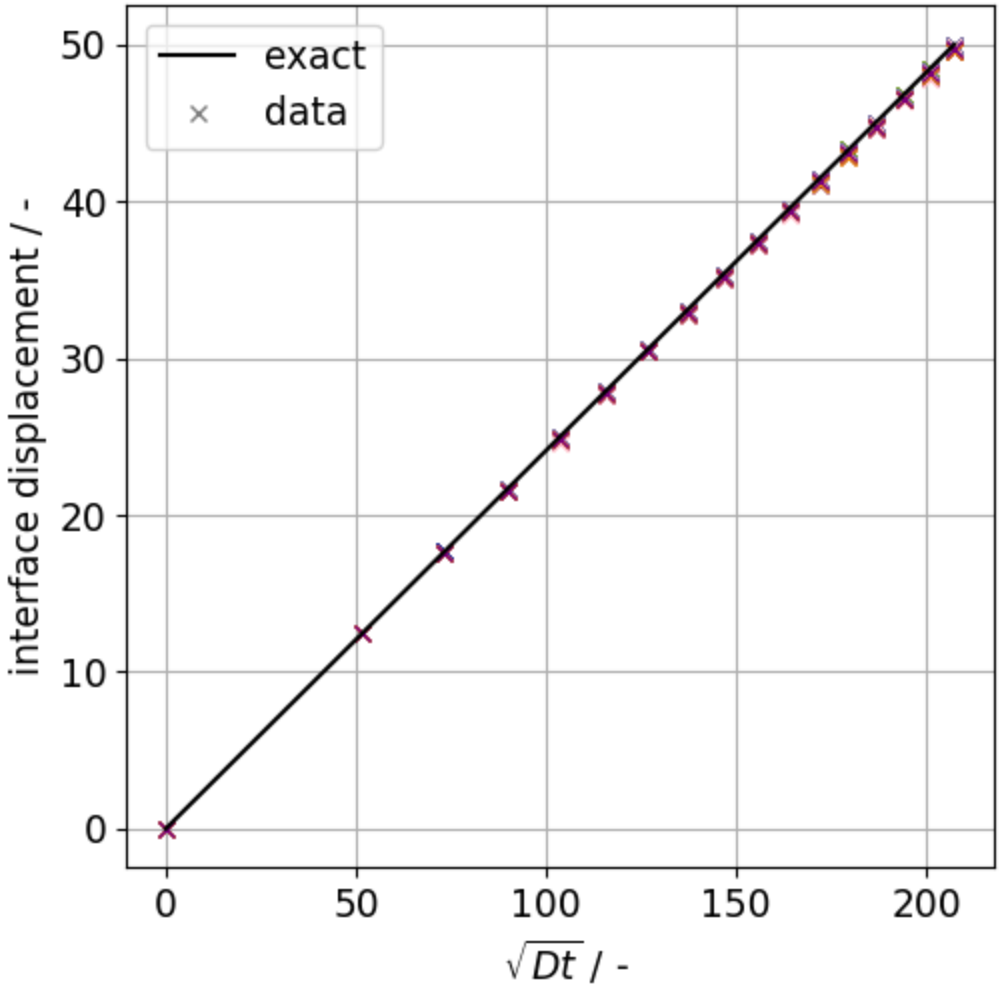
In this notebook the solutal Stefan problem results as described in section 4.4. of the paper are analyzed.

First some useful things from the exact solution...

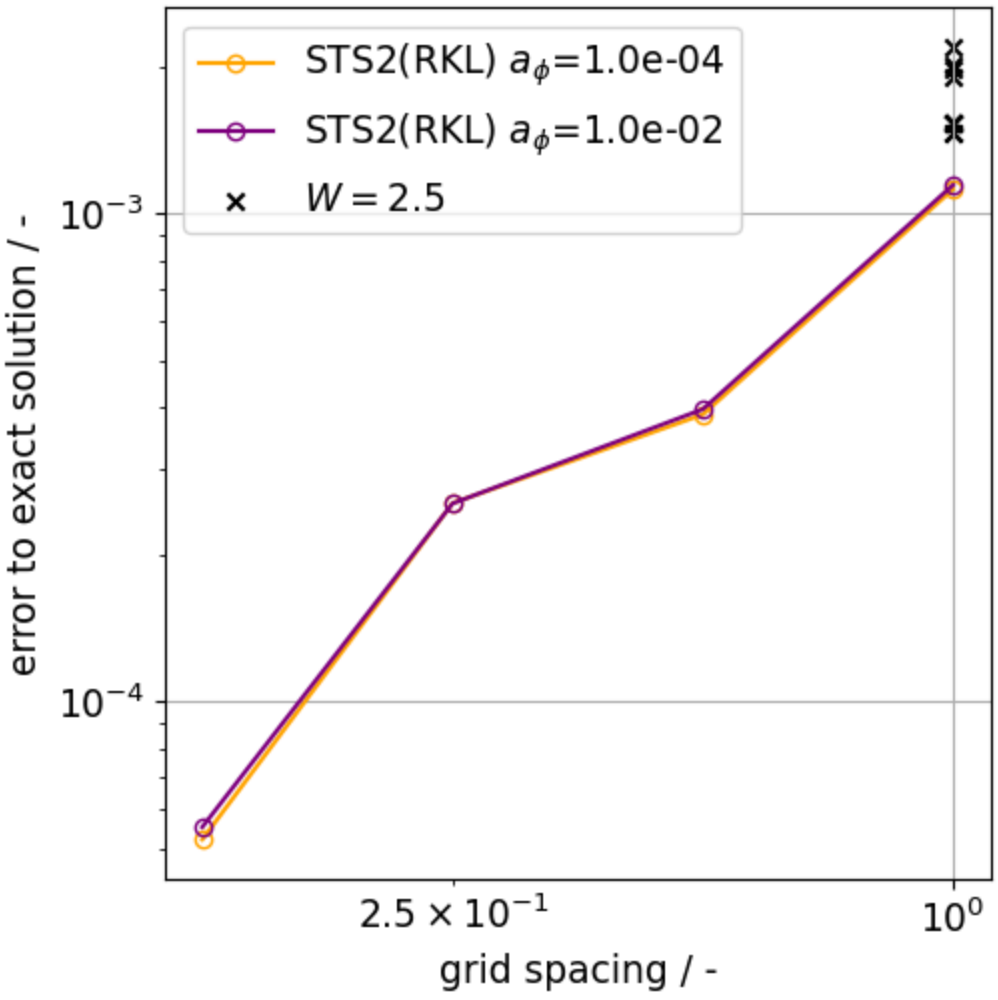
$$A = \frac{2\sqrt{D}\left(\left(\operatorname{erf}\left(\frac{A}{2\sqrt{D}}\right)-1\right)\left(c_{\beta,\alpha}-c_{\beta,\infty}\right)+\left(\operatorname{erf}\left(\frac{A}{2\sqrt{D}}\right)+1\right)\left(c_{\alpha,\beta}-c_{\alpha,\infty}\right)\right)e^{-\frac{A^2}{4D}}}{\sqrt{\pi}\left(\operatorname{erf}\left(\frac{A}{2\sqrt{D}}\right)-1\right)\left(\operatorname{erf}\left(\frac{A}{2\sqrt{D}}\right)+1\right)\left(c_{\alpha,\beta}-c_{\beta,\alpha}\right)}$$

We observe that the exact solution is also matched quite well when concentration coupling is introduced.

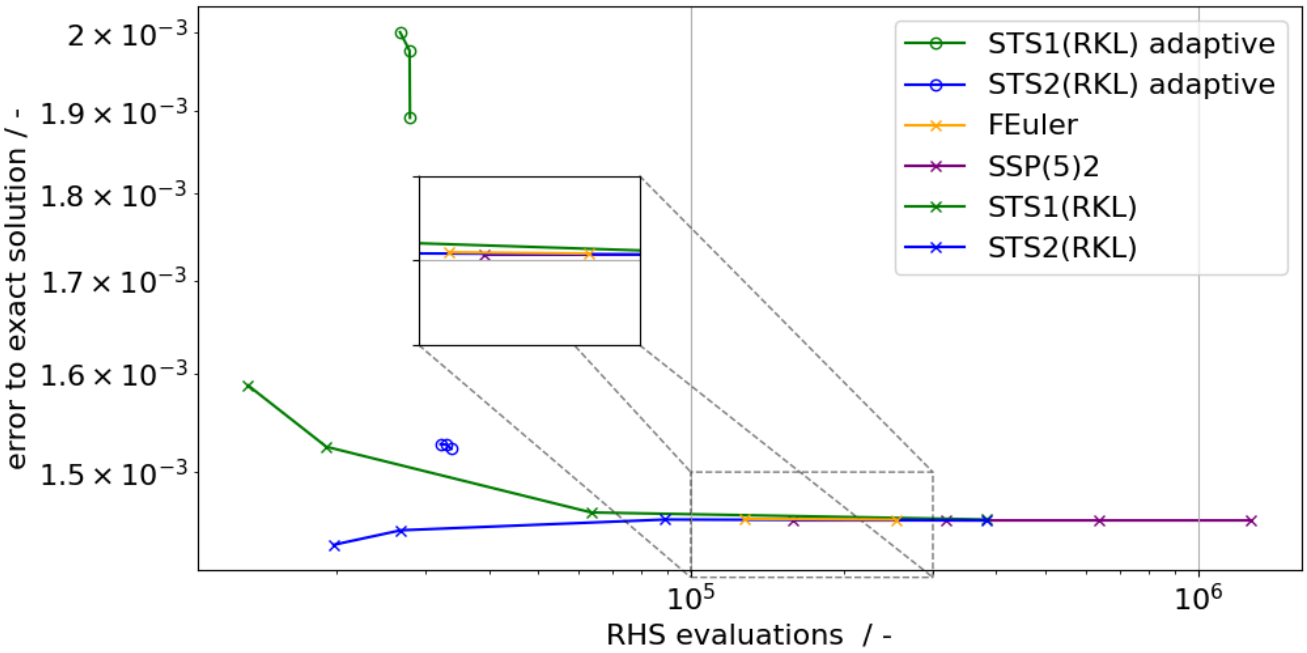



```
Text(0, 0.5, 'interface displacement / -')
```

And convergence is achieved throughout, for any tolerance in the phase-field. This is likely a combined effect of the phase-field effectively following the concentration evolution (which is why changing the tolerance doesn't change the RHS evaluations much as shown below) as well as most likely the error being mostly a function of the interface width.

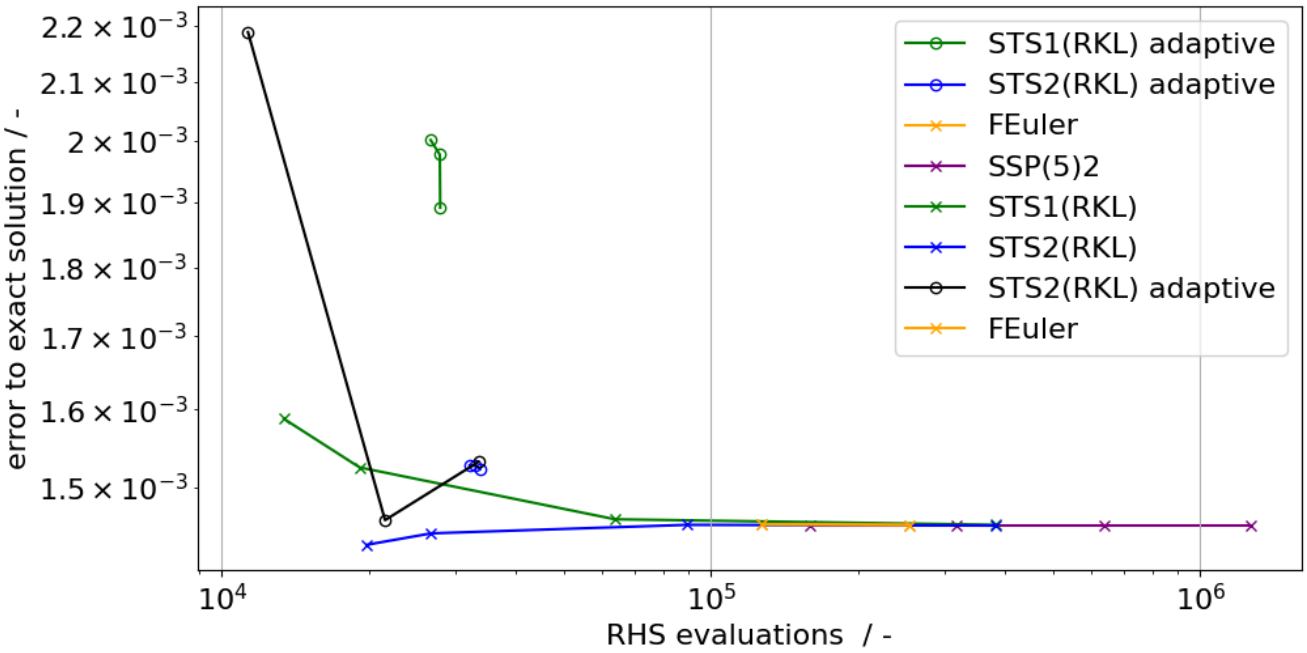


We can again observe that the STS schemes beat out the forward Euler scheme. Due to the problem being insensitive to the phase-field tolerance, there is not much change in the RHS count. Test simulations with the absolute concentration tolerance being equal to the absolute phase-field tolerance yielded a speedup as shown in the next plot.




```
adaptive STS1(RKL)(MPF::Obstacle) speedup 4.7621684312918555 rejection ratio 0.336724
3133265514 0.336529242569511 0.3464419475655431
adaptive STS2(RKL)(MPF::Obstacle) speedup 3.962395716598182 rejection ratio 0.3325942
350332594 0.333692142088267 0.3363544813695871
fixed FEuler(MPF::Obstacle) speedup 1.0
fixed SSP(5)2(MPF::Obstacle) speedup 0.7995477386934673
fixed STS1(RKL)(MPF::Obstacle) speedup 9.493436754176612
fixed STS2(RKL)(MPF::Obstacle) speedup 6.436488673139158
```

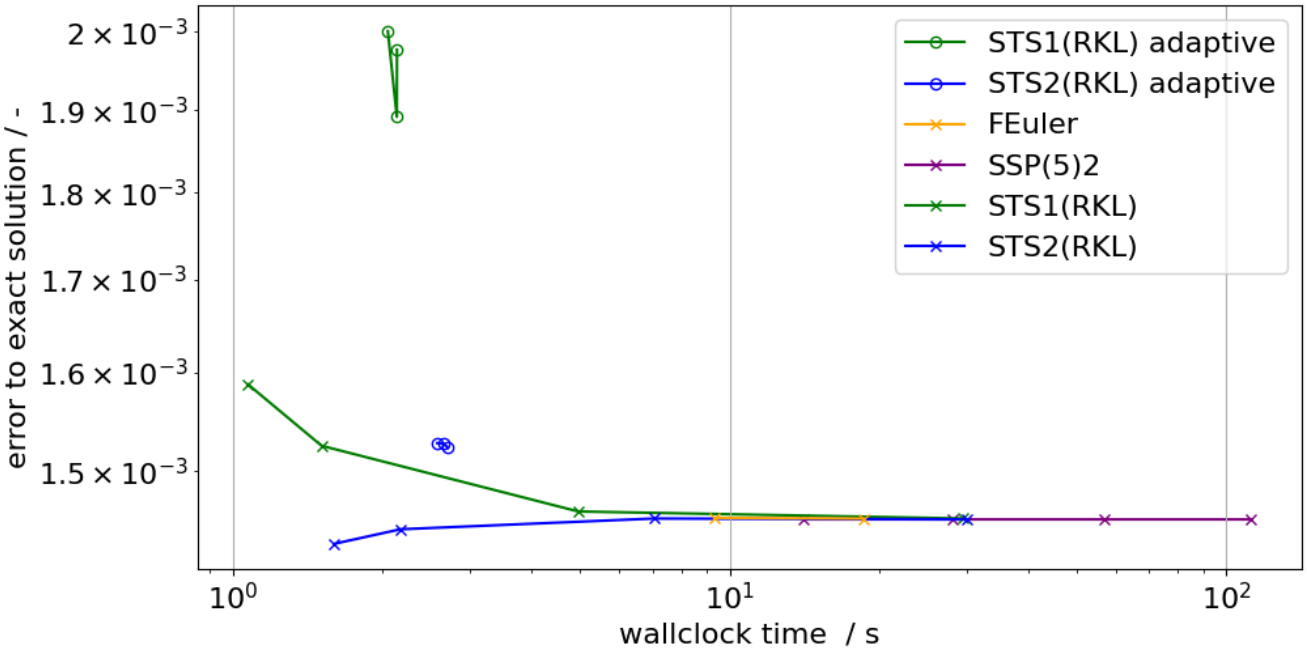
Here we jury-rig the changed concentration tolerance into the plot together with the old data. The black line with hollow circles represents the new data, getting a speedup of up to 11.4 without changing the error much in an absolute sense, but signifcantly in a relative sense. Interestingly, if we put $1 + \epsilon, |\epsilon| \ll 1$ in front of the eigenvalue estimate for the diffusion equation (the dominant one), the error decreases at the loosest tolerance (for $a_c = a_\phi = 10^{-2}$) without much change in the RHS count. E.g. for $\epsilon = 10^{-4}$ (slightly larger eigenvalue), the error in fact decreases to 1.46e-3 with a speedup of 11.6, though without an appreciable change in the rejected ratio. For $\epsilon = -10^{-4}$, an error of 1.6e-3 is obtained at a speedup of 11.3. One reason for the apparently tiny change making a large (relative) effect may be that the time integration error is already in a regime where the noise due to a different time step sequence makes a significant difference.




```
adaptive STS2(RKL)(MPF::Obstacle) speedup 11.248497702368327 rejection ratio 0.243589
74358974358 0.314410480349345 0.33166833166833165
fixed FEuler(MPF::Obstacle) speedup 1.0
```

```
Text(0, 0.5, 'error to exact solution / -')
```

Unsurprisingly there is no large difference between the performance evaluated with the RHS or the raw runtimes here either.




```
adaptive STS1(RKL)(MPF::Obstacle) speedup 4.542108093023411 rejection ratio 0.3367243
133265514 0.336529242569511 0.3464419475655431
adaptive STS2(RKL)(MPF::Obstacle) speedup 3.6123003736670865 rejection ratio 0.332594
2350332594 0.333692142088267 0.3363544813695871
fixed FEuler(MPF::Obstacle) speedup 1.0
fixed SSP(5)2(MPF::Obstacle) speedup 0.6595671252522007
fixed STS1(RKL)(MPF::Obstacle) speedup 8.687422960081209
fixed STS2(RKL)(MPF::Obstacle) speedup 5.822494256726312
```