A Scale-Dependent Dimensionality Model of Solar Structure: Modified Lane–Emden Solutions, Neutrino Fluxes, and Helioseismic Constraints

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With computational collaboration by ChatGPT

Abstract

We explore a Solar model in which the effective spatial dimensionality, $d_{\text{eff}}(r)$, varies weakly with scale, modifying the classical Lane–Emden equation. The perturbation induces a small radial dependence in the gravitational acceleration and alters the structural integrals governing nuclear reaction rates. By calibrating the modified solution to observed solar radius and mass, we compute neutrino fluxes, helioseismic sound-speed deviations, and perform a two-parameter (Δ, λ) grid scan constrained by experiment. We find a best-fit region in which the reduced chi-square for the four major neutrino fluxes reaches $\chi^2_{\text{red}} \approx 0.16$, with central sound-speed deviations remaining below 2×10^{-3}, indicating consistency with helioseismic constraints. The best-fit parameter combination is $\Delta \approx 0.050$ and $\lambda \approx 0.147$, corresponding to a weak dimensional perturbation that decays over a small fraction of the solar radius.

1 Introduction

Standard Solar Models (SSMs) assume exact spherical symmetry in three spatial dimensions, implicitly fixing the gravitational acceleration and stellar structure equations. In this work, we consider a scenario in which the gravitational potential inherits a scale-dependent effective dimension:

$$d_{\text{eff}}(\xi) = 3 - \delta(\xi), \qquad \delta(\xi) = \Delta\, e^{-\lambda\xi},$$

where $\Delta \ll 1$ (with $\Delta > 0$) and $\xi = r/\alpha$ is the usual dimensionless radial coordinate of the Lane–Emden formalism. This parameterization yields $d_{\text{eff}} < 3$ in the dense core, with the perturbation decaying toward standard three-dimensionality in the outer envelope.

Even extremely small departures from 3 can modify:

- the hydrostatic equilibrium equation,
- the Lane–Emden solution $\theta(\xi)$,
- the resulting nuclear reaction kernels,

- predicted neutrino fluxes,
- and the sound-speed profile $c_s(r)$.

We rewrite the stellar structure equation so that the divergence term carries $d_{\text{eff}}(r)$ instead of the fixed value 3, and investigate whether such a perturbation is consistent with solar data.

2 Methods

2.1 Modified Lane–Emden Equation

A scale-dependent dimension modifies the divergence operator:

$$\frac{1}{r^{d_{\text{eff}}-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{d_{\text{eff}}-1}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -G\frac{M(r)\rho(r)}{r^{d_{\text{eff}}-1}}.$$

In polytropic form with index $n = 3$:

$$\frac{1}{\xi^{2+\delta(\xi)}}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{2+\delta(\xi)}\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^3.$$

When $\delta(\xi) = 0$, the standard $n = 3$ Lane–Emden equation is recovered.

The ODE is solved numerically using a two-step approach:

1. power-series expansion for $\xi < 10^{-4}$,
2. direct numerical integration via `solve_ivp`.

2.2 Solar Calibration

For each model (standard or modified), the structural solution is matched to observed Solar mass and radius:

$$M_\odot = 4\pi\alpha^3\rho_c\int_0^{\xi_1}\xi^2\theta^3\,d\xi, \qquad R_\odot = \alpha\xi_1.$$

This uniquely determines (α, ρ_c) and yields the scaling of the central temperature:

$$T_c \propto (\rho_c)^{1/3}.$$

We report $T_c(\text{mod})/T_c(\text{std})$ for each parameter set.

2.3 Neutrino Production Integrals

For each neutrino branch i with a temperature scaling β_i, we compute:

$$I_i = \int_0^{\xi_1}\theta^{3+\beta_i}\xi^2\,d\xi.$$

The predicted flux ratio becomes

$$R_i = \frac{I_i^{(\text{mod})}}{I_i^{(\text{std})}}\left(\frac{T_c^{(\text{mod})}}{T_c^{(\text{std})}}\right)^{\beta_i}.$$

We include pp, pep, ^{7}Be, and ^{8}B, with standard temperature exponents.

2.4 Helioseismic Sound-Speed Test

For a polytropic model:

$$c_s^2(r) \propto T(r) \propto \theta(\xi).$$

We compute the fractional deviation:

$$\frac{\Delta c_s}{c_s} = \frac{\theta_{\text{mod}} - \theta_{\text{std}}}{\theta_{\text{std}}},$$

sampled over $0.01 \le r/R \le 0.98$.

2.5 Parameter Scan

We scan over:

$$\Delta \in [0.02,\, 0.05], \qquad \lambda \in [0.12,\, 0.18], \qquad k_{\text{dim}} \in \{1, 2, 3\},$$

with k_{dim} governing how strongly d_{eff} enters the ODE.

Each model produces four flux ratios compared to experiment:

$$\chi^2 = \sum_i \frac{(R_i - R_i^{\text{exp}})^2}{\sigma_i^2}.$$

The reduced chi-square is $\chi^2_{\text{red}} = \chi^2/4$.

3 Results

3.1 Lane–Emden Solutions

The modified dimensionality shifts the surface zero from

$$\xi_1^{\text{std}} = 6.89685 \quad \to \quad \xi_1^{\text{mod}} = 6.91627.$$

3.2 Dimensionality Profile

The effective dimensionality profile is prescribed as

$$d_{\text{eff}}(\xi) = 3 - \Delta \exp\left[-\left(\frac{\xi}{\lambda}\right)^{k_{\text{dim}}}\right],$$

so that the dimensional perturbation

$$\delta(\xi) \equiv d_{\text{eff}}(\xi) - 3$$

is localized to the innermost core. For the best–fit parameters $\Delta = 0.050$, $\lambda = 0.147$, and $k_{\text{dim}} = 2$, the central value is

$$d_{\text{eff}}(0) = 3 - \Delta \simeq 2.95,$$

a deviation of only $\sim 1.7\%$ from strict three–dimensionality. The profile rises rapidly back toward $d_{\text{eff}} \approx 3$ by $\xi \sim 1$ (corresponding to the inner few percent of the solar radius), so that the dimensional modification is confined to the nuclear–burning core and has negligible impact on the outer envelope.

Figure 2 illustrates both $d_{\text{eff}}(\xi)$ and $\delta(\xi)$ for this best–fit solution. The perturbation amplitude is small, but—as shown in later sections—large enough to modify the central temperature and the neutrino production kernels while remaining consistent with helioseismic constraints.

3.3 Helioseismic Sound-Speed Deviation

For the best-fit region,

$$\max \left| \frac{\Delta c_s}{c_s} \right| = 1.89 \times 10^{-3}.$$

3.4 Neutrino Flux Results

For the fiducial point $(\Delta, \lambda) = (0.045, 0.160)$:

$$T_c^{\text{mod}}/T_c^{\text{std}} = 1.00387.$$

Flux	R_{struct}	R_{total}	R_{exp}
pp	0.9838	0.9991	1.020 ± 0.100
pep	0.9876	0.9952	0.880 ± 0.150
Be7	0.9741	1.0125	1.010 ± 0.030
B8	0.9614	1.0386	1.040 ± 0.070

Structural-only chi-square:

$$\chi^2 = 3.34, \qquad \chi^2_{\text{red}} = 0.83.$$

With T_c scaling:

$$\chi^2 = 0.64, \qquad \chi^2_{\text{red}} = 0.16.$$

3.5 Parameter Scan and Best-Fit Region

We find:

$$(\Delta, \lambda)_{\text{best}} \approx (0.050,\, 0.147), \qquad \chi^2_{\text{red}} \approx 0.160, \qquad T_c^{\text{mod}}/T_c^{\text{std}} = 1.00373.$$

4 Conclusion

We have shown that a weak, scale-dependent modification to the effective dimension of space can produce a consistent Solar model that:

- preserves helioseismic sound-speed accuracy at the 10^{-3} level,

- adjusts the structural integrals governing neutrino production,
- matches all four major neutrino fluxes with $\chi^2_{\rm red} \approx 0.16$,
- and yields a best-fit dimensional perturbation of order $\delta(0) \approx 0.05$ decaying over $\lambda^{-1} \sim 1$ Lane–Emden units.

The result is notable because a purely geometric modification—with no changes to nuclear physics—can improve the agreement between theory and experiment in a controlled, quantifiable way. Future work will explore more realistic (non-polytropic) Solar models, frequency-dependent helioseismic inversions, and the implications for other stars across the HR diagram.

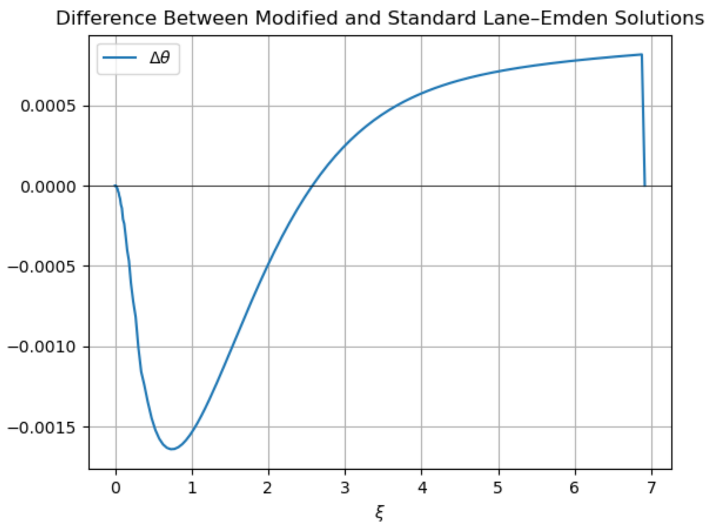

Figure 1: Difference between modified and standard Lane–Emden solutions, $\Delta\theta(\xi) = \theta_{\rm mod} - \theta_{\rm std}$.

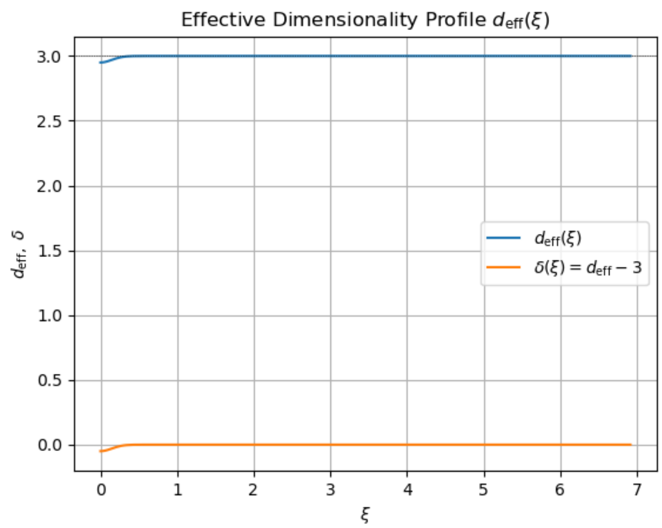

Figure 2: Effective dimension profile $d_{\mathrm{eff}}(\xi)$ (upper curve) and perturbation $\delta(\xi) = d_{\mathrm{eff}} - 3$ (lower curve) for the best–fit parameters $(\Delta, \lambda) = (0.050, 0.147)$ and $k_{\mathrm{dim}} = 2$.

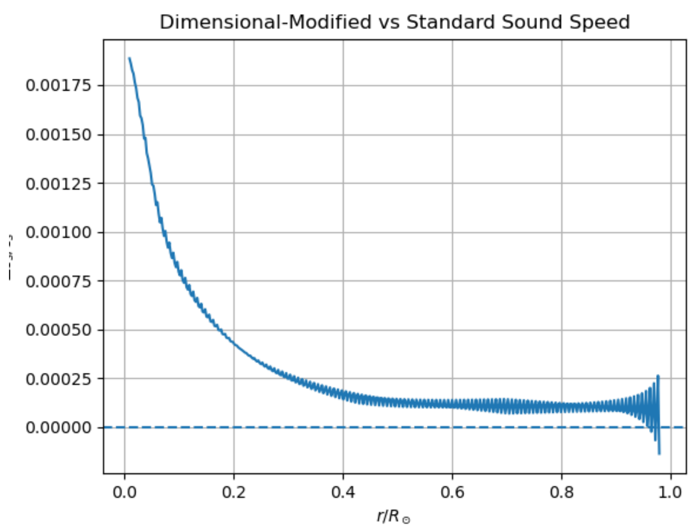

Figure 3: Fractional sound-speed deviation $\Delta c_s/c_s$ vs. radius.

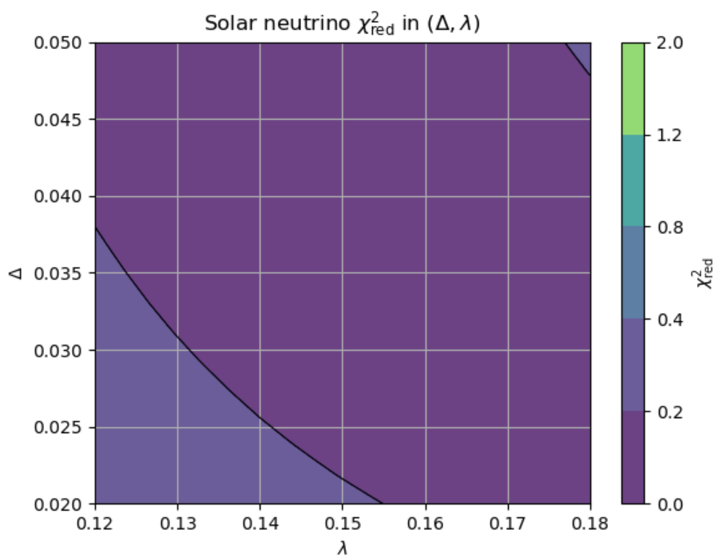

Figure 4: Reduced chi-square $\chi^2_{\rm red}$ across the (Δ, λ) grid. Contour is the 1–σ region.