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A Dynamic Diagnostic Method for Consecutive Faults in Nonlinear Uncertain Systems

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Abstract— **This paper proposes a robust fault diagnosis method for nonlinear uncertain systems with multiple faults, addressing the possible occurrence of two consecutive faults in each state equation. A model-based monitoring module with two submodules is developed, enabling the diagnosis of both faults. The first submodule incorporates decision-making schemes for detecting and isolating the primary fault, enabling its partial or full isolation. The second submodule is introduced to detect the secondary fault while simultaneously determining the partial or full isolation of both the primary and secondary faults. A key design aspect of the proposed fault diagnosis method is that it effectively uses the information from the primary fault isolation process, particularly when partial isolation occurs, to detect and isolate a secondary fault in each system state equation. The boundedness of the system variables and the robustness of the proposed fault diagnosis scheme are analytically shown. Finally, the effectiveness and applicability of the developed framework is demonstrated through numerical simulations.**

I. INTRODUCTION

Motivation and literature review: Over the past three decades, researchers have developed fault diagnosis techniques, specifically fault detection and isolation methods, using quantitative model-based schemes. A key motivation is to minimize the costs and complexity associated with physical redundancy, by developing suitable analytical models to detect and isolate faults, [1], [2]. Generally, the discrepancy between the actual dynamical system and a model-based estimated system is quantified using the so-called residual errors and suitable analysis. These residual errors can be formulated through associated thresholds to consider different fault diagnosis scenarios.

Initially, fault diagnosis results were limited to detecting and isolating faults in linear systems, see [3] and the references therein. For the case of nonlinear systems, early fault diagnosis methods were restricted to bilinear systems [4], nonlinear systems with completely known dynamics [5], or uncertain nonlinear systems, where faults and modeling uncertainties can be decoupled, [6]. To address some of these limitations, an adaptive approximation framework was developed in [7] to deal with the fault diagnosis problem for a general class of nonlinear uncertain systems. Then, the scheme presented in [7] was further extended to study fault diagnosis for nonlinear systems without full state measurements in [8].

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Moreover, [9] considered the problem of fault diagnosis for nonlinear uncertain systems using a suitably designed sliding mode observer. In [10], a rapid oscillation fault diagnosis algorithm with deterministic learning is proposed for a class of nonlinear uncertain systems.

Despite these notable achievements, most of the existing literature has focused on the diagnosis of a single fault within each system state equation. Recently, some fault diagnosis methods for multiple faults in dynamical systems have been proposed; see, for example, [11]–[13]. These approaches use data-based methods, that require large and high-quality datasets for training. To address this issue, initial model-based fault diagnosis results were presented for considering more than one faults in each system state equation, e.g., [14], [15]. However, these works focus only on detecting a single fault, without providing any isolation process.

Contribution: As practical systems become more complex, it becomes more important to consider multiple faults and develop algorithms for detecting and isolating them. This paper introduces a fault diagnosis scheme for a general class of nonlinear uncertain systems, with up to two possible consecutive faults in each system state equation. Specifically, we consider the case of multiple system faults, but with each state equation having up to two consecutive faults. We design a monitoring module with two submodules for each system state equation. Each submodule consists of both detection and isolation components. The first submodule aims to detect and isolate the primary fault, by utilizing the diagnostic methodology presented in [7], [8]. Unlike existing schemes, we do not rely on the fault mismatch function concept; instead, we introduce the idea of partial fault isolation by excluding the non-occurred faults to deal with the fault functions with similar behavior. This process enables two outcomes regarding the primary or secondary fault diagnosis procedure for each state equation: excluding all faults except one of them, i.e., achieving full fault isolation, or partial exclusion of faults. The second submodule, which includes both detection and isolation components, activates following a predefined time interval after the primary fault detection instant. The purpose of the detection component of this submodule is to detect the secondary fault, leveraging the information obtained from the primary fault isolation process. This is achieved by monitoring the residual and the threshold signals associated with the non-excluded primary faults. The isolation component of the second submodule focuses on isolating both primary and possible secondary faults by utilizing information from non-excluded faults in both the primary and secondary isolation phases. Unlike existing model-based fault diagnosis methods with static (predetermined) structures, such as those presented in [6]–[10], [14], [15], the second submodule, specifically its detection and isolation components, features a dynamic (non-predetermined) architecture. This enables the detection and isolation of a secondary fault in each state equation, even when the primary fault is only partially isolated. The boundedness of all system variables, specifically states and control inputs, and the resulting robust performance at the presence of model uncertainty, are analytically guaranteed. Finally, the validity of the presented theoretical results is demonstrated through a numerical simulation.

To the authors' best knowledge, this work is the first to study multiple faults in nonlinear uncertain systems by developing a model-based dynamic diagnostic methodology that accounts for more than one fault in each state equation. Although the methodology can be extended to assess more than two faults, this introduces significant computational complexity and has therefore been omitted from this study.

Notation: $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^n$ denote the sets of real numbers, non-negative real numbers, and real vectors of dimension n, respectively. $\mathcal{A} \triangleq \text{diag}\{a_i\}$, for $i = 1, 2, \dots, n$, refers to the square diagonal matrix of dimension n. The symbols $\cup$ and $\cap$ denote the union and intersection of two sets, respectively, and $|\mathcal{F}|$ represents the cardinality of the set $\mathcal{F}$. Finally, $\mathbb{P}_\Omega$ denotes the parameter projection operator over the compact set Ω, see e.g. [16]. The states, sets, and functions are referred to without the time argument when there is no ambiguity.

II. Problem Formulation

A. System and fault descriptions

Consider a general form of nonlinear systems with unknown dynamics, where up to two consecutive faults can occur in each state equation, as described below:

$$\dot{x} = h(x, u) + \omega(x, u, t) + \sum\nolimits_{k=1}^{2} \beta_k(t - T_k^0) f_k(x, u), \quad (1)$$

where $x \triangleq [x_1, x_2, \dots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ is the state vector, and $u \triangleq [u_1, u_2, \dots, u_m]^{\mathrm{T}} \in \mathbb{R}^m$ represents the control input vector. The nonlinear term $\omega(x, u, t) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^+ \mapsto \mathbb{R}^n$ denotes the unknown piecewise-continuous vector field of modeling uncertainty representing modeling errors and external disturbances, and $h(x, u) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is the continuous vector field that models the known nominal dynamics. Hereafter, the indexes i and k are subject to the sets $i \in \mathcal{I} \triangleq \{1, 2, \dots, n\}$ and $k \in \mathcal{K} \triangleq \{1, 2\}$. The term $\beta_k(t - T_k^0) \triangleq \text{diag}\{\beta_{i,k}(t - T_{i,k}^0)\}$ is the unknown matrix characterizing the time profile of the k-th possible fault vector, where $\beta_{i,k}(t - T_{i,k}^0) : \mathbb{R}^+ \mapsto [0, 1]$ denotes the time profile of the k-th fault in the state equation i, and $T_{i,k}^0 \in \mathbb{R}^+$ represents the k-th unknown fault-occurrence time in the i-th state equation, for $i \in \mathcal{I}$ and $k \in \mathcal{K}$. The considered faults in each state equation occur sequentially, specifically $T_{i,1}^0 < T_{i,2}^0$, for $i \in \mathcal{I}$. The vector field $f_k(x, u) \triangleq [f_{1,k}(x, u), f_{2,k}(x, u), \dots, f_{n,k}(x, u)]^{\mathrm{T}} \in \mathbb{R}^n$ denotes the unknown change of the system due to the occurrence of the k-th fault, and $f_{i,k}(x, u) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$, for $i \in \mathcal{I}$ and $k \in \mathcal{K}$, represents the k-th partially known, continuous fault function that may occur in state equation i. Here, we assume that $f_{i,k}(x, u)$ belongs to the finite time-invariant set

$$\mathcal{F}_{i,k} \triangleq \left\{ f_{i,k}^s(x, u; \theta_{i,k}^s) : \ s \in \{1, 2, \dots, |\mathcal{F}_{i,k}|\} \right\}, \quad (2)$$

where $i \in \mathcal{I}$ and $k \in \mathcal{K}$. Each component of $\mathcal{F}_{i,k}$ is characterized by a linear-in-parameter structure as follows:

$$f_{i,k}^s(x, u; \theta_{i,k}^s) = (\theta_{i,k}^s)^{\mathrm{T}} g_{i,k}^s(x, u), \quad (3)$$

where $g_{i,k}^s(x, u) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^{w_{i,k}^s}$ is the known nonlinear vector field and $\theta_{i,k}^s \in \mathbb{R}^{w_{i,k}^s}$ is the unknown constant vector, assumed to be limited in a known compact set $\Theta_{i,k}^s$, that implies $\theta_{i,k}^s \in \Theta_{i,k}^s \subset \mathbb{R}^{w_{i,k}^s}$. In real-world applications modeled by (1), where faults arise from diverse sources (such as hardware, software, or external factors), representing faults using different distinct sets, as described in (2), is crucial for considering the multiple fault diagnosis problem.

In this study, a general form of the time profile of faults is employed for $i \in \mathcal{I}$ and $k \in \mathcal{K}$ as follows:

$$\beta_{i,k}(t - T_{i,k}^0) \triangleq \begin{cases} 0, & \text{if } t < T_{i,k}^0, \\ 1 - e^{(-p_{i,k}(t - T_{i,k}^0))}, & \text{if } t \geq T_{i,k}^0, \end{cases} \quad (4)$$

where the unknown constant $p_{i,k} > 0$ denotes the fault evolution rate for the k-th fault within state equation i. This model covers both incipient (slowly developing) faults when $p_{i,k}$ is small and abrupt (sudden) faults when $p_{i,k}$ is large.

Hereafter, the index s is subject to the set $\{1, 2, \dots, |\mathcal{F}_{i,k}|\}$, but we omit its set in the presentation of formulas.

B. Main assumptions

To present the fault diagnosis methodology, a set of assumptions is imposed on (1), as described below.

Assumption 1: There exists compact sets $\mathcal{U}$ and $\mathcal{X}$ such that $x \in \mathcal{X} \subset \mathcal{R}^n$ and $u \in \mathcal{U} \subset \mathcal{R}^m$ for all $t \geq 0$.

Assumption 2: Each unstructured uncertain component satisfies $|\omega_i(x, u, t)| \leq \bar{\omega}_i(x, u, t)$ for all $(x, u) \in \mathcal{X} \times \mathcal{U}$ and $t \geq 0$, where $\bar{\omega}_i(x, u, t)$ is a known, positive, and continuous function, uniformly bounded in time t.

Assumption 3: There exists some known minimum time duration between the detection of the primary fault at $t = T_{i,1}^d$ and the occurrence of the secondary fault at $t = T_{i,2}^0$ in each faulty state equation of (1), specifically $T_{i,2}^0 \geq T_{i,1}^d + \Psi_i$.

Since this work deals with the problem of fault detection and isolation, the first assumption requires that the state and control variables remain bounded during fault diagnosis. The second assumption imposes a bounding condition on the modeling uncertainty, which is a commonly adopted approach in existing fault diagnosis schemes [7], [8], [14], [15]. The last assumption requires some known minimum time between the detection of the primary fault and the occurrence of the secondary fault.We would like to clarify that the proposed approach does not necessarily require Assumption 3, (this fact is elaborated in Remark 1). However, Assumption 3 enables the exclusion of non-occurring primary faults for each state equation during the $t \in [T_{i,1}^d + \Psi_i, T_{i,2}^d)$ under less conservative conditions, as presented below.

C. Problem statement

This study aims to design a robust scheme for detecting and isolating multiple faults in (1), allowing up to two faults per state equation, through the following problem:

Problem 1: Consider system (1) and let Assumptions 1-3 hold. Design a dynamic monitoring module that:

(i) determines the primary fault occurrence in state equation i, and enables the exclusion of possible faults from the set $\mathcal{F}_{i,1}$ for all $i \in \mathcal{I}$,

(ii) enables the detection of the secondary fault occurrence in state equation i, and the exclusion of fault functions from the class of possible faults $\mathcal{F}_{i,2}$ for all $i \in \mathcal{I}$.

Property (i) plays a crucial role in the detection and isolation of the primary fault in the i-th state equation, which serves as a prerequisite for diagnosing the secondary fault that may occur. Once the primary fault has been diagnosed, Property (ii) requires a diagnostic scheme that can detect and isolate the secondary fault. Furthermore, Property (ii) must be achieved based on the partial or full isolation information provided by the primary diagnostic architecture.

Before proceeding, we first define the following times for the i-th state equation of (1), where $i \in \mathcal{I}$ and $k \in \mathcal{K}$: $t = T^0_{i,k}$: the occurrence time of k-th fault in state equation i; $t = T^d_{i,k}$: the detection time of the k-th occurred fault in state equation i, where $T^d_{i,k} > T^0_{i,k}$; $t = T^d_{i,k} + \Psi_i$: the maximum time, after the detection of the primary fault, where there is knowledge that a secondary fault has not occurred in state equation i, $t = T^s_{i,k}$: the exclusion time of the s-th fault from the k-th possible set of faults within state equation i, when one of the isolation components determines that the s-th fault from $\mathcal{F}_{i,k}$ has not occurred, where $T^s_{i,k} > T^d_{i,k}$; $t = T^{isol}_{i,k}$: the time of full fault isolation from the k-th possible set of faults within state equation i, when one of the isolation components determines which fault from $\mathcal{F}_{i,k}$ has actually occurred, where $T^{isol}_{i,k} > T^d_{i,k}$.

Then, we introduce a time-varying bi-partition of the possible faults, by defining the time-varying sets $\mathcal{B}_{i,k}(t)$ and $\mathcal{C}_{i,k}(t)$ such that $\mathcal{B}_{i,k}(t) \cup \mathcal{C}_{i,k}(t) = \mathcal{F}_{i,k}$ and $\mathcal{B}_{i,k}(t) \cap \mathcal{C}_{i,k}(t) = \emptyset$, where $\mathcal{C}_{i,k}(t)$ and $\mathcal{B}_{i,k}(t)$ denote the sets of the excluded and non-excluded faults for the k-th possible fault in state equation i at time instant t for all $i \in \mathcal{I}$ and $k \in \mathcal{K}$, respectively. Specifically, we have:

$$\mathcal{B}_{i,k}(t) \triangleq \left\{ f^b_{i,k}(x, u; \theta^b_{i,k}) : b \in \upsilon_{i,k}(t) \right\}, \tag{5a}$$

$$\mathcal{C}_{i,k}(t) \triangleq \mathcal{F}_{i,k} \setminus \mathcal{B}_{i,k}(t), \tag{5b}$$

where $\upsilon_{i,k}(t)$ denotes the index set that define which faults, from the set of faults $\mathcal{F}_{i,k}$, have not been excluded by time t.

The fault diagnosis time profile is depicted in Fig. 1.

III. Dynamic Fault Diagnosis Design

The proposed fault diagnosis scheme consists of two sequential monitoring submodules. The first submodule detects and, when possible, isolates the primary fault occurring in the ith state equation of the system. The second submodule is activated to detect and, if feasible, isolate a consecutive

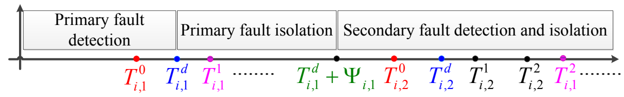

Fig. 1: Fault diagnosis time profile for each system state equation. The time instants coloring is as follows: red represents the fault occurrence time; blue the fault detection time; magenta the primary fault exclusion time; black the secondary fault exclusion time; and green the time instant after which the secondary fault may occur.

fault in the same ith state equation. Detailed descriptions of each submodule's operation are presented in the following.

A. First submodule design

1) First fault detection component (Task 1): In this component, we focus on the design of the primary fault detection architecture, whose main objective is to determine the faulty and non-faulty state equations in (1), (Task 1). As this first detection result does not represent a main contribution of this paper, the corresponding formulation and the fault detection policy are omitted here. For more details, we refer the reader to [7].

2) First fault isolation component (Task 2): The main objective in designing this component for the faulty state equation i is to exclude non-occurring faults from the fault set $\mathcal{F}_{i,1}$ during the time interval $t \in [T^d_{i,1},\ T^d_{i,1} + \Psi_i)$, using $|\mathcal{F}_{i,1}|$ isolation estimators (Task 2), where $T^d_{i,1}$ denotes the first fault detection time instant in state equation i, determined by the first fault detection component. Similar to the first fault detection component, we refer the reader to [7] for some discussions on the formulation and the first fault isolation policy utilized this component.

B. Second monitoring submodule formulation

1) Second fault detection component (Task 3): In the following, considering Assumption 3, based on the obtained information from the primary fault isolation procedure for state equation i, we aim to use a bank of adaptive estimators, activated at $t = T^d_{i,1} + \Psi_i$, to detect the occurrence of the secondary fault, (Task 3).

Accordingly, a bank of adaptive estimators is designed, where the number of estimators is $|\mathcal{B}_{i,1}(T^d_{i,1} + \Psi_i)|$. Each adaptive estimator corresponds to a specific type of non-excluded fault included in the set $\mathcal{B}_{i,1}(T^d_{i,1} + \Psi_i)$. Specifically, the following adaptive estimators are employed for the purpose of secondary fault detection:

$$\dot{\hat{x}}^b_{i,2} = h_i(x, u) + \hat{f}^b_{i,1}(x, u; \hat{\theta}^b_{i,1}) + l^b_{i,2}(x_i - \hat{x}^b_{i,2}), \tag{6}$$

for $b \in \upsilon_{i,1}(T^d_{i,1} + \Psi_i)$, where $\hat{x}^b_{i,2}$ is the i-th state estimation in the presence of the b-th non-excluded fault form the set $\mathcal{B}_{i,1}(T^d_{i,1} + \Psi_i)$, $l^b_{i,2} > 0$ is the i-th estimator gain, and $\hat{f}^b_{i,1}(x, u; \hat{\theta}^b_{i,1}) = (\hat{\theta}^b_{i,1})^{\mathrm{T}} g^b_{i,1}(x, u)$ is the estimation of the b-th non-excluded fault form the set $\mathcal{B}_{i,1}(T^d_{i,1} + \Psi_i)$. Moreover, we define the i-th state estimation error (or the secondary fault detection residual error for the state equation i) as $e^b_{i,2} \triangleq x_i - \hat{x}^b_{i,2}$.

To estimate the unknown constant vector $\theta^b_{i,1}$, we design the adaptive law as follows:

$$\dot{\hat{\theta}}^b_{i,1} = \mathbb{P}_{\Theta^b_{i,1}}\Big\{\gamma^b_{i,1} e^b_{i,2} g^b_{i,1}(x,u)\Big\}, \tag{7}$$

where $\gamma^b_{i,1} > 0$ is a learning rate. Then, the error dynamics of the proposed fault detection estimator using (1) and (6) is given by:

$$\begin{aligned}\dot{e}^b_{i,2} = &- l^b_{i,2} e^b_{i,2} + \omega_i(x,u,t) + \beta_{i,1}(t - T^0_{i,1})\tilde{f}^b_{i,1}(x,u;\tilde{\theta}^b_{i,1}) \\ &- e^{(-p_{i,1}(t-T^0_{i,1}))}\hat{f}^b_{i,1}(x,u;\hat{\theta}^b_{i,1}),\end{aligned} \tag{8}$$

where $\tilde{f}^b_{i,1}(x,u;\tilde{\theta}^b_{i,1}) = (\tilde{\theta}^b_{i,1})^{\mathrm{T}} g^b_{i,1}(x,u)$ and $\tilde{\theta}^b_{i,1} \triangleq \theta^b_{i,1} - \hat{\theta}^b_{i,1}$ denotes the parameter estimation error vector.

Below, we demonstrate the stability properties of the proposed secondary fault detection estimators.

Proposition 1: Consider the b-th, ($b \in \upsilon_{i,1}(T^d_{i,1} + \Psi_i)$), fault detection error dynamics in (8) for the i-th, ($i \in \mathcal{I}$), state equation of system (1) under Assumptions 1-3. Then, the b-th detection estimator designed in (6) with learning law (7) ensures that:

(i) the error variables $e^b_{i,2}$ and $\tilde{\theta}^b_{i,1}$ are uniformly bounded,
(ii) the i-th detection residual errors satisfy:

$$|e^b_{i,2}(t)| \le \bar{e}^b_{i,2}(t), \qquad \forall t \in [T^d_{i,1} + \Psi_i, T^0_{i,2}), \tag{9}$$

where

$$\begin{aligned}\bar{e}^b_{i,2}(t) \triangleq &\int_{T^d_{i,1}}^{t} e^{-(l^b_{i,2}(t-\tau))}\big[\Sigma^b_i(\tau) + \Xi^b_i(\tau)\big]\mathrm{d}\tau \\ &+ |e^b_{i,2}(T^d_{i,1})| e^{-(l^b_{i,2}(t-T^d_{i,1}))},\end{aligned} \tag{10}$$

$$\begin{cases}\Sigma^b_i(\tau) \triangleq |\tilde{f}^b_{i,1}(x(\tau),u(\tau);\tilde{\theta}^b_{i,1}(\tau)| + \bar{\omega}_i(x(\tau),u(\tau),\tau), \\ \Xi^b_i(\tau) \triangleq e^{(-\bar{p}_{i,1}(\tau-T^d_{i,1}))}|\hat{f}^b_{i,1}(x(\tau),u(\tau);\hat{\theta}^b_{i,1}(\tau))|.\end{cases}$$

If for every $b \in \upsilon_{i,1}(T^d_{i,1} + \Psi_i)$, it holds that $|e^b_{i,2}(t)| > \bar{e}^b_{i,2}(t)$ for some time instant $t > T^d_{i,1} + \Psi_i$, then the secondary fault detection is guaranteed in the i-th state equation.

Proof: Due to page limitations, the proof of this result, which is analogous to that of Lemma 3.1 in [7], is omitted. ■

Based on Proposition 1, we now proceed to define the secondary fault detection policy for each faulty state equation.

Secondary fault detection policy: The secondary fault occurrence in the i-th faulty state equation of system (1) is detected after time instant $t = T^d_{i,1} + \Psi_i$, when the fault detection residual errors $|e^b_{i,2}(t)|$ for all $b \in \upsilon_{i,1}(T^d_{i,1} + \Psi_i)$ exceed their respective adaptive threshold functions $\bar{e}^b_{i,2}(t)$. The secondary absolute fault detection time in the faulty state equation i is defined as follows:

$$T^d_{i,2} \triangleq \max_{b \in \upsilon_{i,1}(T^d_{i,1}+\Psi_i)} \inf_t \Big\{ t \ge T^d_{i,1} + \Psi_i \Big| |e^b_{i,2}(t)| > \bar{e}^b_{i,2}(t)\Big\}.$$

Remark 1: Assumption 3 enables to exclude faults from the set $\mathcal{F}_{i,1}$ for the time interval $t \in [T^d_{i,1},\ T^d_{i,1} + \Psi_i)$. However, even when Assumption 3 does not hold, the presented process is still operational by going directly from Task 1 to Tasks 3 and 4 and neglecting Task 2, considering $\Psi_i = 0$, i.e., trying to detect the secondary fault while simultaneously doing the primary fault exclusion procedure $\forall t \ge T^d_{i,1}$. Therefore, Assumption 3 should not be considered as a restriction in the proposed methodology but rather that as a path for less restrictive primary fault exclusion, through the process described in Task 2.

2) Secondary fault isolation component (Task 4): Following the time instant $t = T^d_{i,1} + \Psi_i$, a secondary fault isolation component is activated within the faulty state equation i with two objectives: 1) to exclude non-occurring primary faults from the set $\mathcal{F}_{i,1}$ after $t = T^d_{i,1} + \Psi_i$ and, 2) to exclude non-occurring secondary faults from the set $\mathcal{F}_{i,2}$ after $t = T^d_{i,2}$ (Task 4).

Before proceeding, we introduce the following notations for simplicity in the design and analysis. For the faulty state equation i, let $y_1 \triangleq b \in \{\upsilon_{i,1}(T^d_{i,1} + \Psi_i)\}$ and $y_2 \triangleq r \in \{1, 2, \ldots, |\mathcal{F}_{i,2}|\}$.

In the scenario of partial primary fault isolation at the time instant $t = T^d_{i,1} + \Psi_i$ in Task 2, it is evident that we cannot determine which specific fault from the set $\mathcal{F}_{i,1}$ has occurred as the primary fault within the i-th system state equation. Consequently, to systematically exclude non-occurring primary faults and potential secondary faults in state equation i, a bank of adaptive estimators is designed for each non-excluded fault from the set $\mathcal{B}_{i,1}(T^d_{i,1} + \Psi_i)$. Each adaptive estimator in these banks is associated with one possible fault from the set $\mathcal{F}_{i,2}$. As a result, the total number of estimators required for this task is $|\mathcal{B}_{i,1}(T^d_{i,1} + \Psi_i)||\mathcal{F}_{i,2}|$.

In the y_1-th bank of isolation estimators, each estimator incorporates two nonlinear adaptive estimation terms. The first term, $\hat{f}^{y_1}_{i,1}(x,u;\hat{\theta}^{y_1}_{i,1})$, provides an online estimate of the primary non-excluded fault within the set $\mathcal{B}_{i,1}(T^d_{i,1} + \Psi_i)$. The second term, $\hat{f}^{y_2}_{i,2}(x,u;\hat{\theta}^{y_2}_{i,2})$, estimates the secondary possible fault from the set $\mathcal{F}_{i,2}$. Specifically, the bank of estimators regarding the y_1-th non-excluded fault is designed as follows:

$$\begin{aligned}\dot{\hat{x}}^{y_1,y_2}_{i,2} = &h_i(x,u) + \sum\nolimits_{k=1}^{2} \hat{f}^{y_k}_{i,k}(x,u;\hat{\theta}^{y_k}_{i,k}) \\ &+ l^{y_1,y_2}_i (x_i - \hat{x}^{y_1,y_2}_{i,2}),\end{aligned} \tag{11}$$

where $\hat{x}^{y_1,y_2}_{i,2}$ is the i-th state estimation in the presence of two unknown possible faults, specifically $f^{y_k}_{i,k}(x,u;\theta^{y_k}_{i,k})$, where estimated as $\hat{f}^{y_k}_{i,k}(x,u;\hat{\theta}^{y_k}_{i,k}) = (\hat{\theta}^{y_k}_{i,k})^{\mathrm{T}} g^{y_k}_{i,k}(x,u)$, for $k \in \mathcal{K}$.

In light of the adaptive estimation model presented in (11), we propose the following parameter learning law:

$$\dot{\hat{\theta}}^{y_k}_{i,k} = \mathbb{P}_{\Theta^{y_k}_{i,k}}\Big\{\gamma^{y_k}_{i,2} e^{y_1,y_2}_{i,2} g^{y_k}_{i,k}(x,u)\Big\}, \quad k \in \mathcal{K}, \tag{12}$$

where $\gamma^{y_k}_{i,k} > 0$ is a learning rate. Moreover, $e^{y_1,y_2}_{i,2}$ denotes the i-th state estimation error (or the i-th residual error of the secondary fault isolation when considering y_1 and y_2 as the primary and secondary faults, respectively), and is defined by $e^{y_1,y_2}_{i,2} = x_i - x^{y_1,y_2}_{i,2}$. Based on (1) and (11), it follows that

$$\begin{aligned}\dot{e}^q_{i,2} = &- k^q_{i,2} e^q_{i,2} + \sum\nolimits_{k=1}^{2} \beta_{i,k}(t - T^0_{i,k})\tilde{f}^{y_k}_{i,k}(x,u;\tilde{\theta}^{y_k}_{i,k}) \\ &+ \omega_i(x,u,t) - \sum\nolimits_{k=1}^{2} e^{(-p_{i,k}(t-T^0_{i,k}))}\hat{f}^{y_k}_{i,k}(x,u;\hat{\theta}^{y_k}_{i,k}).\end{aligned} \tag{13}$$

We are now ready to present the main result for the developed bank of isolation estimators in (11).

Theorem 1: Consider the isolation error dynamics in (13) for the i-th, ($i \in \mathcal{I}$), state equation of system (1) under Assumptions 1-3. Then, the y_1-th bank of isolation estimators in (11) with learning law (12) ensures that:

(i) the error variables $e_{i,2}^{y_1,y_2}$, $\tilde{\theta}_{i,k}^{y_k}$ are uniformly bounded for $y_1 \in \{\upsilon_{i,1}(T_{i,1}^d + \Psi_i)\}$ and $y_2 \in \{1, 2, \ldots, |\mathcal{F}_{i,2}|\}$,

(ii) the fault isolation residual errors $e_{i,2}^{y_1,y_2}$ in all of banks of isolation estimators, specifically $\forall y_1 \in \{\upsilon_{i,1}(T_{i,2}^d)\}$ and $y_2 \in \{1, 2, \ldots, |\mathcal{F}_{i,2}|\}$, satisfy:

$$|e_{i,2}^{y_1,y_2}(t)| \leq \bar{e}_{i,2}^{y_1,y_2}(t), \qquad \forall t \in [T_{i,2}^d, +\infty),$$

if the y_1-th and y_2-th faults occur in the i-th state equation for $t \geq T_{i,2}^d$, respectively, where

$$\bar{e}_{i,2}^{y_1,y_2}(t) \triangleq \int_{T_{i,2}^d}^{t} e^{(-l_{i,2}^{y_1,y_2}(t-\tau))}\left[\Phi_i^{y_1,y_2}(\tau) + \Xi_i^{y_1,y_2}(\tau)\right]\mathrm{d}\tau + |e_{i,2}^{y_1,y_2}(T_{i,2}^d)|e^{(-l_{i,2}^{y_1,y_2}(t-T_{i,2}^d))}, \tag{14}$$

and $\Phi_i^{y_1,y_2}(\tau) \triangleq \bar{\omega}_i(x(\tau), u(\tau), \tau) + \sum_{k=1}^{2} |\tilde{f}_{i,k}^{y_k}(x(\tau), u(\tau); \tilde{\theta}_{i,k}^{y_k}(\tau))|$, $\Xi_i^{y_1,y_2}(\tau) \triangleq \sum_{k=1}^{2} e^{(-\bar{p}_{i,k}(\tau - T_{i,k}^d))}|\hat{f}_{i,k}^{y_k}(x(\tau), u(\tau); \hat{\theta}_{i,k}^{y_k}(\tau))|$, and $\bar{p}_{i,k} > 0$ denotes a known lower bound on the unknown fault evolution rate for the k-th possible fault for the state equation i, i.e., $p_{i,k}^s > \bar{p}_{i,k}$ for all $s \in \{1, 2, \ldots, |\mathcal{F}_{i,k}|\}$.

Hence, if $|e_{i,2}^{y_1,y_2}(t)| > \bar{e}_{i,2}^{y_1,y_2}(t)$ holds at some time $t \geq T_{i,2}^d$, then the simultaneous existence of the primary fault y_1 and the secondary fault y_2 in the state equation i is excluded.

Proof: Omitted due to page limitations. The proof can be established by following the procedure outlined in Lemma 3.1 of [7]. Full details will be provided in the extended journal version of this paper.

The primary fault exclusion policy for $t \in [T_{i,1}^d + \Psi_i,\ T_{i,2}^d)$, deduced from the results of Proposition 1 and Theorem 1, is presented below.

(i) Consider some specific non-excluded fault during the primary fault isolation process, specifically non-excluded primary faults in the set $\mathcal{B}_{i,1}(T_{i,1}^d + \Psi_i)$ with index $y_1 \in \upsilon_{i,1}(T_{i,1}^d + \Psi_i)$. If the primary isolation residual error generated considering non-excluded primary fault y_1 exceeds its respective adaptive threshold, i.e., $|e_{i,1}^{y_1}(t)| > \bar{e}_{i,1}^{y_1}(t)$ within $t \in [T_{i,1}^d + \Psi_i, T_{i,2}^d)$ and, the secondary isolation residual errors generated considering the non-excluded primary fault y_1 and all the possible faults from the set $\mathcal{F}_{i,2}$ exceed their respective adaptive thresholds at some time instants $t \in [T_{i,1}^d + \Psi_i, T_{i,2}^d)$, i.e., $|e_{i,2}^{y_1,y_2}(t)| > \bar{e}_{i,2}^{y_1,y_2}(t)$ within $t \in [T_{i,1}^d + \Psi_i, T_{i,2}^d)$ for $y_1 \in \upsilon_{i,1}(T_{i,1}^d + \Psi_i)$ and for all $y_2 \in \{1, 2, \ldots, |\mathcal{F}_{i,2}|\}$, then the occurrence of the y_1-th possible primary fault from the set $\mathcal{F}_{i,1}$ is excluded.

The primary fault exclusion policy for $t \in [T_{i,2}^d, +\infty)$, deduced from the results Theorem 1, is presented below.

(ii) Consider some specific non-excluded fault during the primary fault isolation process, specifically non-excluded faults in the set $\mathcal{B}_{i,1}(T_{i,2}^d)$ with index $y_1 \in \upsilon_{i,1}(T_{i,2}^d)$. If the secondary isolation residual errors generated considering non-excluded primary fault y_1 and all the possible faults from the set $\mathcal{F}_{i,2}$ exceed their respective thresholds after the secondary fault detection time at $t = T_{i,2}^d$, i.e., $|e_{i,2}^{y_1,y_2}(t)| > \bar{e}_{i,2}^{y_1,y_2}(t)$ for some $t \in [T_{i,2}^d, +\infty)$, for $y_1 \in \upsilon_{i,1}(T_{i,1}^d + \Psi_i)$, and for any $y_2 \in \{1, 2, \ldots, |\mathcal{F}_{i,2}|\}$, then the occurrence of the y_1-th primary fault from the set $\mathcal{F}_{i,1}$ is excluded.

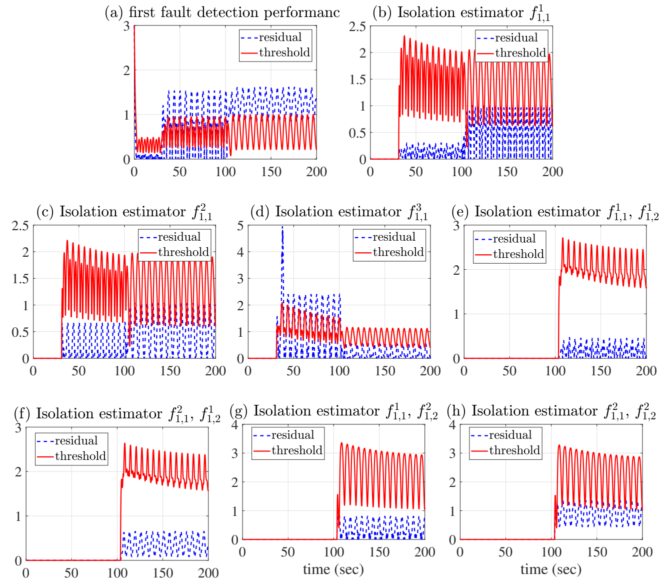

Fig. 2: (a): Time responses of the primary fault detection residual error (dash line) and its threshold (solid line). (b)-(d): Time responses of the primary isolation residual errors (dash lines) and respective thresholds (solid line) with three isolation estimators. (e)-(h): Time responses of the secondary fault isolation residual errors (dash lines) and respective thresholds (solid line) with four isolation estimators.

The secondary fault exclusion policy for $t \in [T_{i,2}^d, +\infty)$, deduced from the results Theorem 1, is presented below.

(iii) Suppose that $y_1 \in \{\upsilon_{i,1}(T_{i,2}^d)\}$ and $y_2 \in \{1, 2, \ldots, |\mathcal{F}_{i,2}|\}$ occur in state equation i at $t = T_{i,1}^0$ and $t = T_{i,2}^0$, and are detected at $t = T_{i,1}^d$ and $t = T_{i,2}^d$, respectively. Furthermore, during the primary fault isolation process by the time instant $t = T_{i,2}^d$, a subset of possible faults from the set $\mathcal{F}_{i,1}$ is excluded, and the remaining non-excluded faults are included in the set $\mathcal{B}_{i,1}(T_{i,2}^d)$ with index $b \in \upsilon_{i,1}(T_{i,2}^d)$. In this context, a set of adaptive threshold functions $\bar{e}_{i,2}^{y_1,y_2}(t)$ for all $b \in \upsilon_{i,1}(T_{i,2}^d)$ exist such that the corresponding isolation errors satisfy $|e_{i,2}^{y_1,y_2}(t)| \leq \bar{e}_{i,2}^{y_1,y_2}(t)$ after the secondary fault detection. For some specific fault y_2 from the set $\mathcal{F}_{i,2}$ within state equation i, if the residual error generated considering fault y_2 and any non-excluded fault from primary fault isolation process exceeds its respective adaptive threshold, i.e., $|e_{i,2}^{y_1,y_2}(t)| > \bar{e}_{i,2}^{y_1,y_2}(t)$ for some $y_1 \in \upsilon_{i,1}(T_{i,2}^d)$, then the occurrence of the y_1-th and y_2-th fault from the set $\mathcal{F}_{i,1}$ and $\mathcal{F}_{i,2}$ are excluded.

IV. Simulation Results

Simulation setup: Consider the nonlinear system given in (1), where $h(x, u) = (1 + \sin^2(x))u + \exp(x)\sin(x)$ and $\omega(x, u, t) = 0.5x\cos(t)$. Specifically, we consider two sets of faults: $\mathcal{F}_{1,1} \triangleq \{\theta_{1,1}^1 \tanh(x), \theta_{1,1}^2 \frac{x}{\sqrt{1+x^2}}, \theta_{1,1}^3 \exp(1.5x)\}$ and $\mathcal{F}_{1,2} \triangleq \{\theta_{1,2}^1 \sin(x), \theta_{1,2}^2 \cos(x)\}$, corresponding to the first and second fault occurrences, respectively, where $\theta_{1,1} \in \Theta_{1,1} = [-1, 3]$, $\theta_{1,2} \in \Theta_{1,2} = [-2, 2]$, $\theta_{1,3} \in \Theta_{1,3} = [-2, 4]$, $\theta_{2,1} \in \Theta_{2,1} = [-1, 3]$, and $\theta_{2,2} \in \Theta_{2,2} = [-1, 3]$. Moreover, we suppose that $p_{i,k} \geq \bar{p}_{i,k} = 0.02$.

To implement the proposed fault diagnosis structure, we select $\gamma_{i,k} = 5$ and $l_{i,k} = 2$, respectively. Moreover, Ψ_i is assumed to be 30 seconds. The upper bound of the modeling uncertainty satisfies $\bar{\omega} = |0.7x|$. To satisfy Assumption 1, the nominal control law, using the feedback linearization scheme, is designed as $u = 1/g(x)[-h(x) - kz + \dot{x}_d]$, where $z = x - x_d$ denotes the tracking error, and $x_d = \sin(t)$ and $\dot{x}_d = \cos(t)$ describe the desired trajectory and its first time-derivative, respectively.

Fig. 2(a) presents the simulation results for the case where an incipient fault from the set $\mathcal{F}_{1,1}$, specifically the first component of $\mathcal{F}_{1,1}$, with an unknown fault evolution rate of $p_{1,1} = 0.2$ and an amplitude of $\theta_{1,1}^1 = 1.5$, occurs at $t = 30$ seconds. The state estimation error (solid line) of the primary fault detection estimator and its corresponding model-based threshold are shown in Fig. 2(a). It can be observed that the primary fault is detected at $t = 31.5$ seconds. Additionally, in Figs. 2(b)-2(d), the isolation residual errors and their corresponding thresholds for each of the three fault isolation estimators employed in the primary fault isolation process are plotted. It is evident that the fault isolation residual of the third estimator exceeds its threshold at approximately $t = 33.2$ seconds, while the residuals of the first and second estimators remain below their thresholds. This allows the exclusion of the third fault, i.e., $f_{1,1}^3$, from the set $\mathcal{F}_{1,1}$, indicating partial isolation of the $f_{1,1}^1$ and $f_{1,1}^2$ within this set. Note that only partial isolation is achieved in this case due to the similar structures of the remaining faults, specifically the similar time profiles of functions $\tanh(x)$ and $\frac{x}{\sqrt{1+x^2}}$, which makes it difficult to distinguish between them.

By monitoring the residual errors and their respective thresholds from the first and second estimators regarding to the partially isolated fault during primary isolation after $t = 61.5$ seconds in Figs. 2(b) and 2(c), based on Assumption 3, the occurrence of the secondary fault, specifically an incipient fault from the set $\mathcal{F}_{1,2}$, with an unknown fault evolution rate of $p_{1,2} = 0.3$ and an amplitude of $\theta_{1,2}^1 = 2$, is detected at approximately $t = 103$ seconds. Considering the two non-excluded faults from $\mathcal{F}_{1,1}$, specifically, $f_{1,1}^1$ and $f_{1,1}^2$, and the two possible faults from $\mathcal{F}_{1,2}$, specifically, $f_{1,2}^1$ and $f_{1,2}^2$, four isolation estimators are implemented after $t = 61.5$ seconds using the cartesian product of the above-mentioned fault functions to isolate the secondary occurred fault from the set $\mathcal{F}_{1,2}$. Furthermore, in Figs. 2(e)-2(f), the residuals and their corresponding thresholds for each of these four isolation estimators are shown. It is clear that the two residuals associated with the first fault of $\mathcal{F}_{1,2}$ remain below their thresholds, in Figs. 2(e) and 2(f), while the two residuals associated with the second fault of $\mathcal{F}_{1,2}$ exceed their thresholds after $t = 103$ seconds, thus enabling the isolation of the second fault, in Figs. 2(g) and 2(h). Hence, the presented numerical simulations validate the theoretical results by demonstrating the detectability and isolability properties of the proposed scheme for up to two faults in system state equation, even when the primary fault is only partially isolated.

V. Conclusion

This paper presented a robust fault diagnosis architecture for detecting and isolating two consecutive faults in nonlinear uncertain systems with multiple faults. We introduce the concept of partial fault isolation which deviates from the widely employed fault mismatch concept. The proposed isolation strategy, utilizing primary fault partial isolation for the secondary fault detection and isolation, features a dynamic structure that represents a key contribution to diagnosing two consecutive possible faults. The robustness of the proposed fault diagnosis architecture is analytically guaranteed. The practicality and applicability of the presented analytic results is validated through numerical simulations.

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