A parametric family of affine-related quadratic polynomials: index pre-sieve, modular partitions, and Bateman–Horn classes

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Abstract

We introduce a parametric family of affine-related quadratic polynomials

$$d_A(n) = n^2 + n + A, \qquad N_A(n) = (A+1)d_A(n) + 1, \qquad A \geq 1,$$

which arises deterministically from a rational transformation applied to an explicit integer sequence. Building on the index pre-sieve developed in [1] and on the modular partition based on the Chinese remainder theorem introduced in [2], we show that all structural results—the density of the filtered set, the partition of the indices into arithmetic progressions, the constancy of modular signatures, and the admissibility dichotomy—extend without modification to the entire parametric family. Assuming the Bateman–Horn conjecture, each admissible class contains infinitely many prime numbers of both forms.

1 Deterministic origin of the parametric family

The results of the present work build upon two previous studies. In the first, *Affine-related quadratic polynomials, an index pre-sieve and finite truncations of Bateman–Horn local factors* [1], an index pre-sieve for integer polynomials was introduced, and the natural density of the filtered set was established, interpreted as a finite truncation of the local factors appearing in the Bateman–Horn heuristic. In the second, *A uniform modular partition of prime candidates generated by a pair of affine-related quadratic polynomials based on an index pre-sieve* [2], the modular structure induced by the sieve was made explicit: a CRT-based partition of the indices, constant modular signatures, and an admissibility dichotomy for the resulting classes. The present work shows that these results extend without modification to an entire parametric family of affine-related quadratic polynomials.

Let $A \geq 1$ be a fixed integer and define

$$d_A(n) = n^2 + n + A, \qquad N_A(n) = (A+1)d_A(n) + 1, \qquad n \in \mathbb{N}_0.$$

This pair of polynomials is not introduced in an ad hoc manner, but arises naturally from a rational transformation applied to an explicit integer sequence.

Definition 1.1 (Generating rational transformation)**.** *For each $A \geq 1$ we define the rational function*

$$f_A(x) = \frac{(A+1)x + A}{x}.$$

Consider the integer sequence

$$x_A(n) = A\,d_A(n) = A\big(n^2 + n + A\big).$$

A direct computation shows that

$$f_A(x_A(n)) = \frac{(A+1)Ad_A(n) + A}{Ad_A(n)} = \frac{(A+1)d_A(n)+1}{d_A(n)} = \frac{N_A(n)}{d_A(n)}.$$

Remark 1.2. *For $A = 5$, this construction coincides exactly with the generating mechanism described in Appendix A of [1]. It follows that the entire family $\{(d_A, N_A)\}_{A\geq 1}$ shares the same deterministic origin.*

2 Index pre-sieve for the family d_A and N_A

Let $g \in \mathbb{Z}[n]$ be a non-constant polynomial and let $\mathcal{P}$ be a finite set of prime numbers. For each $p \in \mathcal{P}$ we set

$$\nu_g(p) = \#\{a \in \mathbb{Z}/p\mathbb{Z} : \; g(a) \equiv 0 \pmod p\}.$$

Definition 2.1 (Filtered set of indices)**.**

$$S_g(\mathcal{P}) = \{n \in \mathbb{N}_0 : \; g(n) \not\equiv 0 \pmod p \text{ for every } p \in \mathcal{P}\}.$$

Proposition 2.2 (Density of the filtered set)**.** *The natural density of $S_g(\mathcal{P})$ exists and is given by*

$$\operatorname{dens}(S_g(\mathcal{P})) = \prod_{p\in\mathcal{P}} \Big(1 - \frac{\nu_g(p)}{p}\Big).$$

Proof. The proof is identical to that of Proposition 3.1 in [1] and relies on the independence of the local conditions and on the Chinese remainder theorem. □

Applying this construction to $g = d_A$ and to $g = N_A$ yields the index pre-sieve for the entire parametric family.

3 Filters, CRT partition, and modular signatures

Fix a finite set of primes $\mathcal{P}$ and set

$$M = \prod_{p\in\mathcal{P}} p.$$

Definition 3.1 (Filter)**.** *A* filter *is a family $C = (c_p)_{p\in\mathcal{P}}$ with $c_p \in \mathbb{Z}/p\mathbb{Z}$. To a filter we associate the set of indices*

$$\mathcal{C}_C = \{n \geq 0 : \; n \equiv c_p \pmod p \text{ for every } p \in \mathcal{P}\}.$$

By the Chinese remainder theorem, there exists a unique residue r mod M such that

$$\mathcal{C}_C = r + M\mathbb{Z}.$$

Definition 3.2 (Modular signatures)**.** *For any $n \in \mathcal{C}_C$ we define*

$$\Sigma_C^{(d_A)} = (d_A(n) \bmod p)_{p\in\mathcal{P}}, \qquad \Sigma_C^{(N_A)} = (N_A(n) \bmod p)_{p\in\mathcal{P}}.$$

Lemma 3.3 (Constancy of the signatures)**.** *The modular signatures* $\Sigma_C^{(d_A)}$ *and* $\Sigma_C^{(N_A)}$ *are constant on* $\mathcal{C}_C$*, that is, they do not depend on the choice of* $n \in \mathcal{C}_C$*.*

Proof. If $n \equiv c_p \pmod p$, then, since d_A and N_A have integer coefficients, we have

$$d_A(n) \equiv d_A(c_p) \pmod p, \qquad N_A(n) \equiv N_A(c_p) \pmod p.$$

□

4 Admissibility and the $\mathcal{P}$-rough dichotomy

Definition 4.1 ($\mathcal{P}$-rough)**.** *An integer is said to be* $\mathcal{P}$*-rough if it is not divisible by any prime in* $\mathcal{P}$*.*

Definition 4.2 (Admissible filter)**.** *A filter* C *is said to be* d_A-admissible *if* $d_A(c_p) \not\equiv 0 \pmod p$ *for every* $p \in \mathcal{P}$*. Analogously,* C *is* N_A-admissible *if* $N_A(c_p) \not\equiv 0 \pmod p$ *for every* $p \in \mathcal{P}$*.*

Lemma 4.3 (Admissibility dichotomy)**.** *If* C *is* d_A*-admissible, then for every* $n \in \mathcal{C}_C$ *the value* $d_A(n)$ *is* $\mathcal{P}$*-rough. If* C *is not* d_A*-admissible, then no* $n \in \mathcal{C}_C$ *produces a* $\mathcal{P}$*-rough value. An analogous statement holds upon replacing* d_A *with* N_A*.*

Proof. By Lemma 3.3, the divisibility of $d_A(n)$ (or $N_A(n)$) by a prime $p \in \mathcal{P}$ is constant throughout the entire class $\mathcal{C}_C$. □

5 A bridge to the Bateman–Horn conjecture

In this section we formulate a conjectural bridge to the Bateman–Horn conjecture [3], applied to the pair of polynomials (d_A, N_A) and to the congruence classes determined by the index pre-sieve.

Let $r \bmod M$ be a residue corresponding to an admissible filter. We set

$$\pi_{d_A,r}(X) = \#\{n \le X : \ n \equiv r \pmod M,\ d_A(n) \text{ is prime}\},$$

and analogously define $\pi_{N_A,r}(X)$.

Proposition 5.1 (Heuristic consequence of Bateman–Horn)**.** *Assuming the Bateman–Horn conjecture for the pair* (d_A, N_A)*, every admissible congruence class* $r \bmod M$ *contains infinitely many prime numbers of both forms* $d_A(n)$ *and* $N_A(n)$*.*

Remark 5.2. *All structural results of the present work are unconditional and rely exclusively on the index pre-sieve and on the CRT-based partition. The infinitude of prime numbers within a single class inevitably requires a conjectural assumption.*

6 Conclusions

We have shown that the index pre-sieve and the modular structure developed in [1, 2] extend naturally to the entire parametric family of affine-related quadratic polynomials $\{(d_A, N_A)\}_{A\ge 1}$. This construction highlights a rigid, deterministic, and verifiable modular structure, which makes it possible to classify prime candidates into congruence classes with fixed modular signatures. Assuming the Bateman–Horn conjecture, each admissible class contains infinitely many prime numbers, providing a unified conceptual framework that generalizes the results obtained in the previously studied special cases.

References

[1] M. Russo, *Affine-related quadratic polynomials, an index pre-sieve and finite truncations of Bateman–Horn local factors: an experimental study*, Zenodo preprint, 2026. doi:10.5281/zenodo.18195623

[2] M. Russo, *A uniform modular partition of prime candidates generated by a pair of affine-related quadratic polynomials based on an index pre-sieve*, Zenodo preprint, 2026. doi:10.5281/zenodo.18208305

[3] P. T. Bateman and R. A. Horn, *A heuristic asymptotic formula concerning the distribution of prime numbers*, Mathematics of Computation **16** (1962), 363–367.