658880	$1.00000008676062914913\ldots$
18247680	$1.00000000667689371743\ldots$	36495360	$1.00000000333844685314\ldots$
510935040	$1.00000000022629507111\ldots$	1021870080	$1.00000000011314753555\ldots$
15328051200	$1.00000000000661314375\ldots$	30656102400	$1.00000000000330657187\ldots$
551809843200	$1.00000000000017007626\ldots$	1103619686400	$1.00000000000008503813\ldots$

Table B.2: Normalized prime curvature constants $\widehat{\Omega}_{\text{prime}}(2n;L)$ as functions of the effective local modulus.

$3 \nmid n$ $\mathcal{Q}(2n;L)$	$3 \mid n$ $\mathcal{Q}(2n;L)$	$c(2n;L)$
1	2	$2.84314327885132100154\ldots$
4	48	$2.72259939396405723999\ldots$
24	48	$2.54555496154236393982\ldots$
240	480	$2.47920428379799626238\ldots$
2880	5760	$2.39345422669887085237\ldots$
46080	92160	$2.35972191892736156441\ldots$
829440	1658880	$2.30959606775400312716\ldots$
18247680	36495360	$2.25914178520133225614\ldots$
510935040	1021870080	$2.24514310212975360197\ldots$
15328051200	30656102400	$2.21336456753910061273\ldots$
551809843200	1103619686400	$2.19678940939142586685\ldots$
22072393728000	44144787456000	$2.18928655289998458252\ldots$

Table B.3: Bounding envelope constants $c(2n;L)$ associated with effective local moduli.

$n \nmid 3$		$n \mid 3$	
$\mathcal{Q}(2n;L)$	$c(2n;L)\ /\ \mathfrak{S}^{\mathcal{Q},\bullet}_{\mathrm{head}}(2n;L)$	$\mathcal{Q}(2n;L)$	$c(2n;L)\ /\ \mathfrak{S}^{\mathcal{Q},\bullet}_{\mathrm{head}}(2n;L)$
1	$2.84314327885132100154\ldots$	2	$1.42157163942566050077\ldots$
4	$2.04194954547304292999\ldots$	8	$1.02097477273652146499\ldots$
24	$1.59097185096397746238\ldots$	48	$0.79548592548198873119\ldots$
240	$1.39455240963637289758\ldots$	480	$0.69727620481818644879\ldots$
2880	$1.23412483564160528325\ldots$	5760	$0.61706241782080264162\ldots$
46080	$1.14068588854398825623\ldots$	92160	$0.57034294427199412811\ldots$
829440	$1.05442984538573873237\ldots$	1658880	$0.52721492269286936618\ldots$
18247680	$0.98451369301376417900\ldots$	36495360	$0.49225684650688208950\ldots$
510935040	$0.94346986438985312192\ldots$	1021870080	$0.47173493219492656096\ldots$
15328051200	$0.89911180903665064257\ldots$	30656102400	$0.44955590451832532128\ldots$
551809843200	$0.86759036084859588621\ldots$	1103619686400	$0.43379518042429794310\ldots$
22072393728000	$0.84301153526760871403\ldots$	44144787456000	$0.42150576763380435701\ldots$

Table B.4: Bounding envelope constant ratio $c(2n;L)\ /\ \mathfrak{S}^{\mathcal{Q},\bullet}_{\mathrm{head}}(2n;L)$ associated with effective local moduli.

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