Relaxation Transform for Iterations of the Sinusoidal Map and Its Physical Interpretation as a Memory-Based Process

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Abstract

Classical relaxation models based on linear differential equations lead to exponential decay and often inadequately describe complex nonlinear systems with memory. This work proposes an alternative approach based on the theory of function iterations. The family of maps $f_a(x) = \sin(x + a)$, $a \in \mathbb{R}$ is considered. A relaxation transform $\Upsilon_a(n, x)$ is introduced, continuously dependent on the nesting parameter $n \in \mathbb{R}_{\geq 0}$ and interpolating the discrete iterations $f_a^{\lfloor n \rfloor}(x)$. Its main properties are proven: consistency with integer iterations, continuity in n, and convergence to the fixed point $U(a)$ of the equation $\sin(x+a) = x$. A physical interpretation of the model as a relaxation process with memory is proposed, where the parameter n is interpreted as a measure of the system's accumulated "experience", and the mapping of physical time t to n is given by a function $n(t)$ defining the relaxation law. Variants of $n(t)$ leading to classical exponential, linear, and power-law decays are considered. The model opens new possibilities for describing non-exponential relaxation in complex systems such as glassy materials and phase synchronization systems.

Keywords: function iterations, relaxation transform, fixed point, nonlinear dynamics, systems with memory, non-exponential decay.

1 Introduction

1.1 The Problem of Non-Exponential Relaxation in Complex Systems

Modeling relaxation processes is a fundamental task in physics, chemistry, and dynamical systems theory. Traditional approaches based on first-order linear differential equations ($\dot{x} = -\gamma x$) lead to the universal exponential decay law $x(t) \propto e^{-\gamma t}$. However, many real systems exhibit significantly more complex behavior [1, 2, 3]:

- **Glass-forming systems** (polymers, silicate glasses) are characterized by stretched exponential laws (Kohlrausch-Williams-Watts function)
- **Geological media** exhibit power-law stress relaxation
- **Biological tissues** show complex viscoelastic relaxation with multiple time scales

Such behavior indicates the presence of memory, temporal nonlocality, or hierarchical structure in the system, which cannot be adequately described within simple linear models.

1.2 Existing Approaches and Their Limitations

Various approaches have been proposed to describe complex relaxation kinetics:

1. **Fractal time models** [4], where time is endowed with a complex structure
2. **Fractional differential equations** [5], accounting for temporal nonlocality
3. **Hierarchical models** [6], assuming a distribution of relaxation times

Although these approaches successfully describe certain classes of phenomena, they are often phenomenological and lack a transparent physical interpretation. Moreover, they may require complex mathematical apparatus for numerical modeling.

1.3 An Alternative Approach Based on Iteration Theory

This work proposes a fundamentally different approach based on the concept of discrete dynamical systems and function iterations. It is known that iterations of contracting maps converge to their fixed points. A classic example is solving the equation $\cos x = x$ by simple iterations: the sequence $x_{n+1} = \cos x_n$ converges to the fixed point $x^* \approx 0.739$. This process can be viewed as discrete relaxation, where each iteration brings the system closer to equilibrium.

This idea allows for substantial generalization. Consider a one-parameter family of nonlinear maps:

$$f_a(x) = \sin(x + a), \quad a \in \mathbb{R}$$

For any initial condition x_0, the sequence of iterations $f_a^k(x_0)$ converges to the unique fixed point $U(a)$, which is the solution to $\sin(x + a) = x$. A natural question arises: can we construct a continuous family of transforms $\Upsilon_a(n, x)$ that interpolates the discrete iterations for non-integer n and describes a smooth relaxation process?

1.4 Aims and Structure of the Work

The main goal is to construct and study the relaxation transform $\Upsilon_a(n, x)$ with its subsequent physical interpretation as a relaxation model with memory. Specific tasks:

1. Constructing the transform $\Upsilon_a(n, x)$ continuous in parameter n
2. Proving its main properties
3. Developing a physical interpretation by introducing a memory function $n(t)$
4. Investigating various relaxation laws corresponding to different forms of $n(t)$
5. Discussing analogies with specific physical systems

The work is organized as follows. Section 2 introduces the mathematical apparatus: the fixed point $U(a)$ and multiplier λ_a are defined, and the transform $\Upsilon_a(n, x)$ is constructed and studied. Section 3 reviews existing methods and compares them with the proposed approach. Section 4 provides a physical interpretation of the model, introduces the concept of the memory function $n(t)$, and discusses analogies with specific physical systems. Section 5 is devoted to numerical methods and examples. Section 6 discusses prospects and directions for further research.

Advantages of the proposed approach:

- Natural mathematical foundation in dynamical systems theory
- Transparent physical interpretation of parameter n as a measure of system "experience"
- Flexibility in describing various relaxation laws via the function $n(t)$
- Relative simplicity of numerical implementation

2 Mathematical Apparatus

2.1 Fixed Point and Its Properties

Definition 2.1. *For a given parameter $a \in \mathbb{R}$, define the function $U(a)$ as the solution to the equation*

$$\sin(x + a) = x \tag{1}$$

The function $U : \mathbb{R} \to \mathbb{R}$ assigns to the parameter a the value of the fixed point x_a^*.

Lemma 2.2. *For any $a \in \mathbb{R}$, equation (1) has a unique real solution x_a^*.*

Proof. The function $g(x) = \sin(x + a) - x$ is continuous, with $g(x) \to -\infty$ as $x \to +\infty$ and $g(x) \to +\infty$ as $x \to -\infty$. By the Intermediate Value Theorem, a solution exists. Uniqueness follows from the derivative $g'(x) = \cos(x + a) - 1 \leq 0$, where equality to zero is achieved only on a set of measure zero (when $\cos(x + a) = 1$). □

Definition 2.3. *The convergence parameter (multiplier) at the fixed point is defined as*

$$\lambda_a = f_a'(x_a^*) = \cos(x_a^* + a) = \cos(U(a) + a) \tag{2}$$

Lemma 2.4. *For all $a \in \mathbb{R}$, the inequality $0 \leq \lambda_a < 1$ holds, and $\lambda_a = 1$ is achieved only for $a = 0$ and $U(0) = 0$.*

Proof. From (1) and (2) it follows that $x_a^* = \sin(U(a) + a)$. Then $\lambda_a = \cos(U(a) + a) = \sqrt{1 - \sin^2(U(a) + a)} = \sqrt{1 - (x_a^*)^2}$ for $x_a^* \in [-1, 1]$. Obviously, $0 \leq \lambda_a \leq 1$. Equality $\lambda_a = 1$ occurs only when $x_a^* = 0$, which according to (1) is equivalent to $\sin a = 0$ and, consequently, $a = 0$ (in the vicinity of zero). For $a = 0$, we have $U(0) = 0$. □

Observation 2.1. *The value of λ_a determines the convergence rate of iterations to the fixed point. The smaller λ_a, the faster the convergence.*

2.2 Relaxation Transform $\Upsilon_a(n, x)$

The discrete dynamics of the system is given by iterations: $x_{k+1} = f_a(x_k)$. For integer $n = k$, we have $x_k = f_a^k(x_0)$. Our goal is to construct a family $\Upsilon_a(n, x)$ continuous in n, consistent with integer iterations.

Motivation for the definition. In the neighborhood of the fixed point $U(a)$, the map linearizes: $f_a(U(a)+\epsilon) \approx U(a)+\lambda_a\epsilon$. Therefore, for integer k and small $\Delta = x_0 - U(a)$, we have $f_a^k(x_0) \approx U(a)+\lambda_a^k\Delta$. It is natural to require that this asymptotics be preserved for intermediate values of n. This leads to the following definition.

Definition 2.5. *For $n \in \mathbb{R}_{\geq 0}$ and $x \in \mathbb{R}$, the relaxation transform is defined by the formula:*

$$\Upsilon_a(n,x) = U(a) + \left(f_a^{\lfloor n \rfloor}(x) - U(a)\right) \cdot \lambda_a^{n-\lfloor n \rfloor} \tag{3}$$

where $\lfloor n \rfloor$ is the integer part of n.

Geometric interpretation. The transform $\Upsilon_a(n,x)$ can be represented as a process consisting of two stages:

1. Performing $\lfloor n \rfloor$ full iterations of the map f_a
2. Bringing the result to a "fractional iteration" by multiplying by a power of the multiplier λ_a

Theorem 2.6 (Main Properties of the Transform Υ_a)**.** *1. **Endpoint consistency:** $\Upsilon_a(0,x) = x$, $\Upsilon_a(1,x) = f_a(x)$*

2. ***Continuity in** n: The function $n \mapsto \Upsilon_a(n,x)$ is continuous on $\mathbb{R}_{\geq 0}$*
3. ***Consistency with integer iterations:** For any $k \in \mathbb{N}$, $\Upsilon_a(k,x) = f_a^k(x)$ holds*
4. ***Limit behavior:** $\lim\limits_{n\to\infty} \Upsilon_a(n,x) = U(a)$ for any x*
5. ***Approximate semigroup property:** For any $s,t \geq 0$, the estimate holds*

$$|\Upsilon_a(s+t,x) - \Upsilon_a(s,\Upsilon_a(t,x))| \leq C \cdot \left|f_a^{\lfloor t \rfloor}(x) - U(a)\right|^2$$

where the constant C depends on a and the derivatives of f_a near $U(a)$.

Proof. Properties 1–4 follow directly from definition (3). To prove property 5, write:

$$\Upsilon_a(t,x) = U + \Delta_t, \quad \text{where } \Delta_t = (f_a^{\lfloor t \rfloor}(x) - U)\lambda_a^{t-\lfloor t \rfloor}$$

Then

$$\Upsilon_a(s,\Upsilon_a(t,x)) = U + \left(f_a^{\lfloor s \rfloor}(U+\Delta_t) - U\right)\lambda_a^{s-\lfloor s \rfloor}$$

Expand $f_a^{\lfloor s \rfloor}$ near U: $f_a^{\lfloor s \rfloor}(U+\epsilon) = U + \lambda_a^{\lfloor s \rfloor}\epsilon + O(\epsilon^2)$. Substituting $\epsilon = \Delta_t$, we get:

$$\Upsilon_a(s,\Upsilon_a(t,x)) = U + \left(\lambda_a^{\lfloor s \rfloor}\Delta_t + O(\Delta_t^2)\right)\lambda_a^{s-\lfloor s \rfloor} = U + \lambda_a^s\Delta_t + O(\Delta_t^2)$$

At the same time,

$$\Upsilon_a(s+t,x) = U + \left(f_a^{\lfloor s+t \rfloor}(x) - U\right)\lambda_a^{s+t-\lfloor s+t \rfloor}$$

Note that $f_a^{\lfloor s+t \rfloor}(x) - U = \lambda_a^{\lfloor s+t \rfloor}(x-U) + O((x-U)^2)$. The leading term of the difference $\Upsilon_a(s+t,x) - \Upsilon_a(s,\Upsilon_a(t,x))$ is determined by the discrepancy between $\lambda_a^{\lfloor s+t \rfloor}$ and $\lambda_a^{\lfloor s \rfloor + \lfloor t \rfloor}$ and nonlinear corrections. The $O(\Delta_t^2)$ estimate follows from the expansion form. □

Advantage of the chosen mathematical apparatus. Formula (3) provides a compact and effective way to interpolate discrete iterations. Unlike methods based on solving functional equations for fractional iterations [7], our approach requires only computing integer iterations and a simple exponentiation operation.

3 Comparison with Existing Methods

3.1 Traditional Approaches to Modeling Relaxation

To contextualize the proposed approach, consider the main existing methods for modeling relaxation processes:

1. **Linear differential equations:**

$$\frac{dx}{dt} = -\gamma x \quad \Rightarrow \quad x(t) = x_0 e^{-\gamma t}$$

Advantages: simplicity, analytical solution
Limitations: only exponential decay, does not account for system memory

2. **Fractional differential equations [5]:**

$$\frac{d^\alpha x}{dt^\alpha} = -\gamma x, \quad 0 < \alpha < 1$$

Advantages: describes non-exponential relaxation
Limitations: complex mathematical apparatus, nontrivial numerical implementation

3. **Integral equations with memory kernels [8]:**

$$x(t) = x_0 - \int_0^t K(t-\tau)x(\tau)d\tau$$

Advantages: explicit accounting of system memory
Limitations: requires specifying the memory kernel, complexity of analytical analysis

3.2 Advantages of the Proposed Approach

The proposed formalism based on the transform $\Upsilon_a(n, x)$ offers several advantages:

1. **Transparent physical interpretation:** Parameter n has a clear meaning — the number of "virtual steps" of relaxation

2. **Natural description of discrete systems:** The approach is especially effective for inherently discrete systems (impact systems, Poincaré maps)

3. **Flexibility in describing temporal dynamics:** Various relaxation laws in physical time are obtained by simple choice of the function $n(t)$

4. **Computational efficiency:** The algorithm requires only computing integer iterations and exponentiation

5. **Connection with dynamical systems theory:** The approach is based on the well-developed theory of function iterations

3.3 Areas of Application

The proposed method is most effective for:

- Systems with discrete dynamics (Poincaré maps)
- Processes with an explicit hierarchy of relaxation times
- Modeling systems with memory where traditional differential models prove inadequate

4 Physical Interpretation as a Relaxation Model with Memory

4.1 Concept of the Memory Function $n(t)$

The key idea of the physical interpretation is the separation of the concepts of *internal system experience* and *physical time*. The parameter n in the transform $\Upsilon_a(n, x)$ is interpreted as a **measure of the accumulated "experience" of relaxation**, the number of "virtual steps" towards equilibrium. Physical time t flows independently, and its connection with experience n is determined by system properties such as memory, hierarchical structure, or dependence of relaxation rate on the current state.

Consider an ensemble of coupled nonlinear oscillators or a distributed system (e.g., a string). The state of the mode with index s at time t is denoted $x(s, t)$. Let the initial state be given by a function $\phi(s)$.

Definition 4.1. *The dynamics of relaxation for such a system is modeled by the equation:*

$$x(s,t) = \Upsilon_a(\, n(t),\, \phi(s)\,), \qquad t \in [0, T] \tag{4}$$

*where $n : [0, T] \to \mathbb{R}_{\geq 0}$ is a monotonically increasing function, the **memory (or "experience") function**, satisfying:*

- $n(0) = 0$ *(no experience at the initial moment)*
- $\lim_{t \to T^-} n(t) = +\infty$ *(as $t \to T$, the system accumulates infinite experience and reaches equilibrium)*

The parameter T is the characteristic time for establishing equilibrium in the system.

4.2 Variants of the Mapping $n(t)$ and Corresponding Relaxation Laws

The choice of a specific form for $n(t)$ determines the **relaxation law in physical time** and reflects the internal properties of the system.

4.2.1 Linear mapping:

$$n_{\text{lin}}(t) = N_{\max} \cdot \frac{t}{T}, \quad t \in [0, T]$$

- **Physical meaning:** System without memory. "Experience" accumulates uniformly in time. Relaxation rate is constant.
- **Relaxation law:**
$$x(t) - U(a) \propto \lambda_a^{t/T}$$
i.e., stretched exponential decay.

4.2.2 Exponential mapping:

$$n_{\exp}(t) = -\frac{\ln\left(1 - \frac{t}{T}\right)}{|\ln \lambda_a|}, \quad t \in [0, T)$$

- **Physical meaning:** A system where the rate of "experience" accumulation is proportional to the remaining "inexperience". This corresponds to the classical assumption that the relaxation rate is proportional to the deviation from equilibrium.
- **Relaxation law:**
$$x(t) - U(a) \propto (1 - t/T)$$
i.e., **linear decay** in physical time.

4.2.3 Power-law mapping:

$$n_{\text{pow}}(t) = \frac{(t/T)^p}{(1 - t/T)^q}, \quad p, q > 0$$

- **Physical meaning:** Models complex systems with a hierarchy of relaxation times or with aging phenomena. Parameters p and q allow describing slowing ($q > 0$) or acceleration ($p > 1$) of the process at the beginning and end of relaxation.
- **Relaxation law:** For $t \ll T$, we have $n(t) \propto t^p$, leading to initial decay according to:
$$x(t) - U(a) \propto \exp(-\text{const} \cdot t^p)$$
which corresponds to a stretched exponential — a typical form for many glassy systems [9].

4.3 Detailed Analogies with Physical Systems

4.3.1 Analogy 1: Nonlinear pendulum with viscous friction and periodic excitation

Consider a physical pendulum subject to:

- Viscous friction with coefficient γ
- Periodic excitation with amplitude A and frequency ω

Equation of motion:

$$\ddot{\theta} + \gamma\dot{\theta} + \omega_0^2 \sin\theta = A\cos(\omega t)$$

The Poincaré map for this system (phase mapping over the excitation period) can often be approximated by a one-dimensional map of the form:

$$\theta_{n+1} = f_a(\theta_n) + \xi_n$$

where θ_n is the pendulum phase at time $t = nT$, ξ_n is small noise, f_a is a nonlinear function.

In this context:

- $\Upsilon_a(n, \theta_0)$ describes the averaged phase evolution over n excitation periods
- The function $n(t)$ relates the number of periods to real time: $n(t) \approx \omega t/(2\pi)$
- The fixed point $U(a)$ corresponds to a synchronous motion regime of the pendulum

4.3.2 Analogy 2: Relaxation in glass-forming systems

Glass-forming systems (polymers, silicate glasses) are characterized by:

1. **Non-exponential relaxation** (Kohlrausch-Williams-Watts law):

$$\phi(t) = \exp\left[-\left(\frac{t}{\tau}\right)^{\beta}\right], \quad 0 < \beta < 1$$

2. **Aging phenomenon:** system properties depend on observation time and preparation history
3. **Hierarchy of relaxation times**

In the proposed model, these features are naturally accounted for:

- Non-exponential decay is ensured by choosing a power-law function $n_{\text{pow}}(t)$
- System memory is encoded in the form of the function $n(t)$
- Parameter n can be interpreted as a measure of progress along the hierarchy of metastable states

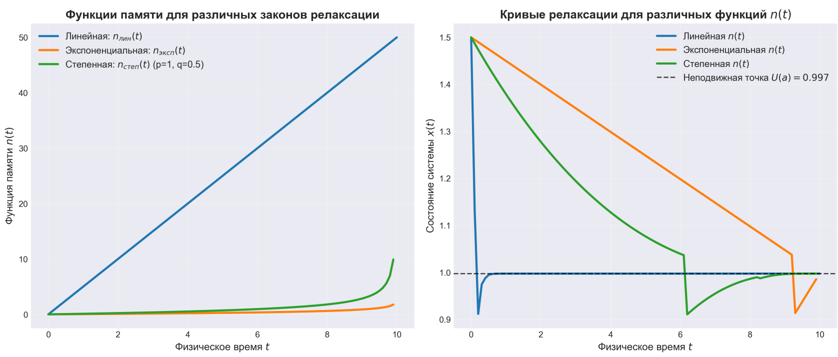

Figure 1: Relaxation curves for different functions $n(t)$: linear (blue), exponential (red), power-law with $p = 1, q = 0.5$ (green)

4.3.3 Analogy 3: Phase-locked loop (PLL) systems

In PLL systems [10], the phase and frequency of one oscillator are adjusted to match a reference signal. The first-order PLL equation:

$$\frac{d\phi}{dt} = \Delta\omega - K \sin\phi$$

where ϕ is the phase difference, $\Delta\omega$ is the frequency detuning, K is the loop gain coefficient.

The discrete version of this equation (in digital implementation):

$$\phi_{n+1} = \phi_n + \Delta\omega T - KT \sin\phi_n$$

This map can be reduced to a form close to $f_a(x) = \sin(x+a)$ with appropriate scaling. In this context:

- The fixed point $U(a)$ corresponds to the steady-state phase shift
- Parameter λ_a characterizes the frequency locking rate
- Model (4) describes a non-ideal transient process in a PLL in the presence of nonlinear distortions

5 Numerical Methods and Examples

5.1 Algorithm for Computing $\Upsilon_a(n,x)$

1. **Finding $U(a)$:** Solving the equation $\sin(x+a) = x$
   - *Simple iteration method:* $x_{k+1} = \sin(x_k + a)$ (converges for any initial guess)
   - *Newton's method:* $x_{k+1} = x_k - \frac{\sin(x_k+a)-x_k}{\cos(x_k+a)-1}$ (faster convergence)
2. **Computing** $\lambda_a = \cos(U(a) + a)$
3. **Computing the integer part iteration:**

$$f_a^{\lfloor n \rfloor}(x) = \underbrace{f_a(f_a(\dots f_a(x)\dots))}_{\lfloor n \rfloor \text{ times}}$$

4. **Applying formula (3)**

Computational complexity: $O(\lfloor n \rfloor)$, which is efficient for moderate values of n.

5.2 Special Cases and Asymptotics

1. **Degenerate case $a = 0$:**

$$U(0) = 0, \quad \lambda_0 = 1$$

   The transform degenerates: $\Upsilon_0(n,x) = \sin^{\lfloor n \rfloor}(x)$, no convergence to a limit, corresponding to a neutral fixed point.

2. **Case of small a (linearization):**

$$U(a) \approx a + \frac{a^3}{6} + O(a^5), \quad \lambda_a \approx 1 - \frac{a^2}{2} + O(a^4)$$

Relaxation occurs slowly.

3. **Limit $a \to \pi/2$:**

$$U(\pi/2) \to 1, \quad \lambda_a \to 0$$

Relaxation becomes practically instantaneous after the first "iteration".

5.3 Visualizing the Relaxation Process

Observations:

- The curve corresponding to $n_{\text{lin}}(t)$ shows smooth, nearly exponential decay
- The curve for $n_{\text{exp}}(t)$ is a straight line, corresponding to the linear law $x(t) - U(a) \propto (1 - t/T)$
- The curve for $n_{\text{pow}}(t)$ shows slowed decay at the initial moment and faster approach to equilibrium at the end of the process

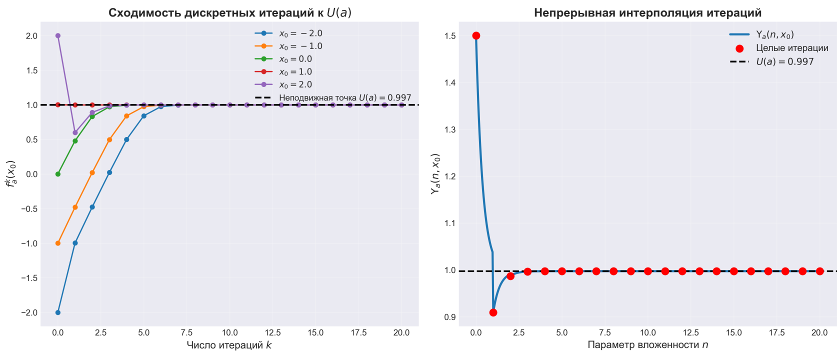

Figure 2: Convergence of discrete iterations (left) and continuous interpolation by the transform Υ_a (right)

6 Discussion and Prospects

6.1 Main Results

1. **Constructed the relaxation transform** $\Upsilon_a(n, x)$, continuously interpolating discrete iterations of the map $f_a(x) = \sin(x + a)$
2. **Proved the main properties** of the transform: consistency with integer iterations, continuity in n, convergence to the fixed point $U(a)$
3. **Proposed a physical interpretation** of the model as a relaxation process with memory, where parameter n is interpreted as a measure of accumulated system "experience"

4. **Introduced the memory function** $n(t)$, linking physical time to the nesting parameter, enabling description of various relaxation laws

5. **Demonstrated analogies** with specific physical systems: nonlinear pendulum, glass-forming systems, PLL systems

6.2 Theoretical Prospects

1. Studying exact conditions for the semigroup property and constructing corrections to it

2. Constructing analytical expressions for fractional iterations $f_a^t(x)$ for non-integer t as solutions to functional equations

3. Investigating bifurcations and emergence of complex dynamics upon modifying the map when $|\lambda_a| > 1$

4. Generalization to multidimensional maps and systems with distributed parameters

6.3 Applied Prospects

1. **Adapting the model to describe specific physical experiments:**
    - Stress relaxation in polymers
    - Magnetic hysteresis in ferromagnets
    - Dielectric relaxation in ferroelectrics

2. **Use in signal processing algorithms** for nonlinear filtering and data smoothing accounting for system memory

3. **Developing numerical methods** for solving integro-differential equations with memory based on the concepts of nesting and the function $n(t)$

4. **Application in control theory** for systems with nonlinear dynamics and memory

6.4 Philosophical-Methodological Aspects

1. The concept of "nesting" as an alternative language for describing evolutionary processes, complementing differential models

2. Interpretation of parameter n as a measure of complexity, memory depth, or system "age" in the context of complex systems theory

3. Connection with fractal time theory and non-standard approaches to describing temporal evolution

7 Conclusion

This work proposed and investigated a relaxation model based on the continuous interpolation of iterations of the nonlinear map $f_a(x) = \sin(x + a)$. The constructed relaxation transform $\Upsilon_a(n, x)$ naturally connects the discrete dynamics of iterations with a continuous process of approaching the stationary state $U(a)$.

A key achievement is the introduction of the memory function $n(t)$, mapping physical time to the nesting parameter. This provides a flexible tool for modeling a wide class of relaxation phenomena — from classical exponential decay to complex non-exponential kinetics characteristic of systems with memory and hierarchical dynamics.

The demonstrated analogies with specific physical systems (nonlinear pendulum, glassy materials, PLL systems) show the practical significance of the proposed approach.

The proposed approach possesses internal integrity, combining elements of dynamical systems theory, functional analysis, and mathematical modeling in physics of complex media. It opens new prospects for both theoretical research and practical applications in physics, chemistry, materials science, and control theory.

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Conflict of Interest

The author declares no conflict of interest.

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