APPENDIX A: NUMERICAL IMPLEMENTATION AND SELF-CONSISTENT SOLUTION

The cosmological evolution of the Plastic Spacetime Fossil (PSF) model requires solving a coupled system in which the background expansion rate ($H(z)$) determines the integration measure for the scalar memory field ($\chi(z)$), while $\chi(z)$ in turn fixes the dark energy density ($\rho_f(\chi)$) that sources $H(z)$. This feedback renders the problem nonlinear. We solve it using a self-consistent shooting method.

1. The Coupled System in Redshift Space

The fundamental memory evolution equation in cosmic time (t) is given by Eq. (9),

$$\dot{\chi} + \frac{\chi}{\tau} = S(t). \tag{A1}$$

To integrate this equation alongside standard cosmological background solvers, we transform to redshift space using

$$dt = -\frac{dz}{H(z)(1+z)}.$$

The memory equation becomes a first-order ordinary differential equation for $\chi(z)$,

$$\frac{d\chi}{dz} = -\frac{1}{H(z)(1+z)}\left[S(z) - \frac{\chi(z)}{\tau}\right]. \tag{A2}$$

The Hubble parameter is determined by the Friedmann equation, including radiation, pressureless matter, and the evolving fossil dark energy density,

$$H(z) = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \frac{\rho_f(\chi(z))}{\rho_{\mathrm{crit},0}}}. \tag{A3}$$

The fossil energy density is specified by the saturating potential,

$$\rho_f(z) = \rho_\Lambda\left(1 - e^{-\chi(z)/\chi_*}\right), \tag{A4}$$

where ρ_Λ sets the asymptotic energy scale.

2. Boundary Conditions and the Shooting Method

The system defined by Eqs. (A2)–(A4) constitutes a boundary value problem.

Initial condition. We integrate from deep in the matter-dominated era, choosing $z_{\mathrm{ini}} = 10$. The initial condition

$$\chi(z_{\mathrm{ini}}) = 0$$

reflects the assumption that the integrated fossil memory is negligible prior to the onset of significant nonlinear structure formation, including star formation and compact-object mergers.

Boundary constraint. The solution must reproduce the observed present-day dark energy density,

$$\rho_f(z=0) = \Omega_{\mathrm{DE},0}\rho_{\mathrm{crit},0} \simeq 0.69\rho_{\mathrm{crit},0}. \tag{A5}$$

Because the source history $S(z)$ (Eq. 18) contains an overall normalization S_{norm}, which encapsulates the effective yield volume (β), this normalization is not a free parameter but an eigenvalue of the coupled system. We determine it iteratively via a shooting procedure:

1. Choose an initial guess for S_{norm}.

2. Integrate the coupled ODE (A2) from $z_{\rm ini}$ to $z = 0$ using a stiff, adaptive solver (e.g., LSODA), which is required when $\tau \ll H_0^{-1}$.

3. Compute the residual
$$\Delta = \rho_f(z=0)_{\rm calc} - \Omega_{\rm DE,0}\rho_{\rm crit,0}.$$

4. Update $S_{\rm norm}$ using a Newton–Raphson root-finding algorithm.

5. Iterate until convergence is achieved, defined by $|\Delta| < 10^{-5}$.

3. Computing the Effective Equation of State

Once a converged solution for $\chi(z)$ and $H(z)$ is obtained, the effective equation of state $w_{\rm eff}(z)$ is computed directly from the continuity equation, avoiding numerical differentiation of noisy data.

Using the relation
$$w_{\rm eff} = -1 + \frac{1+z}{3\rho_f}\frac{d\rho_f}{dz},$$
and substituting $d\chi/dz$ from Eq. (A2), we obtain the numerically stable expression used for the forecasts in Sec. IV,
$$w_{\rm eff}(z) = -1 - \frac{\Upsilon(z)}{3H(z)\rho_f(z)}\left[S(z) - \frac{\chi(z)}{\tau}\right], \tag{A6}$$
where
$$\Upsilon(z) \equiv \frac{\rho_\Lambda}{\chi_*}e^{-\chi(z)/\chi_*}$$
is the analytic derivative of the saturating potential.

This formulation guarantees that the sign of $1 + w_{\rm eff}$ is numerically exact and is controlled entirely by the sign of the balance term $[S - \chi/\tau]$, directly reflecting the underlying Plasticity Sign Law.

APPENDIX B: MULTI-MESSENGER DIMENSIONAL ANALYSIS AND YIELD SCALING

In Sec. III, we introduced the phenomenological identification
$$S(z) = \beta\,\mathcal{R}_{\rm GW}(z),$$
which links the coarse-grained fossil source rate to the compact binary merger rate density. In this appendix, we demonstrate the dimensional consistency of this mapping and derive the expected scaling of the effective coupling parameter β from Schwarzschild geometry.

1. Dimensional Consistency

The evolution of the memory field is governed by Eq. (9),
$$\dot{\chi} = S - \frac{\chi}{\tau}.$$

We adopt a normalized field definition in which χ is dimensionless, measuring the accumulated deformation relative to the saturation scale χ_*. The source term $S(t)$ therefore must carry units of inverse time,
$$[S] = {\rm T}^{-1}. \tag{B1}$$

The gravitational-wave merger rate density $\mathcal{R}_{\rm GW}$ is defined as the number of merger events per unit comoving volume per unit source-frame time, with dimensions

$$[\mathcal{R}_{\rm GW}] = \mathrm{L}^{-3}\mathrm{T}^{-1}. \tag{B2}$$

Consistency of the linear mapping $S = \beta\mathcal{R}_{\rm GW}$ then requires the coupling constant β to have dimensions of volume,

$$[\beta] = \mathrm{L}^3. \tag{B3}$$

Physically, β represents the effective spacetime yield volume associated with a single merger event, weighted by the efficiency of fossilization.

2. Microphysical Scaling of the Yield Volume

We may estimate the magnitude of β by examining the geometry of the yield region surrounding a compact-object merger. Plasticity is triggered when the local curvature invariant

$$K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

exceeds the critical threshold K_c.

For a Schwarzschild black hole of mass M, the Kretschmann scalar scales as

$$K(r) = \frac{48G^2M^2}{c^4r^6} = K_{\rm hor}\left(\frac{r_s}{r}\right)^6, \tag{B4}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius and $K_{\rm hor} \propto r_s^{-4}$ is the curvature evaluated at the horizon.

The boundary of the plastic region is defined by the condition $K(r_{\rm yield}) = K_c$, which yields

$$r_{\rm yield} = r_s\left(\frac{K_{\rm hor}}{K_c}\right)^{1/6}. \tag{B5}$$

The proper volume of the yielding region then scales as

$$V_{\rm yield} \sim \frac{4\pi}{3}r_{\rm yield}^3 \propto r_s^3\left(\frac{K_{\rm hor}}{K_c}\right)^{1/2}, \tag{B6}$$

where we have assumed either $r_{\rm yield} \gg r_s$ so that the geometry is approximately Euclidean, or equivalently that the full relativistic volume element $\sqrt{\gamma}$ has been integrated.

The effective macroscopic coupling β is therefore related to the microscopic physics via

$$\beta \simeq \langle\Delta\chi_{\rm event}\rangle\, V_{\rm yield}, \tag{B7}$$

where $\langle\Delta\chi_{\rm event}\rangle$ is the average dimensionless strain (memory increment) produced per merger event.

3. Renormalization and Constraints

While Eq. (B6) provides the expected microphysical scaling of the yield volume, the precise values of the critical curvature K_c and the fossilization efficiency $\Delta\chi_{\rm event}$ depend on the ultraviolet completion of the theory—namely, the detailed microphysics governing spacetime plasticity.

Within the Effective Field Theory framework, we therefore treat β as a renormalized coupling parameter to be fixed by observation rather than derived ab initio. In practice, β is not chosen arbitrarily but is determined by the requirement that the integrated fossil energy density reproduces the observed dark energy abundance,

$$\rho_{\rm DE,0} \simeq \rho_\Lambda \int_{-\infty}^{t_0} \left(\beta\,\mathcal{R}_{\rm GW}(t')\right) e^{-(t_0-t')/\tau}\, dt'. \tag{B8}$$

This condition anchors the overall amplitude of dark energy to observation, while the *shape* of its evolution—encoded in $w(z)$—is dictated by the redshift dependence of the merger rate $\mathcal{R}_{\rm GW}(z)$ and the memory timescale τ.

APPENDIX C: ROBUSTNESS OF THE SIGN LAW

The central prediction of this work—the **Plasticity Sign Law** governing the crossing of $w = -1$—was derived in the main text using a specific benchmark realization: a Heaviside yield trigger, an exponential saturation potential, and a parameterized source history. In this appendix, we demonstrate that the qualitative features of the dark energy evolution—most importantly the phantom-to-quintessence transition—are **structural consequences** of the plasticity framework and remain robust under variations of these modeling choices.

1. Smoothing the Yield Trigger

In Sec. III, we modeled the onset of plasticity using a sharp Heaviside activation,

$$\mathcal{Y} \propto \Theta(K - K_c).$$

While convenient, physical phase transitions generally exhibit finite width. Consider instead a smooth sigmoid trigger,

$$\mathcal{Y}_{\rm smooth}(K) = \frac{1}{2}\left[1 + \tanh\left(\frac{K - K_c}{\Delta K}\right)\right]\left(\frac{K}{K_c}\right)^n, \tag{C1}$$

where ΔK characterizes the transition width.

The cosmological source term $S(t)$ is obtained by averaging this local production over the cosmic volume. Because this averaging samples a broad statistical distribution of curvature values—arising from mergers with varying masses, spins, and impact parameters—the detailed shape of the activation edge is effectively smoothed even before cosmological integration.

Moreover, the memory equation,

$$\dot{\chi} = S - \frac{\chi}{\tau},$$

acts as a **low-pass filter** on the source term. Sharp features or high-frequency variations in $S(t)$ are exponentially suppressed by the kernel $e^{-(t-t')/\tau}$. As a result, replacing a sharp Heaviside trigger with a smooth sigmoid alters only the overall normalization β required to match $\Omega_{\rm DE}$, while leaving the *time dependence* of $S(z)$—and hence the evolution of $w(z)$—unchanged.

2. Universality with Respect to the Potential Choice

The main text adopts a specific saturating potential,

$$V(\chi) = \rho_\Lambda\left(1 - e^{-\chi/\chi_*}\right).$$

Here we show that the Plasticity Sign Law holds for **any** potential $V(\chi)$ satisfying two physically motivated conditions:

1. **Monotonicity:** $V'(\chi) > 0$ (energy increases with deformation),
2. **Boundedness / Concavity:** $V''(\chi) < 0$ (saturation or diminishing returns).

Recall the definition of the effective equation of state (Eq. 19),

$$w_{\rm eff} = -1 - \frac{\dot{\rho}_f}{3H\rho_f} = -1 - \frac{V'(\chi)}{3HV(\chi)}\dot{\chi}. \tag{C2}$$

Substituting the memory equation $\dot{\chi} = S - \chi/\tau$, we obtain

$$w_{\rm eff} + 1 = -\left[\frac{V'(\chi)}{3HV(\chi)}\right]\left(S(t) - \frac{\chi}{\tau}\right). \tag{C3}$$

The prefactor in brackets is strictly positive for any potential with $V > 0$ and $V' > 0$ in an expanding universe ($H > 0$). Consequently, the **sign** of $w_{\text{eff}} + 1$ is determined *exclusively* by the sign of the balance term $(S - \chi/\tau)$.

Conclusion. The prediction that $w < -1$ during the creation phase ($S > \chi/\tau$) and $w > -1$ during relaxation ($S < \chi/\tau$) is independent of the detailed functional form of $V(\chi)$. It is a kinematic consequence of open-system thermodynamics rather than a model-dependent artifact.

3. Topological Guarantee of the Crossing

Finally, we examine the dependence of the phantom crossing on the source history $S(z)$. We assume only that the astrophysical source rate of strong-field events is **unimodal**: it rises from zero, peaks at some redshift z_p, and decays toward zero as $t \to \infty$.

– **Early times** ($t \ll t_{\text{peak}}$): The source term $S(t)$ is increasing. Since χ is the time-integral of S and $\chi(0) = 0$, initially

$$S > \frac{\chi}{\tau} \quad \Rightarrow \quad w < -1 \quad \text{(phantom phase).}$$

– **Late times** ($t \to \infty$): Astrophysical activity shuts off as star formation and mergers decline, implying $S \to 0$. A residual memory field remains ($\chi > 0$), so eventually

$$S < \frac{\chi}{\tau} \quad \Rightarrow \quad w > -1 \quad \text{(quintessence phase).}$$

Define the continuous function

$$\Delta(t) \equiv S(t) - \frac{\chi(t)}{\tau}.$$

Since $\Delta(t)$ evolves continuously from positive to negative values, the **Intermediate Value Theorem** guarantees the existence of a time $t_\times$ such that $\Delta(t_\times) = 0$. At this moment,

$$w(t_\times) = -1.$$

Crossing Theorem. The existence of a phantom divide crossing is not an artifact of any specific parameterization of $S(z)$ (e.g., Madau–Dickinson or LVK-inspired fits). It is a *topological inevitability* for any universe in which dark energy emerges from a finite, transient epoch of astrophysical activity.