A COMPACT NOTATION FOR PECULIAR PROPERTIES CHARACTERIZING INTEGER TETRATION

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Abstract. By initially working in the decimal numeral system, we introduce a compact notation to express the congruence speed of an integer tetration base a, along with the cycle of the rightmost non-stable digits of ${}^{b}a$ for unit increments of b. The resulting discrete function provides a useful tool for efficiently computing the exact number of frozen digits that characterize the right tail of each nontrivial integer tetration. We also establish an improved upper bound for the minimum hyperexponent $\bar{b}(a)$ that guarantees the constancy of the congruence speed of a for all heights $b \geq \bar{b}(a)$. Moreover, we prove that the minimum between the constant congruence speeds of any two integers greater than 1, whose product is not divisible by 10, is always less than or equal to the constant congruence speed of their product. Additionally, still assuming radix-10, we give examples of infinitely many perfect powers whose degree matches their constant congruence speed at every height above 2, emphasizing the peculiar recurrence relations of hyper-4. Finally, Appendix B generalizes the described radix-10 framework to all squarefree numeral systems, showing that only in such systems the congruence speed stabilizes to a fixed (positive) value for all $a > 1$ not divisible by the radical of the radix. Furthermore, we derive compact formulas for all prime-radix numeral systems and for the composite squarefree senary case.

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1. Introduction

Let $a \geq 2$ and $b \geq 1$ be integers, and assume radix-10. Consider the tetration

$$ {}^{b}a := \begin{cases} a & \text{if } b = 1 \\ a^{\left({}^{(b-1)}a\right)} & \text{if } b \geq 2 \end{cases} $$

(with the convention ${}^{0}a = 1$ and ${}^{-1}a = 0$, adopted solely for notational consistency with [7]) which, for each positive integer n, is known to eventually become periodic modulo 10^n (see [10]). This result generally holds for sufficiently large values of the hyperexponent b.

For example, for $a = 27057$ and $n = 25$, we must reach height 6 since

$$ \begin{aligned} {}^{1}27057 &\equiv (00000000000000000000)\mathbf{27057} \pmod{10^{25}}, \\ {}^{2}27057 &\equiv 477797198828352\mathbf{14952}27057 \pmod{10^{25}}, \\ {}^{3}27057 &\equiv (0)545271142\mathbf{05036}1495227057 \pmod{10^{25}}, \\ {}^{4}27057 &\equiv 63448\mathbf{63520}050361495227057 \pmod{10^{25}}, \\ {}^{5}27057 &\equiv 6\mathbf{4499}63520050361495227057 \pmod{10^{25}}, \end{aligned} $$

and finally we get ${}^{6}27057 \pmod{10^{25}} = {}^{7}27057 \pmod{10^{25}} = 1449963520050361495227057$.

In detail, we observe that the number of "new" rightmost frozen digits of ${}^{b}27057$ is equal to 5 for $b \leq 4$ and decreases to 4 for all $b \geq 5$.

In the 2011 book "La strana coda della serie $n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}$", the concept of *congruence speed* was introduced to quantify the growth rate of the number of stable digits that appear at the end of ${}^{b}a$ for each unit increase in b. That work also established a connection between the congruence speed and the periodic behavior of the least significant digit of ${}^{b}a$ that is not yet stable at height b. Accordingly, we track the cycle of the rightmost non-stable digit of ${}^{b}a$ at successive heights $b, b+1, b+2, \ldots$ via the *phase shift* array (see Section 3 of [8]).

With this preprint, we propose a new notation system that efficiently describes the congruence speed of nonnegative integer tetration bases at each height, along with its associated phase shift.

Building upon previous studies in the field, we refine and extend key results, including bounds for the minimum hyperexponent $\bar{b}(a)$ that ensures stability in the congruence speed of a. Our approach relies on modular arithmetic and p-adic analysis to establish rigorous characterizations of congruence speed behaviors, particularly in the context of bases not divisible by 10.

To this end, we include Appendix A proving that the constant congruence speed of any integer greater than 1 and not divisible by 10 is greater than or equal to the lowest constant congruence speed among its factors.

Then, we give infinitely many perfect powers whose degree equals their congruence speed at every height greater than 2 (in these cases, the congruence speed remains stable for all hyperexponents greater than 1).

Lastly, Appendix B extends the decimal framework to all squarefree numeral systems, providing explicit formulas for each prime case and, in addition, for the senary numeral system. It also formulates the conjecture that squarefreeness is the structural criterion governing when each tetration base coprime to the radix has a constant congruence speed and, in the subsequent Remarks 5 and 6, asserts that only in such numeral systems do all bases greater than 1 and not divisible by the radical of the selected numeral system exhibit a (strictly positive) constant congruence speed.

Taken together, these results establish a unified framework for the stabilization of rightmost digits in tetration, showing that the constant congruence speed is a characteristic property of all and only squarefree numeral systems.

2. Congruence speed and phase shift of tetration

The present section provides rigorous definitions of the terminology introduced earlier, establishing a formal structure for a more compact and efficient notation to describe the recurrence properties characteristic of the right tail of ${}^{b}a$.

For clarity, we denote $\mathbb{N}_0$ as the set of nonnegative integers (including zero) and $\mathbb{N}$ as the set of strictly positive integers (i.e., $\{1, 2, 3, \ldots\}$).

In [5], the congruence speed of each nonnegative integer a was originally defined as the function

$$\begin{gathered} V : \mathbb{N}_0 \times \mathbb{N} \longrightarrow \mathbb{N}_0 \\ (a, b) \mapsto V(a, b) \end{gathered} ,$$

and the author specified that if the constraint $10 \nmid a$ is imposed along with the sufficient condition $b > a$, the congruence speed no longer depends on b, so we obtain a *constant* congruence speed, the one entry function

$$\begin{gathered} V : \mathbb{N} \setminus \{\text{multiples of } 10\} \longrightarrow \mathbb{N}_0 \\ a \mapsto V(a) \end{gathered}$$

peculiar of hyper-4 and fully described by Equations (3) and (16) of [6].

Since a casual reader could easily be misled by the use of $V(a, b)$ and $V(a)$, and given that describing the congruence speed of a as $V(a, 1)$, $V(a, 2)$, …, $V(a, a)$, and $V(a, a+1) = V(a)$ is not very elegant, let us rewrite the original definitions of congruence speed and constant congruence speed as follows.

Definition 2.1. Let $n \in \mathbb{N}_0$ and assume that $a \in \mathbb{N} \setminus \{1\}$ is not a multiple of 10. Then, given ${}^{b-1}a \equiv {}^{b}a \pmod{10^n} \wedge {}^{b-1}a \not\equiv {}^{b}a \pmod{10^{n+1}}$, for each $b = 1, 2, 3, \ldots$, $v_b(a)$ returns the strictly positive integer such that ${}^{b}a \equiv {}^{b+1}a \pmod{10^{n+v_b(a)}} \wedge {}^{b}a \not\equiv {}^{b+1}a \pmod{10^{n+v_b(a)+1}}$, and we define $v_b(a)$ as the congruence speed of the base a at height b.

Definition 2.2. Let $a \in \mathbb{N} \setminus \{1\}$ not be a multiple of 10. Let $\bar{b}(a) := \min\{b \in \mathbb{N} : v_b(a) = v_{b+k}(a), \forall k \in \mathbb{N}_0\}$. We define as the constant congruence speed of a the nonnegative integer $v_{\bar{b}}(a) := v_{\bar{b}(a)}(a)$.

Assuming $b > 2$, we note that $v_{\bar{b}}(a) = v_{\bar{b}+1}(a) = v_{\bar{b}+2}(a) = \cdots$ and $v_b(a) \geq v_{b+1}(a) \geq v_{b+2}(a) \geq \cdots$ hold for each a greater than 1 and not divisible by 10 (see Equations (17) and (18) of [9]).

Now, let us compactly denote $\boldsymbol{\mathfrak{V}}(a)$ as the congruence speed of the integer tetration base a.

Consequently, if a is greater than 1 and is not a multiple of 10, we can write

$$\boldsymbol{\mathfrak{V}}(a) := (v_1(a), v_2(a), \ldots, v_{\bar{b}-1}(a); v_{\bar{b}}(a))$$

since we indicate $(v_1(a), v_2(a), \ldots, v_{\bar{b}-1}(a), v_{\bar{b}}(a), v_{\bar{b}}(a), v_{\bar{b}}(a), \ldots)$ as $(v_1(a), v_2(a), \ldots, v_{\bar{b}-1}(a); v_{\bar{b}}(a))$.

Then, trivially, $\boldsymbol{\mathfrak{V}}(0) = (;0)$ and $\boldsymbol{\mathfrak{V}}(1) = (1; 0)$.

On the other hand, we know that the congruence speed of every tetration base divisible by 10 never becomes stable, so we can proceed as follows.

Let $\nu_p(\ldots)$ be the p-adic valuation of the argument. We have that the congruence speed of a is given by

$$\boldsymbol{\mathfrak{V}}(a) := \begin{cases} (v_1(a), v_2(a), \ldots, v_{\bar{b}-1}(a); v_{\bar{b}}(a)) & \text{if} \quad a > 1 \wedge 10 \nmid a \\ (v_1(a), v_2(a), \ldots; +\infty) & \text{if} \quad a > 1 \wedge 10 \mid a \\ (; v_{\bar{b}}(a)) & \text{if} \quad a = 0 \\ (1; v_{\bar{b}}(a)) & \text{if} \quad a = 1 \end{cases} , \tag{2.1}$$

where (see Equation (3) of [6])

$$
v_{\bar{b}}(a) = \begin{cases} 0 & \text{if} \quad a \in \{0, 1\} \\ \min\{\nu_2(a-1), \nu_5(a-1)\} & \text{if} \quad a \equiv 1 \pmod{20} \wedge a \neq 1 \\ \min\{\nu_2(a+1), \nu_5(a-1)\} & \text{if} \quad a \equiv 11 \pmod{20} \\ \nu_5(a^2+1) & \text{if} \quad a \equiv 2, 8 \pmod{10} \\ \min\{\nu_2(a+1), \nu_5(a^2+1)\} & \text{if} \quad a \equiv 3, 7 \pmod{20} \\ \min\{\nu_2(a-1), \nu_5(a^2+1)\} & \text{if} \quad a \equiv 13, 17 \pmod{20} \\ \nu_5(a+1) & \text{if} \quad a \equiv 4 \pmod{10} \\ \nu_2(a-1) & \text{if} \quad a \equiv 5 \pmod{20} \\ \nu_2(a+1) & \text{if} \quad a \equiv 15 \pmod{20} \\ \nu_5(a-1) & \text{if} \quad a \equiv 6 \pmod{10} \\ \min\{\nu_2(a-1), \nu_5(a+1)\} & \text{if} \quad a \equiv 9 \pmod{20} \\ \min\{\nu_2(a+1), \nu_5(a+1)\} & \text{if} \quad a \equiv 19 \pmod{20} \end{cases} . \tag{2.2}
$$

Equation (2.2) follows from the nonzero solutions of the fundamental equation $y^5 = y$ in the commutative ring of 10-adic integers $\mathbb{Z}_{10} \coloneqq \varprojlim_n \frac{\mathbb{Z}}{10^n\mathbb{Z}}$ (see (2.3) below).

$$
y_{i=1,2,\ldots,14} = \begin{cases} \alpha'_1 = 1 - 2 \cdot \left\{5^{2^n}\right\}_\infty = \ldots 15487480163574218751 & \text{iff} \quad i = 1 \\ \alpha'_2 = \left\{2^{5^n}\right\}_\infty = \ldots 07839804103263499879186432 & \text{iff} \quad i = 2 \\ \alpha'_3 = \left\{5^{2^n}\right\}_\infty - \left\{2^{5^n}\right\}_\infty = \ldots 52996418333704193 & \text{iff} \quad i = 3 \\ \alpha''_3 = -\left\{5^{2^n}\right\}_\infty - \left\{2^{5^n}\right\}_\infty = \ldots 476581907922943 & \text{iff} \quad i = 4 \\ \alpha'_4 = \left\{5^{2^n}\right\}_\infty - 1 = \ldots 7392256259918212890624 & \text{iff} \quad i = 5 \\ \alpha'_5 = \left\{5^{2^n}\right\}_\infty = \ldots 19977392256259918212890625 & \text{iff} \quad i = 6 \\ \alpha''_5 = -\left\{5^{2^n}\right\}_\infty = \ldots 022607743740081787109375 & \text{iff} \quad i = 7 \\ \alpha'_6 = 1 - \left\{5^{2^n}\right\}_\infty = \ldots 2607743740081787109376 & \text{iff} \quad i = 8 \\ \alpha'_7 = -\left\{5^{2^n}\right\}_\infty + \left\{2^{5^n}\right\}_\infty = \ldots 003581666295807 & \text{iff} \quad i = 9 \\ \alpha''_7 = \left\{5^{2^n}\right\}_\infty + \left\{2^{5^n}\right\}_\infty = \ldots 59523418092077057 & \text{iff} \quad i = 10 \\ \alpha'_8 = -\left\{2^{5^n}\right\}_\infty = \ldots 160195896736500120813568 & \text{iff} \quad i = 11 \\ \alpha'_9 = 2 \cdot \left\{5^{2^n}\right\}_\infty - 1 = \ldots 84512519836425781249 & \text{iff} \quad i = 12 \\ \alpha''_9 = -1 = \ldots 99999999999999999999999999999 & \text{iff} \quad i = 13 \\ \alpha''_1 = 1 = \ldots 0000000000000000000000000000001 & \text{iff} \quad i = 14 \end{cases} . \tag{2.3}
$$

Now, as we pick each element of the set $\{\alpha'_1, \alpha'_2, \alpha'_3, \alpha''_3, \alpha'_4, \alpha'_5, \alpha''_5, \alpha'_6, \alpha'_7, \alpha''_7, \alpha'_8, \alpha'_9, \alpha''_9, \alpha''_1\}$ and multiply it by each element of the same set (including the original element itself), we obtain Table 1.

TABLE 1. Multiplicative recurrences of the solutions of the 10-adic equation $y^5 = y$.

$\cdot$	$\boldsymbol{\alpha_1''}$	$\boldsymbol{\alpha_1'}$	$\boldsymbol{\alpha_2'}$	$\boldsymbol{\alpha_3'}$	$\boldsymbol{\alpha_3''}$	$\boldsymbol{\alpha_4'}$	$\boldsymbol{\alpha_5'}$	$\boldsymbol{\alpha_5''}$	$\boldsymbol{\alpha_6'}$	$\boldsymbol{\alpha_7'}$	$\boldsymbol{\alpha_7''}$	$\boldsymbol{\alpha_8'}$	$\boldsymbol{\alpha_9'}$	$\boldsymbol{\alpha_9''}$
$\boldsymbol{\alpha_1''}$	α_1''	α_1'	α_2'	α_3'	α_3''	α_4'	α_5'	α_5''	α_6'	α_7'	α_7''	α_8'	α_9'	α_9''
$\boldsymbol{\alpha_1'}$	α_1'	α_1''	α_2'	α_3''	α_3'	α_4'	α_5''	α_5'	α_6'	α_7''	α_7'	α_8'	α_9''	α_9'
$\boldsymbol{\alpha_2'}$	α_2'	α_2'	α_4'	α_6'	α_6'	α_8'	0	0	α_2'	α_4'	α_4'	α_6'	α_8'	α_8'
$\boldsymbol{\alpha_3'}$	α_3'	α_3''	α_6'	α_9'	α_9''	α_2'	α_5'	α_5''	α_8'	α_1'	α_1''	α_4'	α_7''	α_7'
$\boldsymbol{\alpha_3''}$	α_3''	α_3'	α_6'	α_9''	α_9'	α_2'	α_5''	α_5'	α_8'	α_1''	α_1'	α_4'	α_7'	α_7''
$\boldsymbol{\alpha_4'}$	α_4'	α_4'	α_8'	α_2'	α_2'	α_6'	0	0	α_4'	α_8'	α_8'	α_2'	α_6'	α_6'
$\boldsymbol{\alpha_5'}$	α_5'	α_5''	0	α_5'	α_5''	0	α_5'	α_5''	0	α_5''	α_5'	0	α_5'	α_5''
$\boldsymbol{\alpha_5''}$	α_5''	α_5'	0	α_5''	α_5'	0	α_5''	α_5'	0	α_5'	α_5''	0	α_5''	α_5'
$\boldsymbol{\alpha_6'}$	α_6'	α_6'	α_2'	α_8'	α_8'	α_4'	0	0	α_6'	α_2'	α_2'	α_8'	α_4'	α_4'
$\boldsymbol{\alpha_7'}$	α_7'	α_7''	α_4'	α_1'	α_1''	α_8'	α_5''	α_5'	α_2'	α_9'	α_9''	α_6'	α_3''	α_3'
$\boldsymbol{\alpha_7''}$	α_7''	α_7'	α_4'	α_1''	α_1'	α_8'	α_5'	α_5''	α_2'	α_9''	α_9'	α_6'	α_3'	α_3''
$\boldsymbol{\alpha_8'}$	α_8'	α_8'	α_6'	α_4'	α_4'	α_2'	0	0	α_8'	α_6'	α_6'	α_4'	α_2'	α_2'
$\boldsymbol{\alpha_9'}$	α_9'	α_9''	α_8'	α_7''	α_7'	α_6'	α_5'	α_5''	α_4'	α_3''	α_3'	α_2'	α_1''	α_1'
$\boldsymbol{\alpha_9''}$	α_9''	α_9'	α_8'	α_7'	α_7''	α_6'	α_5''	α_5'	α_4'	α_3'	α_3''	α_2'	α_1'	α_1''

Table 2 (also referenced in Appendix A) synthesizes the behavior of the last digit/two digits of the congruence classes modulo 10/modulo 20 (respectively) enumerated in (2.2). In detail, we note that all the even entries in Table 2 (i.e., 2, 4, 6, and 8) are assumed modulo 10, while the remaining entries (i.e., 1, 11, 3, 13, 7, 17, 5, 15, 9, and 19) are considered modulo 20.

TABLE 2. Transformation through the product of all pairs of congruence classes modulo 10 or 20 considered in (2.2).

$\cdot$	**1**	**11**	**2, 8**	**3, 7**	**13, 17**	**4**	**5**	**15**	**6**	**9**	**19**
1	1	11	2, 8	3, 7	13, 17	4	5	15	6	9	19
11	11	1	2, 8	13, 17	3, 7	4	15	5	6	19	9
2, 8	2, 8	2, 8	4, 6	4, 6	4, 6	2, 8	0	0	2, 8	2, 8	2, 8
3, 7	3, 7	13, 17	4, 6	9, 1	19, 11	2, 8	15	5	2, 8	3, 7	13, 17
13, 17	13, 17	3,7	4, 6	19, 11	9,1	2, 8	5	15	2, 8	13, 17	3, 7
4	4	4	2, 8	2, 8	2, 8	6	0	0	4	6	6
5	5	15	0	15	5	0	5	15	0	5	15
15	15	5	0	5	15	0	15	5	0	15	5
6	6	6	2, 8	2, 8	2, 8	4	0	0	6	4	4
9	9	19	2, 8	3, 7	13, 17	6	5	15	4	1	11
19	19	9	2, 8	13, 17	3, 7	6	15	5	4	11	1

As a general result, we note that, assuming $q, \frac{a}{q} \in \mathbb{N} \setminus \{1\} : a \not\equiv 0 \pmod{10}$,

$$v_{\bar{b}}\left(q \cdot \frac{a}{q}\right) \geq \min\left\{v_{\bar{b}}(q), v_{\bar{b}}\left(\frac{a}{q}\right)\right\} \tag{2.4}$$

always holds (see Appendix A for the proof).

Thus, for each tetration base a as above, $q \mid a$ implies $v_{\bar{b}}(a) \geq \min\left\{v_{\bar{b}}(q), v_{\bar{b}}\left(\frac{a}{q}\right)\right\}$.

In this regard, we note that if $a := 999\ldots999$ (repunit 9's), $q := 111\ldots111$ (repunit 1's), and $k \in \mathbb{N}$ are such that $10^{k-1} < q < a < 10^k$, then the difference $v_{\bar{b}}(a) - \min\left\{v_{\bar{b}}(q), v_{\bar{b}}\left(\frac{a}{q}\right)\right\}$ is always equal to $k-1$ (since $v_{\bar{b}}(999\ldots999) = v_1(10^k - 1) = k$ while $v_{\bar{b}}(9) = v_1(10^1 - 1) = 1 = v_2(111\ldots111) = v_{\bar{b}}(111\ldots111)$).

In the end, the constant congruence speed of every integer greater than 1 and not divisible by 10 is necessarily greater than or equal to the minimum of the constant congruence speeds of its factors.

Remark 1. Despite the asymmetrical nature of relation (2.4), which defines only a (weak) link between the constant congruence speed of a given integer (greater than 1 and not a multiple of 10) and its factorization, Section 3 of [6] shows that the only prime numbers greater than 5 with a unit constant congruence speed are necessarily congruent to 2, 3, 4, 6, 8, 9, 11, 12, 13, 14, 16, 17, 19, 21, 22, or 23 modulo 5^2. Furthermore, Theorem 3 of [5] establishes the existence of infinitely many prime numbers characterized by each positive value of constant congruence speed, so the constraint $v_{\bar{b}}(a) = 1$ is neither a sufficient nor a necessary condition for the primality of a.

As a result, if we combine the popular primality criterion that every prime greater than 3 is congruent to 1 or 5 modulo 6 with the additional requirement that the constant congruence speed must equal 1, the observed increase in the frequency of primes (within a range of natural numbers) is merely a trivial consequence of the fact that we are also excluding all multiples of 5 from the residual list of candidate primes.

We do not explore this aspect further, as the aim of the present remark is to outline a brief application tip of the constant congruence speed formula introduced in [5], beyond its use in recreational mathematics contexts – such as proving that the exact number of stable digits of the well-known Graham's number, g_{64}, is precisely $\operatorname{slog}_3(g_{64}) - 1$, where $\operatorname{slog}_3(g_{64})$ denotes the base-3 super-logarithm of Graham's number itself [8].

As stated in Definition 3.2 of [8], "$(\ldots)$ we call phase shift of a at height b the congruence class modulo 10 of the difference between the rightmost non-stable digit of ${}^{b}a$".

For brevity purposes, similarly to Definition we set $s_{\bar{b}}(a) \coloneqq s_{\bar{b}(a)}(a)$ and then we can compactly indicate the phase shift of each integer tetration base $a \in \mathbb{N}_0$ as $\text{ס}(a)$, where $\text{ס}(a)$ is defined as follows:

$$\text{ס}(a) \coloneqq \begin{cases} (s_1(a), s_2(a), \ldots, s_{\bar{b}-1}(a); [s_{\bar{b}}(a), s_{\bar{b}+1}(a), s_{\bar{b}+2}(a), s_{\bar{b}+3}(a)]) \\ \quad \text{if} \quad s_{\bar{b}}(a) \neq s_{\bar{b}+2}(a) \\ (s_1(a), s_2(a), \ldots, s_{\bar{b}-1}(a); [s_{\bar{b}}(a), s_{\bar{b}+1}(a)]) \\ \quad \text{if} \quad (s_{\bar{b}}(a) = s_{\bar{b}+2}(a) \wedge s_{\bar{b}}(a) \neq s_{\bar{b}+1}(a)) \\ (s_1(a), s_2(a), \ldots, s_{\bar{b}-1}(a); [s_{\bar{b}}(a)]) \\ \quad \text{if} \quad s_{\bar{b}}(a) = s_{\bar{b}+1}(a) \\ (s_1(a), s_2(a); [s_{\bar{b}}(a)]) \\ \text{if} \quad 10 \mid a \\ (; [9, 1]) \\ \quad \text{if} \quad a = 0 \\ (; [0]) \\ \quad \text{if} \quad a = 1 \end{cases} . \tag{2.5}$$

In particular, we call *asymptotic* phase shift the array $[s_{\bar{b}}(a), s_{\bar{b}+1}(a), s_{\bar{b}+2}(a), s_{\bar{b}+3}(a)]$ (or $[s_{\bar{b}}(a), s_{\bar{b}+1}(a)]$, or $[s_{\bar{b}}(a)]$) of $\text{ס}(a)$ (see Section 3 of [8]).

In (2.5), the statement "$10 \mid a \Rightarrow \text{ס}(a) = (s_1(a), s_2(a); [s_{\bar{b}}(a)])$", arises from the trivial consideration that $\bar{b}(a) \leq 3$ holds for all $a \in \{\text{multiples of } 10\}$ (see the comments of the sequence A377124 of the OEIS [3, 11]).

Assuming that $a \in \mathbb{N} \setminus \{1\}$ is such that $10 \nmid a$, from [6], we have that $\bar{b}(a) - 2 \leq \tilde{\nu}(a)$, where

$$\tilde{\nu}(a) \coloneqq \begin{cases} \nu_5(a-1) & \text{if} \quad a \equiv 11 \pmod{20} \\ \nu_5(a^2+1) & \text{if} \quad a \equiv 3, 7 \pmod{10} \\ \nu_5(a+1) & \text{if} \quad a \equiv 9 \pmod{20} \\ 2 & \text{if} \quad a = 5 \\ 1 & \text{if} \quad (a \equiv 5 \pmod{10} \wedge a \neq 5) \\ 1 & \text{if} \quad a \equiv 2, 6, 16, 18, 19 \pmod{20} \\ 0 & \text{if} \quad a \equiv 1, 4, 8, 12, 14 \pmod{20} \end{cases} . \tag{2.6}$$

Accordingly, let $\tilde{b}(a) \coloneqq \tilde{\nu}(a) + 2$ and $\bar{b}(a) \leq \tilde{b}(a)$ follows.

Remark 2. In (2.6), we distinguish between the case $a = 5$ (characterized by $\bar{b}(a) = 4$) and the case $a : (a \equiv 5 \pmod{10} \wedge a \neq 5)$ (where $\bar{b}(a) = 3$ holds). Then, we observe that 5 is the only integer greater than 1 whose third tetration is congruent to its second tetration modulo 1 plus the number of the digits of its second tetration (i.e., the only solution in $\mathbb{N} \setminus \{1\}$ of ${}^{3}a \equiv {}^{2}a \pmod{10^{\lfloor \log_{10}({}^{2}a) \rfloor + 2}}$ is 5 – see [8], page 3). Thus, for each a congruent to 5 modulo 10, a sufficient but not necessary condition that ensures the perfect match between the congruence speed and the number of new stable digits of ${}^{b}a$ is given by $b > 2$, so we can finally write $\text{ע}(5) = (1, 4, 3; 2)$ (instead of $(1, 3, 4; 2)$) and conclude that ${}^{b}a$ exhibits exactly $1 + 4 + 3 + (b - 3) \cdot 2$ stable digits at heights $3, 4, 5, \ldots$ (since $v_1(5) + v_2(5) + v_3(5) = 1 + 4 + 3 = 1 + 3 + 4$, $\bar{b}(5) = 4$, and $v_{\bar{b}}(5) = 2$).

Now, if we are not able to determine the exact value of $\bar{b}(a)$ for some given integer tetration base a above 1, we can still write the congruence speed of a as

$$\tilde{\text{ע}}(a) \coloneqq \begin{cases} (v_1(a), v_2(a), \ldots, v_{\tilde{b}-1}(a); v_{\tilde{b}}(a)) & \text{if} \quad 10 \nmid a \\ (v_1(a), v_2(a), \ldots; +\infty) & \text{if} \quad 10 \mid a \end{cases} . \tag{2.7}$$

Similarly, we can express the phase shift of every $a \in \mathbb{N} \setminus \{1\}$ in the form

$$\tilde{\mathbf{\Im}}(a) := \begin{cases} (s_1(a), s_2(a), \ldots, s_{\tilde{b}-1}(a); [[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a), s_{\tilde{b}+2}(a), s_{\tilde{b}+3}(a)]]) \\ \quad \text{if } \ s_{\tilde{b}}(a) \neq s_{\tilde{b}+2}(a) \\ (s_1(a), s_2(a), \ldots, s_{\tilde{b}-1}(a); [[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a)]]) \\ \quad \text{if } \ (s_{\tilde{b}}(a) = s_{\tilde{b}+2}(a) \wedge s_{\tilde{b}}(a) \neq s_{\tilde{b}+1}(a)) \\ (s_1(a), s_2(a), \ldots, s_{\tilde{b}-1}(a); [s_{\tilde{b}}(a)]) \\ \quad \text{if } \ s_{\tilde{b}}(a) = s_{\tilde{b}+1}(a) \\ (s_1(a), s_2(a); [s_3(a)]) \\ \quad \text{if } \ 10 \mid a \end{cases}, \tag{2.8}$$

where $[[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a), s_{\tilde{b}+2}(a), s_{\tilde{b}+3}(a)]]$ is always one of the four circular permutations of the asymptotic phase shift of a (i.e., $[s_{\bar{b}}(a), s_{\bar{b}+1}(a), s_{\bar{b}+2}(a), s_{\bar{b}+3}(a)]$ is necessarily equal to $[[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a), s_{\tilde{b}+2}(a), s_{\tilde{b}+3}(a)]]$, or $[[s_{\tilde{b}+1}(a), s_{\tilde{b}+2}(a), s_{\tilde{b}+3}(a), s_{\tilde{b}}(a)]]$, or $[[s_{\tilde{b}+2}(a), s_{\tilde{b}+3}(a), s_{\tilde{b}}(a), s_{\tilde{b}+1}(a)]]$, or $[[s_{\tilde{b}+3}(a), s_{\tilde{b}}(a), s_{\tilde{b}+1}(a), s_{\tilde{b}+2}(a)]]$, and similarly $[s_{\bar{b}}(a), s_{\bar{b}+1}(a)] = [[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a)]]$ or $[s_{\bar{b}}(a), s_{\bar{b}+1}(a)] = [[s_{\tilde{b}+1}(a), s_{\tilde{b}}(a)]]$), while it is trivial to point out that $s_{\tilde{b}}(a) = s_{\tilde{b}+1}$ implies $[s_{\tilde{b}}(a)] = [s_{\bar{b}}(a)]$ (so choosing to write $[[s_{\tilde{b}}(a)]]$ instead of $[s_{\tilde{b}}(a)]$ or even $[s_{\bar{b}}(a)]$ would be pointless).

In this case, we call *modular* phase shift the array $[[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a), s_{\tilde{b}+2}(a), s_{\tilde{b}+3}(a)]]$ (or $[[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a)]]$, or $[s_{\tilde{b}}(a)]$) of $\tilde{\mathbf{\Im}}(a)$.

Lastly, for each $a := j \cdot 10^c$ such that $j, c \in \mathbb{N} \setminus \{1\}$ (which implies $10 \mid a$ by construction), the congruence speed of such tetration bases can be compactly rewritten (for each $b \in \mathbb{N}$) as

$$v_b(j \cdot 10^c) = \begin{cases} 0 & \text{iff } \ j = 0 \\ c \cdot \left({}^{b-1}(j \cdot 10^c) - {}^{b-2}(j \cdot 10^c)\right) & \text{otherwise} \end{cases}, \tag{2.9}$$

by [7], and thus, for positive integers c and j, we can finally state that

$$\mathbf{\Im}(a) := \begin{cases} (v_1(a), v_2(a), \ldots, v_{\bar{b}-1}(a); v_{\bar{b}}(a)) \\ \quad \text{if } \ \ a > 1 \wedge 10 \nmid a \\ (c, c \cdot \left({}^{0}(j \cdot 10^c) - {}^{-1}(j \cdot 10^c)\right), c \cdot \left({}^{1}(j \cdot 10^c) - {}^{0}(j \cdot 10^c)\right), \\ c \cdot \left({}^{2}(j \cdot 10^c) - {}^{1}(j \cdot 10^c)\right), \ldots; +\infty) \\ \quad \text{if } \ \ a = j \cdot 10^c \ \ (j, c \in \mathbb{N}) \\ (; v_{\bar{b}}(a)) \\ \quad \text{if } \ \ a = 0 \\ (1; v_{\bar{b}}(a)) \\ \quad \text{if } \ \ a = 1 \end{cases}. \tag{2.10}$$

Here, for any integer $r > 1$, we use the notation $\nu_r(d)$ to denote the largest integer $q \geq 0$ such that $r^q \mid d$. This notation is purely exponent-counting and does not define a valuation unless r is a prime number.

Thus, $\nu_r(d)$ denotes the exponent of the highest power of r dividing d.

For each $b > 2$, we have

$$\left({}^{b}a \equiv {}^{b+1}a \pmod{10^{\sum_{k=1}^{b} v_k(a)}}\right) \wedge \left({}^{b}a \not\equiv {}^{b+1}a \pmod{10^{\left(\sum_{k=1}^{b} v_k(a)\right)+1}}\right).$$

Consequently, if $b > \bar{b}(a)$ and $1 < a : a \not\equiv 0 \pmod{10}$ are given,

$$\left({}^{b}a \equiv {}^{b+1}a \pmod{10^{v_1(a)+v_2(a)+\cdots+v_{\bar{b}(a)-1}(a)+\left(b(a)-\left(\bar{b}(a)-1\right)\right)\cdot v_{\bar{b}}(a)}}\right) \wedge$$
$$\left({}^{b}a \not\equiv {}^{b+1}a \pmod{10^{v_1(a)+v_2(a)+\cdots+v_{\bar{b}(a)-1}(a)+\left(b(a)-\left(\bar{b}(a)-1\right)\right)\cdot v_{\bar{b}}(a)+1}}\right)$$

follows by construction.

Under the constraints above, the integer tetration ${}^{b}a$ exhibits exactly $\sum_{k=1}^{\bar{b}(a)-1} v_k(a) + \left(b - \left(\bar{b}(a) - 1\right)\right) \cdot v_{\bar{b}}(a)$ stable digits, and then the bound $(b-2) \cdot v_{\bar{b}}(a) \leq v_1(a) + v_2(a) + \cdots + v_b(a) \leq (b+1) \cdot v_{\bar{b}}(a)$ holds for all such pairs (a, b) (e.g., ${}^{400}2$ has exactly $0 + 0 + (400 - 2) \cdot 1$ frozen digits).

For example, let c, k, and t be positive integers such that $t \geq 2 + \nu_{10}(c)$. Then, the congruence speed of $\left(10^{t+k} + 10^{t-\nu_{10}(c)} + 1\right)^c$ is

$$\mathbf{\Im}\left(\left(10^{t+k} + 10^{t-\nu_{10}(c)} + 1\right)^c\right) = (2 \cdot t; t), \tag{2.11}$$

which does not depend on k, so (2.11) shows the existence of infinitely many c-th perfect powers whose constant congruence speed is also equal to c (for this purpose, it is sufficient to assume $t = c$ and then $v_{\bar{b}}\left(\left(10^{k+t} + 10^{t-\nu_{10}(c)} + 1\right)^c\right) = v_2\left(\left(10^{k+c} + 10^{c-\nu_{10}(c)} + 1\right)^c\right) = c$ follows – see [9]).

An improved version of this result is provided in Remark 3.

Remark 3. Let $c, k, m+1, t \in \mathbb{N}$ be such that $t \geq 2 + \nu_{10}(c)$ (i.e., t greater than 1 plus the number of trailing 0's at the end of the radix-10 expansion of c, if any), $k \geq t$, and assume $m \equiv 0, 1 \pmod{3}$. Then, $\left(3 \cdot m \cdot 10^{t+k} + 10^{t+k} + 10^{t-\nu_{10}(c)} + 1\right)^c$ is always a c-th perfect power (i.e., exactly a c-th perfect power by the *digit sum divisible by* 3 *and not by* 3^2 argument) characterized by a constant congruence speed of t.

In particular, we observe that, for each positive integer c, also $v_{\bar{b}}\left(\left(10^{c-\nu_{10}(c)} + 1\right)^c\right) = c$, and then $\left(10^{c-\nu_{10}(c)} + 1\right)^c$ is always a perfect power of degree exactly c with the same constant congruence speed (by Mihăilescu's theorem [4] – since $10^{c-\nu_{10}(c)}$ is a perfect power different from 2^3).

The preceding example is not peculiar to the decimal numeral system, but is simply the radix-10 instance of a more general identity.

Let $\text{rad}(r)$ denote the product of the distinct prime factors of r. For every integer $r \geq 3$ (squarefree, or non-squarefree except when $\text{rad}(r) \mid c$ and $r \nmid c$), and every $c \geq 1$, $t \geq 2 + \nu_r(c)$, and $z \geq 0$, we have

$$v_{\bar{b}}^{[r]}\left(\left(z \cdot r^{t+1} + r^{t-\nu_r(c)} \pm 1\right)^c\right) = v_{\bar{b}}^{[r]}\left(\left(r^{t-\nu_r(c)} \pm 1\right)^c\right) = v_2^{[r]}\left(\left(r^{t-\nu_r(c)} \pm 1\right)^c\right) = t, \tag{2.12}$$

where $v_{\bar{b}}^{[r]}$ denotes the constant congruence speed in radix-r (see Appendix B for the general framework in squarefree numeral systems).

3. Conclusion

The notation we have introduced for the congruence speed and phase shift of tetration may facilitate more sophisticated investigations of these properties in the future.

We expect that this specialized nomenclature has not only supported our proof in Appendix A, which shows how the relationship between the constant congruence speed and the corresponding multiplicative group ensures that

$$v_{\bar{b}}(a) \geq \min\left\{v_{\bar{b}}(q), v_{\bar{b}}\left(\frac{a}{q}\right)\right\}$$

holds for each pair $\left(q, \frac{a}{q}\right)$ of integers greater than 1 and not divisible by 10, but may also inspire further research aimed at providing a comprehensive description of the set M of all pairs of positive integers whose minimum constant congruence speed equals that of their product. Such a classification would allow us to determine in advance whether $v_{\bar{b}}\left(q \cdot \frac{a}{q}\right) = \min\left\{v_{\bar{b}}(q), v_{\bar{b}}\left(\frac{a}{q}\right)\right\}$ is satisfied for the given pair $\left(q, \frac{a}{q}\right)$ (at present, for example, we can confirm that $(624, 22943) \in M$ only after directly computing $v_{\bar{b}}(14316432) = 4$ and verifying that $\min\{v_{\bar{b}}(624), v_{\bar{b}}(22943)\} = \min\{4, 5\} = 4$ also holds).

In the previous sections, we have focused our study of congruence speed and phase shift entirely on radix-10; however, Appendix B extends the analysis of the constant congruence speed to all squarefree numeral systems, so that proving the general Conjecture 3.1, stated there, will remain a key objective for future research in the field.

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Appendix A

Proof of (2.4). This appendix is devoted to proving that the constant congruence speed of each integer greater than 1 and not a multiple of 10 is greater than or equal to the minimum of the constant congruence speeds of every subset of its factors whose product equals the given tetration base (so that $v_{\bar{b}}\left(q \cdot \frac{a}{q}\right) \geq \min\left\{v_{\bar{b}}(q), v_{\bar{b}}\left(\frac{a}{q}\right)\right\}$ holds for all $q, \frac{a}{q} \in \mathbb{N} \setminus \left(\{\text{ multiples of } 10\,\} \cup \{1\}\right)$).

For this purpose, given $m := \frac{a}{q}$, we consider (2.2) to show that (2.4) holds for each pair of positive integers (q, m) whose product $q \cdot m$ is greater than 1 and not a multiple of 10. Since, in (2.4), q and m (i.e., $\frac{a}{q}$) can independently span the entire set $\mathbb{N} \setminus \left(\{\text{ multiples of } 10\,\} \cup \{1\}\right)$. This result implies the general property that the constant congruence speed of every integer greater than 1 (whose last digit is not 0) cannot be lower than the lowest constant congruence speed among its factors (as we can ideally iterate the same process as many times as needed). Consequently, we will proceed case by case to fulfill all the lines of (2.2).

Since all the cases mentioned above flow in the 7 types of multiplicative classes listed in Table 3, we write the symbol (r) next to some class products to make clear that, in the corresponding case of the present proof, the variables q and m should be swapped with each other.

Table 3. The seven fundamental multiplicative classes enumerated in (2.2).

·	**1**	**11**	**2, 8**	**3, 7**	**13, 17**	**4**
1	I	I; II	III	III; II	III; I	II
11	I; II	I; IV	III	III; IV	III; II(r)	II
2, 8	III	III	V; VI	V; VI	V; VI	VII
3, 7	III; I	III; IV	V; VI	IV; V; VI	V; II(r); VI	VII
13, 17	III; I	III; II(r)	V; VI	V; II(r); VI	V; I; VI	VII
4	II	II	VII	VII	VII	IV
5	I	II(r)	/	II(r)	I	/
15	II	IV	/	IV	II	/
6	I	I	III(r)	III(r)	III(r)	II(r)
9	II; I	II; II(r)	VII	VII; II(r)	VII; I	IV
19	II	II; IV	VII	IV; VII	VII; II	IV

Table 3. The seven fundamental multiplicative classes enumerated in (2.2), continued.

·	**5**	**15**	**6**	**9**	**19**
1	I	II	I	II; I	II
11	II(r)	IV	I	II; II(r)	II; IV
2, 8	/	/	III(r)	VII	VII
3, 7	II(r)	IV	III(r)	VII; II(r)	IV; VII
13, 17	I	II	III(r)	VII; I	VII; II
4	/	/	II(r)	IV	IV
5	I	II	/	I	II
15	II	IV	/	II(r)	IV
6	/	/	I	II	II
9	I	II(r)	II	IV; I	IV; II
19	II	IV	II	IV; II	IV

CLASS I PRODUCTS

Let $q \equiv m \equiv 1 \pmod{20}$. We consider the number $q \cdot m$, which is also congruent to 1 modulo 20, aiming to evaluate its constant congruence speed (we are interested in comparing $v_{\bar{b}}(q \cdot m)$ with $v_{\bar{b}}(q)$ and $v_{\bar{b}}(m)$).

With reference to (2.2), we can first suppose that $\min\{\nu_5(q \cdot m - 1), \nu_2(q \cdot m - 1)\} = \nu_5(q \cdot m - 1)$, $\min\{\nu_2(q-1), \nu_5(q-1)\} = \nu_5(q-1)$, and $\min\{\nu_2(m-1), \nu_5(m-1)\} = \nu_5(m-1)$. If so, $\nu_5(q \cdot m - 1)$ can be rewritten as $\nu_5(q - 1 + (m-1) \cdot q)$ and then, by the properties of p-adics, we get

$$\nu_5(q - 1 + (m-1) \cdot q) \geq \min\{\nu_5(q-1), \nu_5((m-1) \cdot q)\}.$$

Using the properties of p-adics on the product,

$$\nu_5((m-1) \cdot q) \geq \nu_5(m-1)$$

easily follows (since $\nu_5((m-1)\cdot q) = \nu_5(m-1) + \nu_5(q)$).

Thus, the 5-adic valuation of $(q \cdot m - 1)$ is greater than or equal to the minimum between $\nu_5(q-1)$ and $\nu_5(m-1)$.

On the other hand, if $\min\{\nu_2(q-1), \nu_5(q-1)\} = \nu_2(q-1)$ and $\min\{\nu_2(m-1), \nu_5(m-1)\} = \nu_2(m-1)$, the proof would remain unchanged, provided that we explicitly state the inequalities $\nu_5(q-1) \geq \nu_2(q-1)$ and $\nu_5(m-1) \geq \nu_2(m-1)$.

Conversely, if $\min\{\nu_5(q\cdot m-1), \nu_2(q\cdot m-1)\} = \nu_2(q\cdot m-1)$, we can assume that $\min\{\nu_2(q-1), \nu_5(q-1)\} = \nu_2(q-1)$ and also that $\min\{\nu_2(m-1), \nu_5(m-1)\} = \nu_2(m-1)$.

In the same way, we observe that $\nu_2(q \cdot m - 1)$ can be rewritten as $\nu_2(q - 1 + (m-1)\cdot q)$ and then, applying the well-known p-adic properties, we find that the 2-adic valuation of $(q \cdot m - 1)$ is greater than or equal to the minimum between $\nu_2(q-1)$ and $\nu_2(m-1)$.

CLASS II PRODUCTS

Here we consider the case where $q \equiv 1 \pmod{20}$ and $m \equiv 11 \pmod{20}$ (see (2.2)), so the product $q \cdot m$ belongs to the congruence class 11 modulo 20.

Then, we assume that $\min\{\nu_2(q\cdot m+1), \nu_5(q\cdot m-1)\} = \nu_2(q\cdot m+1)$, $\min\{\nu_2(q-1), \nu_5(q-1)\} = \nu_2(q-1)$, and $\min\{\nu_2(m+1), \nu_5(m-1)\} = \nu_2(m+1)$.

At this point, we note how $\nu_2(q \cdot m + 1)$ can be rewritten as $\nu_2(m + 1 + m \cdot (q-1))$ so that, from the properties of p-adic valuations, we get

$$\nu_2(m + 1 + m \cdot (q-1)) \geq \min\left\{\nu_2(m+1), \nu_2((q-1)\cdot m)\right\}.$$

Hence, from $\nu_2((q-1)\cdot m) = \nu_2(q-1) + \nu_2(m)$, it follows that $\nu_2((q-1)\cdot m) \geq \nu_2(q-1)$.

Thus, the 2-adic valuation of $(q \cdot m + 1)$ is greater than or equal to the minimum between $\nu_2(m+1)$ and $\nu_2(q-1)$.

On the contrary, if we had supposed that $\min\{\nu_2(q-1), \nu_5(q-1)\} = \nu_5(q-1)$ and $\min\{\nu_2(m+1), \nu_5(m-1)\} = \nu_5(m-1)$, the proof of this case would not have changed substantially, provided that we had explicitly stated that $\nu_2(q-1) \geq \nu_5(q-1)$ and $\nu_2(m-1) \geq \nu_5(m-1)$.

CLASS III PRODUCTS

Let $q \equiv 1 \pmod{20}$ and $m \equiv 2, 8 \pmod{10}$. We evaluate $q \cdot m$ knowing that it is congruent to 2 or 8 modulo 10.

With reference to (2.2), we consider $\nu_5((q\cdot m)^2+1)$ and assume that $\min\{\nu_2(q-1), \nu_5(q-1)\} = \nu_5(q-1)$. The other value to be taken into account is $\nu_5(m^2+1)$, which is associated with the congruence classes 2 and 8 modulo 10.

We note that $\nu_5((q\cdot m)^2+1)$ can be rewritten as $\nu_5((q^2-1)\cdot(m^2-1) + q^2 + m^2)$, which is greater than or equal to

$$\min\left\{\nu_5((q^2-1)\cdot(m^2-1)), \nu_5(q^2+m^2)\right\}.$$

Since $\nu_5((q^2-1)\cdot(m^2-1)) = \nu_5((q-1)\cdot(q+1)\cdot(m-1)\cdot(m+1))$, by the properties of the p-adic valuations,

$$\nu_5((q-1)\cdot(q+1)\cdot(m-1)\cdot(m+1)) \geq \nu_5(q-1)$$

easily follows.

At the same time, from $\nu_5(q^2+m^2) = \nu_5(m^2+1+q^2-1)$, we have that $\nu_5(m^2+1+q^2-1) \geq \min\left\{\nu_5(m^2+1), \nu_5(q^2-1)\right\}$ and, since $\nu_5(q^2-1) = \nu_5((q-1)\cdot(q+1)) = \nu_5(q-1) + \nu_5(q+1)$, it follows that $\nu_5(q^2-1) \geq \nu_5(q-1)$.

Thus, the 5-adic valuation of $((q\cdot m)^2+1)$ is greater than or equal to the minimum between $\nu_5(q-1)$ and $\nu_5(m^2+1)$.

In contrast, if $\min\{\nu_2(q-1), \nu_5(q-1)\} = \nu_2(q-1)$ had been the case, the proof would be analogous by noting that $\nu_5(q-1) \geq \nu_2(q-1)$.

CLASS IV PRODUCTS

Let $q \equiv m \equiv 11 \pmod{20}$. We consider the product $q \cdot m$, which belongs to the congruence class 1 modulo 20.

Following (2.2), we suppose that $\min\{\nu_2(q\cdot m-1), \nu_5(q\cdot m-1)\} = \nu_2(q\cdot m-1)$, $\min\{\nu_2(q+1), \nu_5(q-1)\} = \nu_2(q+1)$, and $\min\{\nu_2(m+1), \nu_5(m-1)\} = \nu_2(m+1)$.

From $\nu_2((q+1)\cdot m - m - 1) = \nu_2((q+1)\cdot m - (m+1))$, we get $\nu_2((q+1)\cdot m - (m+1)) \geq \min\left\{\nu_2((q+1)\cdot m), \nu_2(m-1)\right\}$.

Thus, $\nu_2((q+1)\cdot m) = \nu_2(q+1) + \nu_2(m)$ so that $\nu_2((q+1)\cdot m) \geq \nu_2(q+1)$.

As a result, the 2-adic valuation of $(q \cdot m - 1)$ is greater than or equal to the minimum between $\nu_2(q+1)$ and $\nu_2(m-1)$.

Lastly, if $\min\{\nu_2(q+1), \nu_5(q-1)\} = \nu_5(q-1)$ and $\min\{\nu_2(m+1), \nu_5(m-1)\} = \nu_5(m-1)$ had been the case, the proof remains the same by noting that $\nu_2(q+1) \geq \nu_5(q-1)$ and $\nu_2(m-1) \geq \nu_5(m-1)$.

CLASS V PRODUCTS

Let $q \equiv 2, 8 \pmod{10}$ and $m \equiv 2, 8 \pmod{10}$. Specifically, when $q \equiv 2 \pmod{10}$ and $m \equiv 2 \pmod{10}$, the product $q \cdot m$ belongs to the congruence class 4 modulo 10. In this case, following (2.2), we consider $\nu_5(q \cdot m + 1)$ to show that it cannot fall below the minimum between $\nu_5(q^2+1)$ and $\nu_5(m^2+1)$.

Let $\beta \coloneqq \min\{\nu_5(q^2+1), \nu_5(m^2+1)\}$. Then, $q^2 + 1 \equiv 0 \pmod{5^\beta}$ implies

$$q^2 \equiv -1 \pmod{5^\beta} \tag{3.1}$$

and, symmetrically, $m^2 + 1 \equiv 0 \pmod{5^\beta}$ implies

$$m^2 \equiv -1 \pmod{5^\beta}. \tag{3.2}$$

By multiplying term by term both sizes of (3.1) and (3.2), we get

$$q^2 \cdot m^2 \equiv 1 \pmod{5^\beta}.$$

Hence,

$$(q \cdot m)^2 \equiv 1 \pmod{5^\beta}.$$

It follows that $q \cdot m \equiv -1 \pmod{5^\beta}$ or $q \cdot m \equiv 1 \pmod{5^\beta}$.

Since here we are considering $q \cdot m \equiv 4 \pmod{10}$, we pick

$$q \cdot m \equiv -1 \pmod{5^\beta} \tag{3.3}$$

so that $q \cdot m + 1 \equiv 0 \pmod{5^\beta}$.

By (3.3), $\nu_5(q \cdot m + 1) \geq \beta$ follows, and so we have proved that

$$\nu_5(q \cdot m + 1) \geq \min\left\{\nu_5(q^2+1), \nu_5(m^2+1)\right\}.$$

CLASS VI PRODUCTS

As $q \equiv 2 \pmod{10}$ and $m \equiv 8 \pmod{10}$ are given, the product $q \cdot m$ belongs to the congruence class 6 modulo 10.

In (2.2), we need to consider $\nu_5(q \cdot m - 1)$ to show that it is always greater than or equal to the minimum between $\nu_5(q^2+1)$ and $\nu_5(m^2+1)$.

Consequently, we can reconstruct the proof of the Class V products by following the same steps, as we choose $q \cdot m \equiv 1 \pmod{5^\beta}$ instead of $q \cdot m \equiv -1 \pmod{5^\beta}$, since here we assume $q \cdot m \equiv 6 \pmod{10}$.

In this way, we easily show that $\nu_5(q \cdot m - 1) \geq \beta$, from which $\nu_5(q \cdot m - 1) \geq \min\{\nu_5(q^2+1), \nu_5(m^2+1)\}$ follows.

CLASS VII PRODUCTS

Lastly, we consider $q \equiv 2, 8 \pmod{10}$ and $m \equiv 4 \pmod{10}$. Then, $q \cdot m \equiv 2, 8 \pmod{10}$ and, from (2.2), we have to evaluate $\nu_5((q \cdot m)^2 + 1)$ by comparing it to the minimum between $\nu_5(q^2+1)$ and $\nu_5(m+1)\}$.

We note that the proof would be almost identical to the one we have provided for the Class III products, but now we need to adopt a different approach, consistent with the new hypotheses. In detail, we focus on the product $\nu_5((q \cdot m)^2 + 1)$ and observe that it can be rewritten as $\nu_5((q^2-1) \cdot (m^2-1) + q^2 + m^2)$, which is greater than or equal to $\min\{\nu_5((q^2-1) \cdot (m^2-1)), \nu_5(q^2+m^2)\}$.

From $\nu_5((q^2-1) \cdot (m^2-1)) = \nu_5((q-1) \cdot (q+1) \cdot (m-1) \cdot (m+1))$, we get $\nu_5((q^2-1) \cdot (m^2-1)) \geq \nu_5(m+1)$ by applying the well-known property of the p-adic product (i.e., $\nu_5((q-1) \cdot (q+1) \cdot (m-1) \cdot (m+1)) = \nu_5(q-1) + \nu_5(q+1) + \nu_5(m-1) + \nu_5(m+1)$).

Similarly, from $\nu_5(q^2+m^2) = \nu_5(m^2 - 1 + q^2 + 1)$, we obtain $\nu_5(q^2+m^2) \geq \min\{\nu_5(q^2+1), \nu_5(m^2-1)\}$, and since

$$\nu_5(m^2-1) = \nu_5((m-1) \cdot (m+1)) = \nu_5(m-1) + \nu_5(m+1),$$

$\nu_5(m^2-1) \geq \nu_5(m+1)$ follows.

Thus, there is no exception to the general rule among all Class I to Class VII products.

Therefore, $v_{\bar{b}}(q \cdot m) \geq \min\{v_{\bar{b}}(q), v_{\bar{b}}(m)\}$ holds for every pair (q, m) of integers greater than 1 and that do not end in 0. This concludes the proof of (2.4) (with $m \coloneqq \frac{a}{q}$). □

Appendix B: Constancy of the congruence speed in squarefree numeral systems

It is natural to investigate whether the concept of constant congruence speed, originally formulated in the decimal numeral system, can be coherently extended to other radices $r > 1$.

Although the choice of radix-10 is historically and biologically motivated by our ten fingers, the underlying (general) property emerges in every squarefree numeral system. In this appendix, we provide compact formulas for the constant congruence speed in all prime-radix numeral systems and, in addition, for the composite squarefree case $r = 6$.

More generally, one may ask, within a given radix-r numeral system, for which tetration bases the congruence speed stabilizes, and how this depends on the prime factorization of r.

For the purposes of this Appendix, we provide the following two definitions.

Definition 3.1. Let $\mathbb{P}$ denote the set of all prime numbers, and let p_j ($j \in \mathbb{N}$) be the j-th smallest prime (i.e., $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\ldots$). We write $\mathbb{P} \setminus \{2\}$ for the set of odd primes and, for any integer $r > 1$, we denote by $\{\, p_j \in \mathbb{P} : p_j \mid r \,\}$ the set of its prime divisors.

Finally, let $a > 1$ be an integer and write the canonical prime factorization of a as

$$a \coloneqq \prod_{i=1}^{\omega(a)} p_i^{\eta_i},$$

where $\omega(a)$ is the number of distinct prime factors of a, each $p_i \in \mathbb{P}$, and $\eta_i \geq 1$; consequently,

$$\mathrm{rad}(a) = \prod_{i=1}^{\omega(a)} p_i.$$

Definition 3.2. We denote by

$$\mathbb{S} \coloneqq \{\, r \in \mathbb{N} \setminus \{1\} : p_j^2 \nmid r, \forall j \in \mathbb{N} \,\}$$

the set of all *squarefree* numeral systems, that is, integers $r > 1$ whose prime factorization contains no repeated prime factor. Equivalently, $r \in \mathbb{S}$ if and only if $\mu(r)^2 = 1$, where μ is the Möbius function.

According to Definition 3.1, let $r > 1$ and $a > 1$ be integers. We say that a has a constant congruence speed in radix-r if there exists $\bar{b}(a) \in \mathbb{N}$ such that $v_b^{[r]}(a)$ stabilizes for all integer hyperexponents $b \geq \bar{b}(a)$. When this occurs, we denote by $v_{\bar{b}}^{[r]}(a)$ (see Definition 2.2) the resulting constant value, called the constant congruence speed of a in radix-r.

Note that, under this more general notation, every quantity $v_{\bar{b}}(a)$ appearing in the main text should be read as $v_{\bar{b}}^{[10]}(a)$.

Remark 4. A simple structural observation already shows that for every non-squarefree radix $r > 1$ there exist infinitely many tetration bases, greater than 1 and not divisible by r, whose radix-r congruence speed never stabilizes. For this purpose, write the prime factorization of r as

$$r \coloneqq \prod_{j=1}^{\omega(r)} \kappa_j^{\gamma_j} \qquad (\kappa_j \in \mathbb{P},\ \gamma_j \in \mathbb{N}).$$

Since r is non-squarefree by hypothesis, there exists at least one index j such that $\gamma_j \geq 2$. Consider now the auxiliary base

$$\mathrm{rad}(r) \coloneqq \prod_{j=1}^{\omega(r)} \kappa_j,$$

namely the product of all the prime divisors of r, each taken with exponent 1. For any index j such that $\gamma_j \geq 2$, one has $\nu_{\kappa_j}\big(\mathrm{rad}(r)\big) = 1 < \gamma_j$, hence the κ_j-adic valuation of the power tower ${}^{b}(\mathrm{rad}(r))$ eventually exceeds γ_j and keeps increasing with b, so that ${}^{b}(\mathrm{rad}(r))$ acquires trailing zeros in radix-r from some height onward, and their number strictly (and rapidly) increases with each unit increment in b [7]. Thus, the radix-r congruence speed of $\mathrm{rad}(r)$ never stabilizes. Moreover, if $u > 1$ is any integer satisfying $\gcd(u, r) = 1$, we may replace $\mathrm{rad}(r)$ by $u \cdot \mathrm{rad}(r)$ to obtain infinitely many positive integers, not divisible by r, whose constant congruence speed does not exist in the given non-squarefree radix-r numeral system.

Conjecture 3.1. Let the radix-r ($r > 1$) numeral system be given and assume that the tetration base $a > 1$ is an integer. A constant congruence speed $v_{\bar{b}}^{[r]}(a)$ exists for all a not divisible by $\mathrm{rad}(r)$ if and only if $r \in \mathbb{S}$. In contrast, for each $r \notin \mathbb{S}$ there exist infinitely many $a > 1$ with $\gcd(r, a) = 1$ (and thus $\mathrm{rad}(r) \nmid a$, hence $r \nmid a$) whose congruence speed never stabilizes to a fixed value.

When it exists for all a not divisible by $\mathrm{rad}(r)$ (i.e., assuming $r \in \mathbb{S}$, which is the maximal class for which such uniform existence can hold), the constant congruence speed satisfies the following classification:

$$v_{\bar{b}}^{[r]}(a) = \begin{cases} \nu_r(a^{r-1}-1) & \text{if } r \in \mathbb{P} \setminus \{2\} \\ \nu_2(a-1)+\nu_2(a+1)-1 & \text{if } r = 2 \\ \geq \min\{v_{\bar{b}}^{[r]}(p_i) : i = 1, 2, \ldots, \omega(a)\} & \text{if } r \in (\mathbb{S} \setminus \mathbb{P}) \end{cases} . \tag{3.4}$$

It is worth noting that, for every prime-radix numeral system ($r \in \mathbb{P}$) and every integer $a > 1$ not divisible by r, the existence of a constant congruence speed $v_{\bar{b}}^{[r]}(a) = \nu_r(a^{r-1}-1)$ (or $\nu_2(a-1)+\nu_2(a+1)-1$ in the special case $r = 2$) follows, in virtue of the classical p-adic stabilization lemma (cf. [2]), from the well-known stabilization of the sequence $(a^{r^n})_{n \in \mathbb{N}}$ modulo $r^{n+\nu_r(a^{r-1}-1)}$, which ensures that any (finite) arbitrarily large number of rightmost digits of a^{r^n} will eventually become fixed in the r-adic expansion of ${}^b a$ (for a sufficiently large value of $b \coloneqq b(a)$). In detail, this r-adic stabilization concerns not only the eventual fixation of the rightmost digits of ${}^b a$, but also the incremental rate at which new digits become stable as b increases, which is precisely the quantity measured by the constant congruence speed of a in radix-r.

Remark 5. If $r > 1$ is squarefree, the constancy of the congruence speed of $a > 1$ holds if and only if r does not divide a. For non-squarefree r, stabilization does not occur uniformly for all such a (see Remark 6). At present, $r = 6$ and $r = 10$ are the only non-prime squarefree numeral systems for which explicit closed formulas for $v_{\bar{b}}^{[r]}(a)$ have been determined for all $a > 1$ such that $r \nmid a$, as described in [6] and illustrated by (3.8)–(3.9) for the case $r = 6$.

Conjecture 3.2. For all integers $r > 2$, $c > 0$, $z \geq 0$, and $t > \nu_r(c) + 1$, the identity

$$v_{\bar{b}}^{[r]}\big((z \cdot r^{t+1} + r^{t-\nu_r(c)} + 1)^c\big) = t = v_{\bar{b}}^{[r]}\big((z \cdot r^{t+1} + r^{t-\nu_r(c)} - 1)^c\big) \tag{3.5}$$

holds if and only if $\mathrm{rad}(r) \nmid c$ or $r \mid c$ (i.e., the identity applies to every squarefree $r > 2$, and to all non-squarefree integers as well, except in the unique case where $\mathrm{rad}(r)$ divides c while r does not). If $r \notin \mathbb{S}$ is given, for every c, z, and t as above, the radix-r congruence speed of $(z \cdot r^{t+1} + r^{t-\nu_r(c)} \pm 1)^c$ never stabilizes to a fixed value, while, in general,

$$v_b{}^{[r]}\big((z \cdot r^{t+1} + r^{t-\nu_r(c)} \pm 1)^c\big) \geq t \tag{3.6}$$

holds for all integers $r > 1$, $b > 1$, $c > 0$, $z \geq 0$, and $t > \nu_r(c) + 1$.

Remark 6. For every integer $r > 1$ and positive integer k with $(r, k) \neq (2, 1)$, we have $v_{\bar{b}}^{[r]}(r^k + 1) = k$. The identity $v_{\bar{b}}^{[r]}(r^k + 1) = k$ shows that each radix-r numeral system admits integers $a > 1$ whose constant congruence speed equals any given (arbitrarily large) positive integer. However, this does not imply that a general constancy of the congruence speed holds for all numeral systems and all $a > 1$ such that $r \nmid a$ (nor such that $\mathrm{rad}(r) \nmid a$). Nevertheless, for any given non-squarefree r, there certainly exist integers a not divisible by $\mathrm{rad}(r)$ whose congruence speeds never stabilize. Furthermore, even restricting the domain of a from $r \nmid a$ to the mere $\gcd(a, r) = 1$, according to Conjecture 3.2, there would exist infinitely many positive integers coprime to r without a constant congruence speed (since $(z \cdot r^{t+1} + r^{t-\nu_r(c)} + 1)^c \equiv 1 \pmod{r^t}$, hence coprime to r), while the minimal configurations (r, a) that do not exhibit a constant congruence speed are significantly smaller: $(4, 7)$, $(8, 3)$, $(9, 2)$, $(12, 55)$, $(16, 3)$, $(18, 5)$, $(20, 7)$, $(24, 5)$, $(25, 2)$, $(27, 2)$, $(28, 215)$, and these examples well illustrate the development of the aforementioned general rule (despite the fact that a different classification was originally suggested by Germain [1], where some non-squarefree moduli – such as $r = 4$, $r = 8$, $r = 12$, and so forth – are included among the bases potentially admitting stabilization). Therefore, the uniform existence for all $a > 1$ such that $\mathrm{rad}(r) \nmid a$ occurs if and only if r is squarefree, whereas for all non-squarefree r we conjecture the existence of infinitely many $a > 1$ coprime to r such that $v_{\bar{b}}^{[r]}(a)$ does not exist).

Conjecture 3.1 generalizes the radix-10 framework developed in Section 2 of the present preprint to all squarefree numeral systems and states that the constancy of the congruence speed, holding for all tetration bases not divisible by r, is an intrinsic property of all squarefree numeral systems.

In particular, we will examine the case $r = 6$, explicitly deriving a compact formula for $v_{\bar{b}}^{[6]}(a)$, as it relies on the nine stationary 6-adic solutions of $y^3 = y$, in the commutative ring of 6-adic integers $\mathbb{Z}_6$.

Remark 7. In this context, for any given radix-r numeral system, we are interested in solving the fundamental equation $y^{\bar{\tau}(r)} = y$ in $\mathbb{Z}_r \coloneqq \varprojlim_n \frac{\mathbb{Z}}{r^n \mathbb{Z}}$, where $\bar{\tau}(r) > 1$ denotes the smallest odd integer such that the equation $y^{\bar{\tau}(r)} = y$ has a maximal set of solutions in $\mathbb{Z}_r$. It is not difficult to show that the cardinality of this solution set equals $\mathrm{rad}(r)$ if and only if $2 \nmid r$, and $3 \cdot \frac{\mathrm{rad}(r)}{2}$ otherwise.

Now, as in Remark 4, write $r = \prod_{j=1}^{\omega(r)} \kappa_j^{\gamma_j}$, so that $\omega(r)$ denotes the number of distinct prime divisors of r. Then, we may compactly state that

$$\bar{\tau}(r) = \begin{cases} 3 & \text{if} \quad \operatorname{rad}(r) = 2 \\ 1 + \operatorname{lcm}\{\, \kappa_j - 1 : 1 \le j \le \omega(r),\ \kappa_j \ne 2 \,\} & \text{if} \quad \operatorname{rad}(r) \ge 3 \end{cases} . \tag{3.7}$$

Similarly to the radix-10 case, where Equation (16) of [6] invokes some of the fifteen stationary 10-adic integers $y_{1,2,\ldots,15}^{[\mathbb{Z}_{10}]}$ solving the fundamental equation $y^5 = y$ to express the modulo 20 sub-equations of the decimal congruence speed, we list below the corresponding nine stationary 6-adic integers, namely the nine stationary solutions of $y^3 = y$ in $\mathbb{Z}_6$, which play the same role for radix-6 as the aforementioned fifteen stationary 10-adic integers do in the decimal case.

Thus, for $r = 6$, we get the following:

$$y_{i=1,2,\ldots,9}^{[\mathbb{Z}_6]} = \begin{cases} \alpha_1'^{[\mathbb{Z}_6]} = \{4^{3^n}\}_\infty - \{3^{2^n}\}_\infty = \ldots 0011252320121111241513 1_6 & \text{iff } i = 1 \\ \alpha_2'^{[\mathbb{Z}_6]} = -\{4^{3^n}\}_\infty = \ldots 355525055521314155152221350212_6 & \text{iff } i = 2 \\ \alpha_3'^{[\mathbb{Z}_6]} = \{3^{2^n}\}_\infty = \ldots 4135552505552131415515222135021 3_6 & \text{iff } i = 3 \\ \alpha_3''^{[\mathbb{Z}_6]} = -\{3^{2^n}\}_\infty = \ldots 200030500034241400403334205343_6 & \text{iff } i = 4 \\ \alpha_4'^{[\mathbb{Z}_6]} = \{4^{3^n}\}_\infty = \ldots 1420003050003424140040333420534 4_6 & \text{iff } i = 5 \\ \alpha_5'^{[\mathbb{Z}_6]} = \{3^{2^n}\}_\infty - \{4^{3^n}\}_\infty = \ldots 5544303235434444314042 5_6 & \text{iff } i = 6 \\ \alpha_5''^{[\mathbb{Z}_6]} = -1_6 = \ldots 55555555555555555555555555555555555_6 & \text{iff } i = 7 \\ \alpha_1''^{[\mathbb{Z}_6]} = 1_6 = \ldots 000000000000000000000000000000000001_6 & \text{iff } i = 8 \\ \alpha_0'^{[\mathbb{Z}_6]} = 0_6 = \ldots 000000000000000000000000000000000000_6 & \text{iff } i = 9 \end{cases} . \tag{3.8}$$

Each of the nine 6-adic integers above is a stationary solution of $y^3 = y$ in $\mathbb{Z}_6$, and serves as an attracting fixed point for the tetration dynamics in radix-6. These stationary points determine the asymptotic behaviour of $v_{\bar{b}}^{[6]}(a)$.

Thus, we obtain the senary constant congruence-speed formula for each integer $a > 1$ not divisible by 6, in the same way as Equation (3) of [6] does in the decimal numeral system:

$$v_{\bar{b}}^{[6]}(a) = \begin{cases} \min\{\nu_2(a^2 - 1) - 1,\ \nu_3(a^2 - 1)\} & \text{if } a \equiv 1, 5 \pmod 6 \\ \nu_2(a^2 - 1) - 1 & \text{if } a \equiv 3 \pmod 6 \\ \nu_3(a^2 - 1) & \text{if } a \equiv 2, 4 \pmod 6 \end{cases} . \tag{3.9}$$

Obeserve that (3.9) reflects the ring isomorphism $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$: for any integer $a > 1$ with $6 \nmid a$, the stabilization of the rightmost digits of ${}^b a$ as $b \to \infty$ decomposes into independent 2-adic and 3-adic components, whose combined effect is expressed by the minimum operator.

To conclude, it remains an open question whether $r \in \mathbb{S}$ represents a necessary and sufficient condition for the existence of $v_{\bar{b}}^{[r]}(a)$ for every integer $a > 1$ coprime to r.

Proving Conjecture 3.2, or even Conjecture 3.1, would fully complete the classification of all (and only) the numeral systems that exhibit stable congruence-speed behaviour, confirming that the squarefreeness of $r > 1$ is the precise structural criterion ensuring that, in radix-r, a constant congruence speed characterizes all tetration bases $a > 1$ coprime to r (and, when $r \in \mathbb{S}$, all $a > 1$ with $r \nmid a$).