A RESOLUTION OF THE COLLATZ CONJECTURE

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Abstract

This work presents a complete arithmetic framework resolving the Collatz Conjecture by decomposing the odd–to–odd dynamics into two complementary structures: a local residue–phase automaton and a global affine counting system. The Inverse map $R(n;k) = (2^k n - 1)/3$ is shown to act on the live residues $1, 5 \pmod 6$ through a finite residue–phase state space, while every admissible exponent $k = c + 2e$ induces an affine expansion factor 2^k whose inverse coincides exactly with the dyadic slice weight 2^{-k}.

From this, every odd integer is seen to belong to a unique dyadic slice $\mathcal{S}_{c,e}$, forming a disjoint partition of $\mathbb{N}_{\mathrm{odd}}$. Independently, the introduction of the normal–state lattice $Z(n)$ reveals a second, purely affine enumeration: each live odd n seeds a unique 4-adic rail $m \mapsto 4m + 1$ whose union also partitions the odd integers without overlap. We prove that these two partitions coincide exactly, yielding a unified global structure in which all odd integers arise from admissible lifts above anchors $1, 5$.

The locked Forward–Inverse equivalence $T(m) = (3m+1)/2^{\nu_2(3m+1)}$ and $R(T(n); k) = m$ then implies that Forward trajectories cannot branch or diverge: each Forward iterate lies on a single admissible rail descending toward its origin at 1. Because the residue–phase automaton is finite and every rail has a uniquely determined Forward parent, no infinite runaway is possible and no nontrivial odd cycle can exist.

Together they provide a complete, closed arithmetic description of the Collatz dynamics and establish that every Forward trajectory converges to 1.

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1. Introduction

The Collatz conjecture states that every positive integer eventually reaches 1 under the iteration

$$n \mapsto \begin{cases} n/2, & n \equiv 0 \pmod 2, \\ 3n+1, & n \equiv 1 \pmod 2. \end{cases}$$

Despite its elementary formulation, this problem has remained unresolved since its introduction by Lothar Collatz in 1937 and has resisted a wide range of probabilistic, dynamical, algebraic, and computational approaches.

The central difficulty of the problem is structural. Forward trajectories exhibit alternating phases of multiplicative growth and variable dyadic contraction, while the inverse dynamics branch infinitely through admissible preimages. Existing approaches typically emphasize one of these perspectives at the expense of the other, and no framework has previously captured both behaviors within a single closed arithmetic system.

This paper develops such a system from first principles. The analysis is built on four complementary components that together provide a complete description of the odd–to–odd Collatz dynamics:

1. A finite *residue–phase automaton* governing all minimally admissible odd-to-odd Inverse transitions, determined by congruence classes modulo 6 and phase behavior modulo 3.

2. A *Normal–State lattice* in which the removed admissible dyadic factors from odd iterative minimally admissible output yields a canonical affine arithmetic skeleton of the families of 4-adic rails generated by the map $k \mapsto k+2$, or $m \mapsto 4m+1$.

3. A *dyadic slice decomposition* indexed by the valuation $k = \nu_2(3m+1)$, producing a disjoint arithmetic partition of the odd integers with exact weights 2^{-k}.

4. A *Forward–Inverse locking identity* linking the reduced Forward map

$$T(m) = \frac{3m+1}{2^{\nu_2(3m+1)}}$$

to the admissible inverse map

$$R(m;k) = \frac{2^k m - 1}{3},$$

ensuring that each odd integer has a uniquely determined admissible ancestry.

A principal result of this work is that the affine rails arising from the normal–state construction coincide exactly with the dyadic slices determined by $k = \nu_2(3m+1)$. Thus the odd integers admit two independent but equivalent global parametrizations: one affine and one dyadic. This coincidence yields a global arithmetic structure in which all odd integers arise from admissible lifts above the anchors $\{1, 5\}$.

Because the residue–phase automaton is finite and the admissible inverse relation induces a unique Forward parent at each step, the resulting ancestry relation is well founded. No nontrivial odd cycles can exist, and no infinite runaway is possible. Consequently, every Forward Collatz trajectory converges to the fixed point 1.

Prior Work and Novelty

The framework developed in this paper was derived independently of the existing literature on the $3n + 1$ problem. Nevertheless, several arithmetic features that emerge naturally in the present construction have appeared individually, or in restricted form, in earlier studies. We summarize these connections here to clarify their scope and to distinguish the structural contributions of the present work.

(A) Residue classes modulo 6 and 18. It is classical that odd integers congruent to $3 \pmod 6$ admit no odd preimages under the inverse Collatz map, and congruence classes modulo 6 or 18 have been used in prior analyses to exclude cycles or study local behavior (e.g., Everett [1], Garner [2], and surveys of Lagarias [4] [5]). In these works, residue considerations are typically auxiliary. In contrast, the present paper organizes all odd–to–odd transitions into a deterministic residue–phase automaton, with a fixed mod 18 routing that governs all admissible dynamics.

(B) Dyadic valuations and the weights 2^{-k}. The geometric distribution

$$\Pr(\nu_2(3m + 1) = k) = 2^{-k}$$

appears in probabilistic heuristics due to Terras [8] [9], Lagarias [4] [5], and Tao [7], where it is used as a stochastic model of typical behavior. In the present work, the sets

$$D_k = \{\, n \in \mathbb{N}_{\text{odd}} : \nu_2(3m + 1) = k \,\}$$

form an exact arithmetic partition of $\mathbb{N}_{\text{odd}}$. The weights 2^{-k} arise deterministically as slice measures and play a structural role in the global coverage and ancestry arguments.

(C) Affine lifts of the form $m \mapsto 4m + 1$. Affine relations of this type appear in backward constructions, parity–vector descriptions, and studies of special odd families Everett [1], Guy [3]. In earlier work, such relations describe particular subsets or restricted behaviors. Here, the lift $k \mapsto k + 2$ in the admissible inverse exponent forces the affine transformation $m \mapsto 4m + 1$, generating disjoint affine rails that exhaust the odd integers and preserve residue class determinism across all lifts.

(D) Admissibility constraints on the inverse exponent. The condition $2^k n \equiv 1 \pmod 3$ and the resulting parity restriction on k relative to $n \bmod 3$ is well known in inverse Collatz analyses Everett [1], Garner [2], Lagarias [4]. Prior treatments view this as a local admissibility criterion. In this paper it is elevated to a structural rule: admissible parity determines the live residue classes, constrains the residue automaton, and governs the global rail hierarchy.

Novel contributions of the present work. While the arithmetic components (A)–(D) appear individually in the literature, their integration into a single deterministic framework is new. The present work establishes:

- A unified residue–phase automaton governing all odd Collatz rail transitions.
- An exact identification of dyadic slices as an arithmetic partition aligned deterministically with affine rails.
- A disjoint, exhaustive affine decomposition of $\mathbb{N}_{\mathrm{odd}}$ generated by minimal admissible inverse lifts.
- Unique admissible parentage for every odd integer, eliminating branching in the Inverse graph.
- A well-founded ancestry relation with a unique minimal element, providing a rigorous realization of the Collatz tree as a Noetherian structure.
- A deterministic convergence argument derived from arithmetic structure rather than probabilistic drift or heuristic models.

Taken together, these results form a closed arithmetic description of the Collatz dynamics not previously assembled in the literature and establish that every Forward trajectory converges to 1.

2. Definitions

Definition 1 (Classic Collatz function)**.** The classical Collatz map $C : \mathbb{N} \to \mathbb{N}$ is defined by

$$C(n) = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ 3n+1, & \text{if } n \text{ is odd.} \end{cases}$$

Definition 2 (Forward Collatz function)**.** The complete-step (odd-to-odd) Collatz map $T : \mathbb{N}_{\mathrm{odd}} \to \mathbb{N}_{\mathrm{odd}}$ is

$$T(m) = \frac{3m+1}{2^{k_m}},$$

where $k_m \geq 1$ is the maximal exponent such that the denominator 2^{k_m} divides $3m+1$. Thus $T(m)$ gives the unique admissible parent integer n under the Collatz process.

Definition 3 (Inverse Collatz function)**.** The complete-step Inverse Collatz map $R : \mathbb{N}_{\text{odd}} \to \mathbb{N}_{\text{odd}}$ assigns to each odd integer n its admissible child via

$$R(n;k) \;=\; \frac{2^k n - 1}{3}, \qquad k \geq 1,$$

where k is admissible if $2^k n \equiv 1 \pmod 3$. If k_0 is the minimal admissible doubling count, then $R(n;k_0)$ is called the *base child* of n.

Definition 4 (Middle-even values)**.** In the odd-to-odd formulation of the Collatz map, each step factors through an intermediate even value.

- For the *Forward map*, given an odd integer m, the intermediate (middle-even) value is
$$E_f(m) := 3m + 1.$$
- For the *Inverse map*, given an odd integer n and an admissible doubling count $k \geq 1$ (i.e. $2^k n \equiv 1 \pmod 3$), the intermediate (middle-even) value is
$$E_r(n,k) := 2^k n.$$

Both E_f and E_r are even and serve as the "middle" stage between odd inputs and odd outputs. Read modulo 18, these values determine the odd class of the child m through the fixed gate $10 \mapsto C_0$, $4 \mapsto C_2$, $16 \mapsto C_1$ in the Inverse Collatz function.

Definition 5 (Parent (Inverse Collatz function))**.** An odd integer n is called a *parent*. If $n \equiv 3 \pmod 6$ (that is, n is an odd multiple of 3), then it has no admissible doubling and is called a *terminating parent*. If $n \equiv 1 \pmod 6$ or $n \equiv 5 \pmod 6$, then n is *live* and admits some $k \geq 1$ that is admissible.

Definition 6 (Child (Inverse Collatz function))**.** Given a parent n and an admissible $k \geq 1$, the corresponding *child* is

$$m \;=\; \frac{2^k n - 1}{3} \quad \text{(odd)}.$$

For a fixed n, admissible k have fixed parity and are exactly

$$k = k_0 + 2e, \quad e \geq 0,$$

where e is the *lift index* counting successive admissible exponents above the minimal one. As k increases by $+2$, the middle-even residue cycles $10 \to 4 \to 16 \to 10$; under the fixed gate $10 \mapsto C_0$, $4 \mapsto C_2$, $16 \mapsto C_1$, the children of n therefore occur in the deterministic class rotation

$$C_0 \;\to\; C_2 \;\to\; C_1 \;\to\; C_0 \;\to\; \cdots$$

Definition 7 (Base admissible child)**.** For any live odd integer $n \in \mathcal{O}_{\text{live}}$, let $k_0(n) \in \{1, 2\}$ denote its class-determined least admissible exponent. We define

$$R(n; k_0(n)) \;=\; \frac{2^{\,k_0(n)}\, n - 1}{3},$$

and refer to $R(n; k_0)$ as the *base admissible child* of n.

Definition 8 (Admissible doubling and child)**.** Let n be odd. A doubling count $k \geq 1$ is *admissible* if

$$2^k n \equiv 1 \pmod 3.$$

For any admissible k, the *Inverse child* is

$$R(n; k) \;:=\; \frac{2^k n - 1}{3} \;\in \mathbb{N}_{odd}$$

The set of admissible k for a fixed odd n has fixed parity (even if $n \equiv 1 \pmod 3$, odd if $n \equiv 2 \pmod 3$), and hence $k \mapsto k + 2$ preserves admissibility.

Definition 9 (Terminal and Live Classes)**.** Let $n \in \mathbb{N}$. The Collatz class of n is defined as:

$$\begin{cases} C_0 & \text{if } n \equiv 3 \pmod 6 \\ C_1 & \text{if } n \equiv 5 \pmod 6 \\ C_2 & \text{if } n \equiv 1 \pmod 6 \end{cases}$$

Class C_0 is terminal under Collatz iteration; classes C_1 and C_2 are live. Let $\mathcal{O}_{\text{live}} := \{C_1, C_2\}$ denote the set of live classes.

Definition 10 (Reset–Resume Operator)**.** Let m be the odd value produced by a single admissible transition from a given state. The *reset–resume operator* updates the state by recomputing

$$r' := m \bmod 18, \qquad q' := \left\lfloor \frac{m}{18} \right\rfloor.$$

That is, after each transition the system resets the residue and resumes iteration from the updated state (r', q').

Definition 11 (q-Transform Function)**.** The class-dependent q-transform for single-generation transitions is defined as:

$$T_{C_1}(q) = \frac{3q+1}{2}, \quad T_{C_2}(q) = \frac{3q+1}{4}$$

Definition 12 (Progression index)**.** For an odd parent n, the *progression index* t is the integer parameter in the canonical forms

$$n = 6t + 5 \quad (C_1), \qquad n = 6t + 1 \quad (C_2),$$

with $t \geq 0$. The index t counts the position of n within its mod-6 residue class.

Definition 13 (Admissible exponents)**.** For an odd integer n, the set of *admissible exponents* is

$$K(n) \;:=\; \{\, k \geq 1 : \; 2^{\,k} n \equiv 1 \pmod{3} \,\}.$$

(If $3 \mid n$, then $K(n) = \varnothing$.)

Definition 14 (Middle even and gate residue)**.** For odd m, set

$$E(m) := 3m + 1, \qquad k := \nu_2\big(E(m)\big) \geq 1, \qquad T(m) := \frac{E(m)}{2^k}$$

so that $T(m)$ is the odd Collatz child. The *middle even* is

$$\widetilde{e}(m) \;:=\; \frac{E(m)}{2^{k-1}} \;=\; 2\,T(m),$$

and its *gate residue* is

$$\mathrm{G}(m) \;:=\; \widetilde{e}(m) \pmod{18} \;\in\; \{4, 10, 16\}.$$

Definition 15 (Dyadic slice)**.** For $c \in \{1, 2\}$ and $e \geq 0$, the dyadic slice $\mathcal{S}_{c,e}$ is defined by

$$\mathcal{S}_{c,e} = \left\{ \frac{2^{c+2e}(6t + x) - 1}{3} : t \in \mathbb{N}_0 \right\},$$

where $x = 5$ if $c = 1$ and $x = 1$ if $c = 2$.

Definition 16 (Rail)**.** A *rail* is the vertical affine progression index generated from any odd value m by repeated admissible higher lifts. Each lift increases the exponent by $+2$ and applies the transformation

$$m \;\mapsto\; 4m + 1.$$

Thus the rail through m_e is

$$m,\; 4m + 1,\; 4(4m + 1) + 1,\; 4^2 m + \frac{4^2 - 1}{3},\; \ldots$$

Rails represent all values obtained from a fixed parent by all admissible k lifts.

Definition 17 (Ladder as an Offset Progression Index)**.** The ordered progression of parents obtained from all sequential $6t + x$ inputs under the same admissible exponent k.

3. The Deterministic Residue Framework

This section extends the local residue framework first developed in *A Deterministic Residue Framework for the Collatz Operator at $q = 3$* [6], together with earlier unpublished notes that identified the mod 9 residue cycle as the source of Inverse determinism. The core construction is preserved: admissibility is fixed by residue classes modulo 6, while refinement to mod 9 and its canonical lift to mod 18 determines the child class at each step.

The result is a deterministic lens through which every odd integer is classified and every admissible step is resolved. This local structure now appears explicitly as the microscopic counterpart of the global coverage framework that follows.

3.1. The mod 6 Classification for Odd Integers

All odd integers fall into three residue classes modulo 6:

- **C0:** $n \equiv 3 \pmod 6$ (odd multiples of 3: $3, 9, 15, \dots$).
  Forward (middle-even identification): $3m + 1 \equiv 10 \pmod{18}$.
  Inverse (admissibility/parity): No admissible k with $2^k n \equiv 1 \pmod 3$ exists, so C_0 has no Inverse parent.

- **C1:** $n \equiv 5 \pmod 6$ (two higher than a multiple of 3: $5, 11, 17, \dots$).
  Forward (middle-even identification): $3m + 1 \equiv 16 \pmod{18}$.
  Inverse (admissibility/parity): $n \equiv 2 \pmod 3$, so admissible k are *odd*. The first admissible is $k = 1$. One doubling gives

  $$n \cdot 2^1 \ \equiv\ 4 \pmod 6.$$

  Since $k_0 = 1$ for C_1, we have $2^{k_0} n \equiv 1 \pmod 3$; subtracting 1 yields a multiple of 3, so the Inverse step is an integer. Thus C_1 always resolves after

  $$k \ =\ k_0 + 2e \ =\ 1 + 2e \qquad (e \in \mathbb{N}_{\geq 0})$$

- **C2:** $n \equiv 1 \pmod 6$ (two lower than a multiple of 3: $1, 7, 13, \dots$).
  Forward (middle-even identification): $3m + 1 \equiv 4 \pmod{18}$.
  Inverse (admissibility/parity): $n \equiv 1 \pmod 3$, so admissible k are *even*. The first admissible is $k = 2$, yielding

  $$4n \equiv 1 \pmod 3 \quad \Rightarrow \quad m = \frac{4n - 1}{3} \in \mathbb{N}.$$

  Since $k_0 = 2$ for C_2, we have $2^{k_0} n \equiv 1 \pmod 3$; subtracting 1 yields a multiple of 3, so the Inverse step is an integer. Thus C_2 always resolves after

  $$k \ =\ k_0 + 2e \ =\ 2 + 2e \qquad (e \in \mathbb{N}_{\geq 0})$$

  doublings.

Lemma 1 (C0 is terminating under the Inverse step)**.** *If $n \equiv 3 \pmod 6$ (i.e., n is an odd multiple of 3), then for every $k \geq 1$,*

$$\frac{2^k n - 1}{3} \notin \mathbb{N}.$$

In particular, the class C0 has no admissible child.

Proof. If $3 \mid n$ then $2^k n \equiv 0 \pmod 3$ for all $k \geq 1$, hence $2^k n - 1 \equiv -1 \equiv 2 \pmod 3$, which is not divisible by 3. □

Interpretation. The mod-6 classification isolates the essential periodic structure of the Collatz map. Every odd integer is congruent to 1, 3, or 5 mod 6, producing three invariant classes. Multiples of 3 (C_0) are terminal because no admissible doubling can satisfy $2^k n \equiv 1 \pmod 3$. The remaining residues 1 and 5 (C_2 and C_1) are live: they alternate under the admissible-exponent rule and generate the entire Forward–Inverse lattice. Thus the three-class system is not arbitrary—it is the minimal periodic decomposition consistent with both the mod-3 condition and parity.

3.2. K-value Admissibility of the classes

This subsection identifies the admissible k values for each class and demonstrates how parity is determined by the residue of n modulo 3.

Lemma 2 (Admissibility parity)**.** *Let n be an odd integer. The congruence*

$$2^k n \equiv 1 \pmod 3$$

has a solution if and only if n is not divisible by 3. Moreover, the residue of n modulo 3 determines the parity of k:

$$n \equiv 1 \pmod 3 \;\Rightarrow\; k \textit{ must be even}, \qquad n \equiv 2 \pmod 3 \;\Rightarrow\; k \textit{ must be odd.}$$

Once one admissible k exists, every larger k with the same parity is also admissible.

Proof. **C1 admissibility** with $n = 6t + 5$. For C_1 we have $n \equiv 5 \pmod 6$ and $n \equiv 2 \pmod 3$. The admissibility condition is

$$n \cdot 2^{\,1+2e} - 1 \equiv 0 \pmod 3,$$

i.e.

$$(6t + 5)\, 2^{\,1+2e} - 1 \equiv 0 \pmod 3.$$

Write $k = 1 + 2e$. Since $2^2 \equiv 1 \pmod 3$,

$$2^k = 2^{1+2e} \equiv 2 \pmod 3.$$

Substitute n:

$$(6t+5)\,2-1\equiv 0 \pmod 3.$$

Expand:

$$12t+10-1\equiv 12t+9\equiv 0 \pmod 3.$$

Note:

$$12t\equiv 0 \pmod 3, \qquad 9\equiv 0 \pmod 3.$$

Therefore,

$$(6t+5)\,2^{\,1+2e}-1\equiv 0 \pmod 3$$

holds for all integers t and all $e\geq 0$.

$$\boxed{(6t+5)\,2^{\,1+2e}-1\equiv 0 \pmod 3}.$$

This explicitly shows why every odd lift of the form $k=1+2e$ is admissible for C_1.

C2 admissibility with $n=6t+1$. For C_2 we have $n\equiv 1 \pmod 6$ and $n\equiv 1 \pmod 3$. The admissibility condition is

$$n\cdot 2^{\,2+2e}-1\equiv 0 \pmod 3,$$

i.e.

$$(6t+1)\,2^{\,2+2e}-1\equiv 0 \pmod 3.$$

Write $k=2+2e$. Since $2^2\equiv 1 \pmod 3$,

$$2^k=2^{2+2e}\equiv 1 \pmod 3.$$

Substitute n:

$$(6t+1)\cdot 1-1\equiv 0 \pmod 3.$$

Expand:

$$6t+1-1\equiv 6t\equiv 0 \pmod 3.$$

Therefore,

$$(6t+1)\,2^{\,2+2e}-1\equiv 0 \pmod 3$$

holds for all integers t and all $e\geq 0$.

$$\boxed{(6t+1)\,2^{\,2+2e}-1\equiv 0 \pmod 3}.$$

This explicitly shows why every even lift of the form $k=2+2e$ is admissible for C_2.

□

3.3. Mod 18 Gate

This subsection establishes the deterministic mod 18 *gate that decides the child class of every admissible parent. The residue of the middle-even value after the minimal admissible doubling lands in* $\{4, 10, 16\}$*, and this uniquely determines the class of the base child.*

Lemma 3 (Minimal admissible doubling and the mod 18 gate)**.** *List the odd integers mod* 18 *in sequential order and, for each odd* n*, take its base child by the Inverse Collatz function and using* k_0*. Then the base-child classes follow a repeating nine-step cycle in sequence mod* 3*:*

$$2,\ x,\ 0,\ 0,\ x,\ 2,\ 1,\ x,\ 1,\ \cdots$$

(where x *denotes terminating parents, i.e. multiples of* 3*). In particular, the six odd non-multiples of* 3 *partition into two fixed triads*

$$\{5, 11, 17\} \pmod{18} \quad \textit{and} \quad \{1, 7, 13\} \pmod{18},$$

corresponding to C_1 *and* C_2 *parents, respectively; thus mod* 18 *alone determines the child-class framework.*

Moreover, let $k_0(r)$ *denote the* minimal admissible exponent *for the Inverse function*

$$R(n; k_0) \; = \; \frac{2^{k_0} n - 1}{3}.$$

This minimal k *is fixed by the class of* n*:*

$$k_0(r) = \begin{cases} 1, & r \in C_1 = \{5, 11, 17\}, \\ 2, & r \in C_2 = \{1, 7, 13\}. \end{cases}$$

Applying the minimal admissible doubling directly to the residue $r = n \bmod 18$ *gives the deterministic gate*

$$\mathrm{G}(m) := 2^{k_0(r)}\, r \pmod{18}.$$

Evaluating this for each residue yields the fixed gate assignment

$C_2:$	$r = 1$	$r = 7$	$r = 13$
$\mathrm{G}(m)$	4	10	16

$C_1:$	$r = 5$	$r = 11$	$r = 17$
$\mathrm{G}(m)$	10	4	16

Thus the minimal admissible doubling maps each odd residue to a unique even gate in $\{4, 10, 16\}$.

Proof. Proof. (i) Odd residue structure modulo 18. Working modulo 18 and restricting to odd residues, the classes

$$\{3, 9, 15\}, \quad \{1, 7, 13\}, \quad \{5, 11, 17\}$$

correspond respectively to terminating parents (C_0), class C_2, and class C_1. The forward odd-to-odd map preserves these classes and induces a fixed cyclic progression within each.

(ii) Lift to middle-even gates. Applying the minimal admissible doubling directly to an odd residue $r \in \{1, 5, 7, 11, 13, 17\}$ yields the deterministic gate

$$1 \mapsto 4, \quad 7 \mapsto 10, \quad 13 \mapsto 16 \quad \text{and} \quad 5 \mapsto 10, \quad 11 \mapsto 4, \quad 17 \mapsto 16,$$

which are precisely the even gates $\{4, 10, 16\}$ claimed. □

Corollary 1 (Linear segment pattern 19–35)**.** *Listed are the odd integers n from 19 to 35. For each n, record its class (mod 6), its residue (mod 9) and (mod 18), the Inverse middle-even at the minimal admissible doubling k_0 ($k_0 = 2$ for C_2, $k_0 = 1$ for C_1, none for C_0), and the class of the base child*

$$m = \frac{2^{k_0} n - 1}{3} \quad (\textit{when defined}).$$

n	*class*(n) (mod6)	n mod 18	($2^{k_0}n$) mod 18	*base-child class*
19	C_2 (1)	1	4	C_2
21	C_0 (3)	3	–	*none (terminating parent)*
23	C_1 (5)	5	10	C_0
25	C_2 (1)	7	10	C_0
27	C_0 (3)	9	–	*none (terminating parent)*
29	C_1 (5)	11	4	C_2
31	C_2 (1)	13	16	C_1
33	C_0 (3)	15	–	*none (terminating parent)*
35	C_1 (5)	17	16	C_1

Explanation. For each n: determine its class by n mod 6 (C_0: 3, C_1: 5, C_2: 1). If $n \in C_0$, no admissible Inverse step exists. If $n \in C_1$ (resp. C_2), take $k_0 = 1$ (resp. $k_0 = 2$) by admissibility parity. Then use the deterministic gate: $(2^{k_0}n) \bmod 18 \in \{10, 4, 16\}$ with the fixed mapping $10 \mapsto C_0,\ 4 \mapsto C_2,\ 16 \mapsto C_1$. Evaluating these nine cases yields the displayed sequence $2, x, 0, 0, x, 2, 1, x, 1$. This finite segment is a repeating cycle. □

These nine odd residues partition into inadmissible and admissible parents:

$$\underbrace{\{3, 9, 15\}}_{\textit{inadmissible (terminated parent)}},$$

$$\underbrace{\{5, 7\}}_{\textit{base child is } C_0} +10, \qquad \underbrace{\{13, 17\}}_{\textit{base child is } C_1} +16, \qquad \underbrace{\{1, 11\}}_{\textit{base child is } C_2} +4.$$

Lemma 4 (Equidistribution of Base-child Classes)**.** *Across every complete* 18*-residue cycle of odd parents, the base-child classes* C_0, C_1, C_2 *appear with exact frequency* 1/3 *each.*

Proof. By Corollary 1, the nine admissible residues modulo 18 yield the child-class sequence

$$C_2,\ -,\ C_0,\ C_0,\ -,\ C_2,\ C_1,\ -,\ C_1,$$

where dashes denote terminating parents. Each 18-step cycle therefore contains precisely two occurrences of each live class, giving equal frequency 1/3 when restricted to C_0, C_1, C_2. □

Lemma 5 (Forward mod-6 lift to mod-18 at the first even)**.** *Let n be odd and define the Forward middle-even value $E_f(m) := 3m + 1$. Then the residue of n modulo* 6 *determines $E_f(n)$ modulo* 18 *via*

$$n \equiv 1, 3, 5 \pmod 6 \ \mapsto\ E_f(n) \equiv 4, 10, 16 \pmod{18} \textit{ respectively.}$$

*In particular, the first Forward step lifts the mod-*6 *classification to a unique gate residue modulo* 18*.*

Proof. Write $n \equiv r \pmod 6$ with $r \in \{1, 3, 5\}$. Then $E_f(m) = 3m + 1 \equiv 3r + 1 \pmod{18}$ since $18 = 3 \cdot 6$. Direct evaluation gives

$$3 \cdot 1 + 1 \equiv 4 \pmod{18}, \quad 3 \cdot 3 + 1 \equiv 10 \pmod{18}, \quad 3 \cdot 5 + 1 \equiv 16 \pmod{18},$$

which proves the three implications and the uniqueness of the lifted gate residue. □

Proposition 1 (Deterministic child-class decision via mod 18)**.** *In the Inverse Collatz function, and for odd n, the residue of the middle even in* $\{4, 10, 16\} \pmod{18}$ *alone determines the child's odd class, both in Forward and Inverse middle-even. This gives a one-step, local rule independent of trajectory history.*

$$10 \mapsto C_0, \qquad 4 \mapsto C_2, \qquad 16 \mapsto C_1,$$

Existence of a Forward–Inverse alignment through the middle-even gate.

Lemma 6 (Middle-even equivalence mod 18)**.** *If* 3 *does not divide n, then there exists an admissible $k \geq 1$ such that*

$$2^k n \equiv 3m + 1 \pmod{18}.$$

Proof. Forward side (mod 6 *lifted to mod* 18*).* For odd n, the Forward middle-even value is $E_f(m) = 3m + 1$. Reducing n modulo 6 and multiplying by 3 lifts the residue to mod 18:

$$n \equiv 1, 3, 5 \pmod 6 \quad \Longrightarrow \quad E_f(n) \equiv 4, 10, 16 \pmod{18},$$

so $E_f(n)$ always lies in $\{4, 10, 16\}$ (mod18).

Inverse side (mod 18 determinism). For odd n not divisible by 3, the residue n mod 18, together with the admissible parity of k_0 (even if $n \equiv 1 \pmod 3$, odd if $n \equiv 2 \pmod 3$), selects exactly one of the two triads of units modulo 18:

$$\{1, 7, 13\} \quad (\text{even } k), \qquad \{5, 11, 17\} \quad (\text{odd } k).$$

Applying 2^{k_0} places n into the middle-even value that belongs to the nine-step cycle of Corollary 1. That middle-even value is already one of $\{10, 4, 16\}$ (mod 18), the Forward gates. □

3.4. Microcycles and lifted k with tables

Lemma 7 (Rotation under $k \mapsto k+2$ in mod 18)**.** *If k is admissible for odd n ($2^k n \equiv 1 \pmod 3$), then*

$$E_r(n, k) = 2^k n \equiv 10,\ 4,\ 16 \pmod{18}.$$

Moreover $E_r(n, k+2) = 4\, E_r(n, k)$, and hence

$$10 \xrightarrow{+2} 4 \xrightarrow{+2} 16 \xrightarrow{+2} 10 \quad (\text{mod}18).$$

Proof. Admissible $E_r(n, k)$ are even and $1 \pmod 3$, so only $10, 4, 16$ occur modulo 18. For admissible k, $E_r(n, k+2) = 2^{k+2} n = 4\, E_r(n, k)$; computing mod 18 gives $4 \cdot 10 \equiv 4$, $4 \cdot 4 \equiv 16$, $4 \cdot 16 \equiv 10$, which establishes the 3-cycle. □

Microcycles: function and reason. Fix a live odd parent n not divisible by 3. For the Inverse Collatz Function, all admissible Inverse doublings for n share the same parity (by admissibility parity), so from the minimal admissible count k_0 we may advance by steps of 2: k_0, k_0+2, $k_0+4, \ldots$. By Lemma 7, each $+2$ step multiplies the Inverse middle-even by 4 modulo 18, sending $10 \mapsto 4 \mapsto 16 \mapsto 10$ and hence rotating the child classes $C_0 \mapsto C_2 \mapsto C_1 \mapsto C_0$.

$$E_r(n, k_0) \bmod 18 \ \in\ \{10, 4, 16\} \quad \Longrightarrow \quad E_r(n, k_0{+}2) \equiv 4 \cdot E_r(n, k_0) \ (\text{mod}18),$$

$$E_r(n, k_0{+}4) \equiv 4 \cdot E_r(n, k_0{+}2) \ (\text{mod}18),$$

cycling through $10 \to 4 \to 16 \to 10 \pmod{18}$. By the common mod-18 gate (Lemma 6), these three middle-even classes deterministically select the child odd classes C_0, C_2, C_1, in that order. Thus every fixed parent n generates a *k-lifted microcycle* of children:
(C_0, C_2, C_1), in cyclic order beginning with the base admissible child, repeating every three admissible lift steps. Moreover, by the Forward–Inverse middle-even equivalence (Lemma 6), there exists an admissible k for which $E_r(n, k) \equiv E_f(m) =$

$3m + 1 \pmod{18}$, so the Inverse microcycle is aligned with the residue one sees on the Forward side.

To display this mechanism explicitly, we present two parallel tables: *(i) the integer view*, which lists specific n and its children at each admissible lift, and *(ii) the residue view*, which reduces n to $r \equiv n \pmod{18}$. Both views coincide in the mod-18 column and the resulting child class.

Reading across the rows of either table shows how each $+2$ lift advances through the microcycle, and how every admissible parent reaches a residue 10 mod 18 within at most two steps, certifying an accessible termination to C_0.

Example $n = 25$ (Inverse step, even k; here n mod $18 = 7$, n mod $6 = 1 \Rightarrow C_2$):

n	k (even)	$2^k n$	$(2^k n)$ mod 18	$\frac{2^k n - 1}{3}$	$\left(\frac{2^k n-1}{3}\right)$ mod 6	class
25	2	100	10	33	3	C_0
25	4	400	4	133	1	C_2
25	6	1600	16	533	5	C_1
25	8	6400	10	2133	3	C_0
25	10	25600	4	8533	1	C_2
25	12	102400	16	34133	5	C_1

r	k (even)	$2^k r$	$(2^k r)$ mod 18	$\frac{2^k r - 1}{3}$	$\left(\frac{2^k r-1}{3}\right)$ mod 6	class
7	2	28	10	9	3	C_0
7	4	112	4	37	1	C_2
7	6	448	16	149	5	C_1
7	8	1792	10	597	3	C_0
7	10	7168	4	2389	1	C_2
7	12	28672	16	9557	5	C_1

Example $n = 29$ (Inverse step, odd k; here $n \bmod 18 = 11$, $n \bmod 6 = 5 \Rightarrow C_1$):

n	k (odd)	$2^k n$	$(2^k n) \bmod 18$	$\frac{2^k n - 1}{3}$	$\left(\frac{2^k n-1}{3}\right) \bmod 6$	class
29	1	58	4	19	1	C_2
29	3	232	16	77	5	C_1
29	5	928	10	309	3	C_0
29	7	3712	4	1237	1	C_2
29	9	14848	16	4949	5	C_1
29	11	59392	10	19797	3	C_0

r	k (odd)	$2^k r$	$(2^k r) \bmod 18$	$\frac{2^k r - 1}{3}$	$\left(\frac{2^k r-1}{3}\right) \bmod 6$	(class)
11	1	22	4	7	1	C_2
11	3	88	16	29	5	C_1
11	5	352	10	117	3	C_0
11	7	1408	4	469	1	C_2
11	9	5632	16	1877	5	C_1
11	11	22528	10	7509	3	C_0

3.5. Mod 54 Refinement: Fixing the Child Residue

The mod-18 gate (Lemma 3, Proposition 1) determines the *child class*. Refining the lens to mod 54 determines, already at the first admissible Inverse step, the child's *odd residue modulo* 18.

Triad map (mod 54). Write every live odd n as

$$n \;=\; 54m + r,$$

$$r \in \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 49, 53\}, \quad m \in \mathbb{N}_{\geq 0}.$$

Set $q \equiv j \pmod 3 \in \mathcal{T}_{54}$, with $j \in \{0, 1, 2\}$. For each $r_{18} \in \{1, 5, 7, 11, 13, 17\}$, the corresponding residues in mod 54 are

$$r_{54} \in \{\, r_{18},\; r_{18} + 18,\; r_{18} + 36 \,\}.$$

Define the lifted triads $\mathcal{T}_{54}(r_{54}) = (t_{r,0}, t_{r,1}, t_{r,2})$ by

$r_{54} \in j\{0,1,2\}$	$t_{r,0}$	$t_{r,1}$	$t_{r,2}$
$1, 19, 37$	1	7	13
$11, 29, 47$	7	1	13
$13, 31, 49$	17	5	11
$17, 35, 53$	11	5	17
$5, 23, 41$	3	15	9
$7, 25, 43$	9	15	3

Each lifted triad row follows the same deterministic pattern as the mod 18 table. The indexing variable r_{54} plays the same role as q_{r18} in selecting the correct column of the triad. Rows for $r_{54} \in \{1, 11, 13, 17\}$ are in C_2 or C_1, and $\{5, 7\}$ remain in C_0.

Lemma 8 (Mod 54 refinement fixes the child residue)**.** *Let*

$$n \;=\; 54m + r_{54},$$

$$r \in \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 49, 53\}, \quad m \in \mathbb{N}_{\geq 0}.$$

Set $j \pmod 3 \equiv \mathcal{T}_{54}$*. Then the first admissible Inverse child of* n *has odd residue*

$$\left(\frac{2^{\,k_0}\, n - 1}{3}\right) \;\equiv\; t_{r,\,j} \pmod{18},$$

where $t_{r_{54},\,j}$ *is determined by the lifted triad* $\mathcal{T}_{54}(r_{54})$*. Equivalently, the pair* (r_{54}, j) *uniquely determines the child's odd residue modulo* 18*.*

Proof sketch. By Lemma 2, the minimal admissible exponent $k_0(n)$ is odd for $n \in C_1$ and even for $n \in C_2$. The mod 18 structure (Lemma 3) partitions the six live residues into deterministic triads, and the admissibility parity lifts each residue canonically to its gate (Proposition 1).

Passing to mod 54, each r_{18} splits into three residues

$$r_{54} \in \{\, r_{18},\; r_{18} + 18,\; r_{18} + 36 \,\},$$

and the index j mod 3 selects one of the three columns of the lifted triad table $\mathcal{T}_{54}$. Evaluating the first admissible Inverse step for $j = 0, 1, 2$ within each r_{54} reproduces exactly the triad outputs listed in Table 1. Thus (r_{18}, j) completely determines the child residue modulo 18. $\square$

Compact 54**-row table.** Because n mod 54 is completely determined by (r, q), the mapping

$$n \bmod 54 \;\longmapsto\; \big(\text{child odd residue mod } 18\big)$$

is obtained by grouping the 27 live residues mod 54 into six blocks by r and subdividing each block by $j \in \{0, 1, 2\}$. For example, the block $r = 1$ contributes residues

$$\{1,\ 19,\ 37\} \pmod{54} \quad \rightsquigarrow \quad \{1, 7, 13\} \pmod{18}$$

in the order $j = \{0, 1, 2\}$.

Table 1: Mod 54 refinement: for odd $n \in [1, 53]$, the residue $r \equiv n \pmod{18}$ and the base child's class and residue (mod18).

n_{54}	r_{18}	parent class	base child class	base child residue (mod18)
1	1	C2	C2	1
3	3	C0	—	—
5	5	C1	C0	3
7	7	C2	C0	9
9	9	C0	—	—
11	11	C1	C2	7
13	13	C2	C1	17
15	15	C0	—	—
17	17	C1	C1	11
19	1	C2	C2	7
21	3	C0	—	—
23	5	C1	C0	15
25	7	C2	C0	15
27	9	C0	—	—
29	11	C1	C2	1
31	13	C2	C1	5
33	15	C0	—	—
35	17	C1	C1	5
37	1	C2	C2	13
39	3	C0	—	—
41	5	C1	C0	9
43	7	C2	C0	3
45	9	C0	—	—
47	11	C1	C2	13
49	13	C2	C1	11
51	15	C0	—	—
53	17	C1	C1	17

Corollary 2 (Periodicity of the Mod 54 Child Mapping)**.** *Let n be an odd integer with*

$$n = 18q + r, \qquad r \in \{1, 5, 7, 11, 13, 17\}, \qquad q \equiv q \pmod{3}.$$

Let $c(n)$ denote the residue modulo 18 of the first admissible Inverse child of n,

$$c(n) := \left(\frac{2^{k_0(n)}\, n - 1}{3}\right) \bmod 18.$$

Then for every integer $m \geq 0$ (period index),

$$c(n + 54m) = c(n).$$

Equivalently, the mapping

$$n \bmod 54 \;\longmapsto\; c(n)$$

is periodic with fundamental period 54*. In particular, the table of base-child residues for odd* $n \in [1, 53]$ *repeats identically on each interval* $[1 + 54m,\ 53 + 54m]$.

Interpretation. The refinement to modulus 54 resolves the residual ambiguity left by the mod-18 gate. At mod-18, each live residue determines only the *class* of its child; lifting to mod-54 records the phase of the quotient $q \bmod 3$, which fixes the child's exact odd residue mod 18. The resulting triads $\mathcal{T}_{54}(r_{54})$ show that every parent residue r_{54} generates three distinct child residues, one for each phase position. Because these triads repeat with period 54, the entire Inverse map becomes periodic at that modular scale. This periodicity demonstrates that the residue–phase system is finite and deterministic: each pair $(r_{54}, q \bmod 3)$ has one unique successor, and every possible parent–child relationship repeats identically on successive 54-blocks.

Lemma 9 (Affine Inverse update law)**.** *Let* $n = 18q + r$ *with* $r \in \{1, 5, 7, 11, 13, 17\}$ *and* $q \in \mathbb{N}_0$*, and set*

$$k_0(r) = \begin{cases} 2, & r \in C_2 = \{1, 7, 13\}, \\ 1, & r \in C_1 = \{5, 11, 17\}. \end{cases}$$

Define

$$m = R(n; k_0(r)) = \frac{2^{k_0(r)}(18q + r) - 1}{3} = A_r\, q + B_r,$$

$$A_r = \frac{2^{k_0(r)} \cdot 18}{3}, \quad B_r = \frac{2^{k_0(r)} r - 1}{3}.$$

Then $m \in \mathbb{N}$ *and the single–step update* $(r, q) \mapsto (r', q')$ *is given by*

$$r' = m \bmod 18, \qquad q' = \left\lfloor \frac{m}{18} \right\rfloor,$$

with the following explicit formulas:

1. *(Slope and intercept)*

$$r \in C_1: \;\; A_r = 12, \;\; B_r = \tfrac{2r-1}{3} \in \{3, 7, 11...\};$$

$$r \in C_2: \;\; A_r = 24, \;\; B_r = \tfrac{4r-1}{3} \in \{1, 9, 17...\}.$$

2. *(Residue update by phase)*

$$\boxed{\begin{array}{lll} r \in C_1: & r' \equiv B_r - 6\,(q \bmod 3) & (\mathrm{mod}\ 18), \\ r \in C_2: & r' \equiv B_r + 6\,(q \bmod 3) & (\mathrm{mod}\ 18). \end{array}}$$

3. (Quotient update)

$$q' = \begin{cases} \left\lfloor \dfrac{12q}{18} \right\rfloor = \left\lfloor \dfrac{2}{3}q \right\rfloor, & r \in C_1, \\ \left\lfloor \dfrac{24q}{18} \right\rfloor = \left\lfloor \dfrac{4}{3}q \right\rfloor, & r \in C_2. \end{cases}$$

Consequently, the pair $(r, q \bmod 3)$ *uniquely determines* $r' \bmod 18$*, and the next phase is* $q' \bmod 3$ *computed from the affine form* $m = A_r q + B_r$*.*

Corollary 3 (Finite Residue–Phase Automaton)**.** *For each step of the Inverse map defined by*

$$F : (r, q \bmod 3) \mapsto (r', q' \bmod 3),$$

the image (r', q') *depends only on* $(r, q \bmod 3)$ *through the valuation of* $3n + 1$*. The quotient component evolves under the induced transformation*

$$q' \bmod 3 = \left\lfloor \frac{2^{k_0(r)}(18q + r) - 1}{54} \right\rfloor \bmod 3,$$

and defines a finite deterministic automaton on the space $\{(r, q \bmod 3)\}$*. The sequence* $\{F_t\}$ *obtained by successive iterations remains bounded within this finite set, generating locally deterministic residue–phase transitions.*

Lemma 10 (Residue–Phase Transition and Reset–Resume Law)**.** *Let* $n = 18q + r$ *with* $r \in \{1, 5, 7, 11, 13, 17\}$ *and* $m = R(n; k_0(r))$ *as above. Then the following properties hold:*

1. *For fixed* r*, as* q *varies modulo* 3*, the residues* $m \bmod 18$ *occupy three distinct elements of* $\{1, 3, 5, 7, 9, 11, 13, 15, 17\}$ *corresponding to the classes* C_0, C_1, C_2*.*
2. *The order of appearance of these residues is determined by* r *and the parity of* $k_0(r)$*, defining a locally unique orientation.*
3. *For each iteration, the next phase and residue* $(r', q' \bmod 3)$ *are re-evaluated from the resulting* m*, establishing a reset and resume transition of the form*

$$(r, q \bmod 3) \mapsto (r', q' \bmod 3),$$

where $r' = m \bmod 18$ *and* $q' = \lfloor m/18 \rfloor$*.*

The residue phase system thereby forms a finite deterministic automaton with terminal residues $C_0 = \{3, 9, 15\}$*, transitional residues* $\{5, 7\}$ *mapping into* C_0*, and active residues* $\{1, 11, 13, 17\}$ *forming the lattice* $\{C_2 \to C_2,\, C_2 \to C_1,\, C_1 \to C_2,\, C_1 \to C_1\}$*.*

r	$k_0(r)$	$B_r = \dfrac{2^{k_0(r)}r - 1}{3}$	$\sigma(r)$	$(m \bmod 18)$ for $q \bmod 3 = 0, 1, 2$
1	2	1	+1	(1, 7, 13)
7	2	9	+1	(9, 15, 3)
13	2	17	+1	(17, 5, 11)
5	1	3	−1	(3, 15, 9)
11	1	7	−1	(7, 1, 13)
17	1	11	−1	(11, 5, 17)

Table 2: Residue classes, minimal exponents, orientation signs, and resulting triads $(m \bmod 18)$ for each live residue r.

Interpretation. The affine Inverse update law converts the inverse Collatz step into a linear rule on the quotient–residue plane. For each live residue r, the minimal admissible exponent $k_0(r)$ fixes the slope A_r and intercept B_r of an affine map $m = A_r q + B_r$. The modulus 18 confines all results to nine possible odd residues, and the quotient modulus 3 serves as a rotating phase selector. Hence every pair $(r, q \bmod 3)$ specifies a unique successor $(r', q' \bmod 3)$.

Geometrically, the system behaves as a finite automaton of six residue rows ($r \in \{1, 5, 7, 11, 13, 17\}$) and three phase columns ($q \bmod 3$). The "reset–resume" rule means that after each Inverse step, the new residue and phase become the parameters of the next affine map. This continual reassignment makes the process locally deterministic but globally adaptive: the governing equation changes with each step while remaining finite. Terminal residues in $C_0 = \{3, 9, 15\}$ close the automaton, ensuring every k_0 only sequence eventually reaches a fixed point of the system (C_0) under k_0.

Markov sieve viewpoint (two transients, one absorbing terminal). Although the update rule is deterministic once the state is fixed, it is useful to regard the induced directed graph on the finite state space

$$S \;=\; \{(r,\ q \bmod 3)\ :\ r \in \{1, 5, 7, 11, 13, 17\}\}$$

as a Markov-style transition diagram: each inverse step sends a state $(r,\ q \bmod 3)$ to a uniquely determined successor $(r',\ q' \bmod 3)$. In this graph, the terminal residue set

$$C_0 \;=\; \{3, 9, 15\} \pmod{18}$$

acts as an absorbing (terminating) sieve: once the orbit enters the terminal region, no further admissible inverse continuation exists (equivalently, the inverse chain closes immediately in the C_0 class). The remaining admissible states decompose into two transient components (the live classes C_1 and C_2), which may transition among themselves but cannot support an infinite forward-avoiding inverse ancestry: the admissibility constraint encoded by $k_0(r)$ governs which transitions exist, and in particular forces every orbit to reach the terminal sieve in finitely many steps.

Theorem 1 (Global Determinism and Finite Termination of the Inverse Automaton)**.** *Let (r_t, q_t) denote the residue and quotient at step t, and define*

$$n_t = 18q_t + r_t, \qquad m_t = \frac{2^{k_0(r_t)} n_t - 1}{3}, \qquad r_{t+1} = m_t \bmod 18, \qquad q_{t+1} = \left\lfloor \frac{m_t}{18} \right\rfloor.$$

Then:

1. *For each step, $(r_t, q_t \bmod 3)$ uniquely determines $(r_{t+1}, \mathit{class}(r_{t+1}))$, forming a finite deterministic mapping.*

2. *The transition structure satisfies*

$$7, 5 \to C_0, \qquad 1, 13, 11, 17 \to \{C_1, C_2\},$$

   producing the four active transition types:

$$1 \to \{C_2 \to C_2\},\ 13 \to \{C_2 \to C_1\},$$
$$11 \to \{C_1 \to C_2\},\ 17 \to \{C_1 \to C_1\}.$$

3. *The system evolves through successive local maps*

$$F_t : (r_t, q_t \bmod 3) \mapsto (r_{t+1}, q_{t+1} \bmod 3),$$

   generating a finite deterministic sequence in the residue phase space.

4. *Each active transition ultimately reaches a terminal residue in C_0 within finitely many steps. The mapping admits no infinite nonterminal orbit.*

Hence the Inverse Collatz dynamics on odd integers forms a finite, locally deterministic reset and resume automaton whose transitions are governed by residue class and phase position at each step.

3.6. Bounded Corridor Dynamics at Fixed Residues

Among the six live residues modulo 18, only

$$r \in \{1, 17\}$$

have the special property that their first admissible Inverse child under k_0 remains in the same residue class. This follows directly from the triadic structure established in Subsection 3.3: all other live residues transition immediately to a different residue upon the first admissible lift, whereas $r = 1$ and $r = 17$ alone form self-contained local corridors under Forward iteration.

Because these two residues can map to themselves under k_0, their Forward dynamics admit chains of arbitrary length determined solely by arithmetic properties of the phase index q. For $r = 1$, the Forward map contracts by a factor of $\frac{3}{4}$ until the 2-power in q is exhausted. For $r = 17$, the Forward map expands by $\frac{3}{2}$ for exactly $\nu_2(q_0 + 1)$ steps, consuming one factor of 2 per iteration.

The results in the following subsections establish the precise structure and length of these corridors: - $r = 1$ admits contraction chains controlled by divisibility of q. - $r = 17$ admits expansion chains controlled by the 2-adic valuation of $q + 1$.

These two cases are the only local residue dynamics that can persist beyond a single step under k_0, and their exhaustion determines the maximal extent of fixed-residue behavior in the entire system.

Inverse map at $r = 1$. Let

$$m = 18q + 1.$$

$$3m + 1 = 54q + 4 = 2\,(27q + 2).$$

If q is divisible by 4, then

$$27q + 2 \equiv 2 \pmod 4 \quad \Rightarrow \quad \nu_2(27q + 2) = 1,$$

$$k_m = 1 + 1 = 2.$$

The Forward update is then

$$m = \frac{54q + 4}{2^2} = \frac{27}{2}\,q + 1.$$

Since we only care about the q-level:

$$q' = \frac{m - 1}{18} = \frac{\frac{27}{2}q}{18} = \frac{3}{4}\,q.$$

$$\boxed{q' = \frac{3}{4}\,q}.$$

This shows that, as long as q remains divisible by 4, the Forward map strictly scales q by a factor of $\frac{3}{4}$ without changing the residue class $r = 1$. *The descent in q continues until the 2-adic factor is exhausted, at which point the residue transition occurs.*

Inverse map at $r = 17$. Let

$$m = 18q + 17.$$

Then

$$3m + 1 = 54q + 52 = 2\,(27q + 26).$$

If q is odd (i.e. $q \equiv 1 \pmod 2$), then $27q + 26$ is odd, so

$$\nu_2(27q + 26) = 0 \quad \Rightarrow \quad k_m = 1.$$

The Forward update is therefore

$$m = \frac{54q + 52}{2} = 27q + 26.$$

Writing $m = 18q' + r'$ gives

$$27q + 26 \;=\; 18\Big(\frac{3q+1}{2}\Big) + 17,$$

so $r' = 17$ and

$$q' = \frac{3q+1}{2}.$$

$$\boxed{q' = \frac{3q+1}{2}} \qquad \text{(valid exactly when } q \text{ is odd).}$$

This map preserves the residue $r = 17$ precisely while q remains odd. Rewriting the recurrence,

$$q_{t+1} = \frac{3q_t + 1}{2} \quad \Longleftrightarrow \quad q_{t+1} + 1 = \frac{3}{2}\,(q_t + 1),$$

gives the explicit evolution

$$q_t + 1 = \Big(\frac{3}{2}\Big)^t (q_0 + 1) = 3^t\, 2^{\,\nu_2(q_0+1)-t}.$$

Hence the number of consecutive $r = 17$ steps is determined entirely by the 2-adic valuation of $q_0 + 1$:

$$\boxed{e = \nu_2(q_0 + 1)}.$$

Remark 1. If $q_0 + 1$ is a pure power of 2, the corridor length equals that power's exponent exactly. If it contains an odd factor $u > 1$, where

$$q_0 + 1 = 2^e u, \quad u \text{ odd},$$

the corridor length still equals e, and the odd factor merely remains as a cofactor during the valid steps. Thus the run length for $r = 17$ is governed entirely by the 2-adic valuation of $q_0 + 1$ and not by any fixed external bound.

Together with the $r = 1$ case, this establishes explicit local corridor dynamics: the $r = 1$ map contracts by a factor $\frac{3}{4}$ until powers of 2 are exhausted, while the $r = 17$ map expands by $\frac{3}{2}$ for exactly e steps, with e determined directly by the factorization of $q_0 + 1$.

Lemma 11 (Higher admissible lifts are strictly ascending and rotate the gate)**.** *Fix a live odd parent n and let $k_0 \in \{1, 2\}$ be its minimal admissible exponent (determined by class). For each $t \geq 0$ define the t-th admissible lift and Inverse child by*

$$k_e := k_0 + 2e, \qquad m_e := R(n; k_e) = \frac{2^{\,k_e} n - 1}{3}.$$

Then:

(a) ***Strict ascent in the Inverse value.*** *The sequence $(m_t)_{t \geq 0}$ is strictly increasing, with the exact increment*

$$m_{t+1} - m_t \;=\; \frac{2^{\,k_t+2} n - 1}{3} - \frac{2^{\,k_t} n - 1}{3} \;=\; 2^{\,k_t} n \;>\; 0.$$

Equivalently,

$$m_e \;=\; \frac{2^{\,k_0}}{3}\, 4^{\,t}\, n \;-\; \frac{1}{3},$$

so m_t grows geometrically in t.

(b) ***Gate rotation (class rotation).*** *The associated Inverse middle-even residues rotate deterministically:*

$$E_r(n, k_e) = 2^{\,k_e} n \equiv 10,\, 4,\, 16 \pmod{18} \quad \textit{with}$$

$$E_r(n, k_{e+1}) \equiv 4\, E_r(n, k_e) \pmod{18},$$

yielding the cycle $10 \to 4 \to 16 \to 10$ (Lemma 7). Consequently the child class rotates $C_0 \to C_2 \to C_1 \to C_0$.

(c) ***Higher lifts are higher transformations.*** *Each increment $e \mapsto e + 1$ multiplies the affine scaling factor by 4 (from $\frac{2^{k_e}}{3}$ to $\frac{2^{k_e+2}}{3}$) while preserving the constant drift $-\frac{1}{3}$. Thus every higher admissible lift is a strictly larger affine transform on n, independent of the gate rotation.*

Proof. (a) Compute directly:

$$m_{e+1} - m_e = \frac{2^{k_e+2} n - 1}{3} - \frac{2^{k_e} n - 1}{3} = 2^{k_e} n > 0,$$

so (m_t) is strictly increasing. The closed form follows from $k_e = k_0 + 2e$.

(b) This is Lemma 7: for admissible k, $E_r(n, k) \equiv 10, 4, 16 \pmod{18}$ and $E_r(n, k+2) \equiv 4E_r(n, k) \pmod{18}$, producing the stated rotation and class cycle.

(c) From $R(n;k) = \frac{2^k}{3}n - \frac{1}{3}$, replacing k by $k+2$ multiplies the linear coefficient by 4 and leaves the drift unchanged, so the transform strictly enlarges the image while the residue gate rotates as in (b). $\square$

Interpretation. Only the residues $r = 1$ and $r = 17$ form self-contained "corridors" in the residue–phase system. All other live residues immediately transition to a different class after one admissible lift. Within these two corridors the Forward dynamics are governed purely by 2-adic properties of the quotient variable q.

For $r = 1$, the Forward map contracts q by a factor of $\frac{3}{4}$ as long as q remains divisible by 4. Each iteration removes one factor of 2, so the chain length equals the 2-adic valuation of q. For $r = 17$, the Forward map expands by $\frac{3}{2}$ while q is odd, and the number of valid steps is exactly $\nu_2(q_0 + 1)$. Thus the persistence of each corridor is determined entirely by local 2-adic content, not by any external bound.

Beyond these corridors, higher admissible lifts always increase the Inverse value and rotate the middle-even gate through $10 \to 4 \to 16$. Each lift multiplies the affine scale by 4 while preserving the constant drift, so the sequence of lifts is a strictly ascending geometric rail. Together these facts show that fixed-residue behavior is finite and bounded, and that all non-terminal paths ultimately exit their local corridors to join the global terminating flow.

4. Consequences of Lens Refinement and Finite Inverse Lifespan

In this section all integers are odd and positive. We retain the classes

$$\mathrm{C0} = \{3, 9, 15\} \pmod{18}, \quad \mathrm{C1} = \{5, 11, 17\} \pmod{18}, \quad \mathrm{C2} = \{1, 7, 13\} \pmod{18},$$

the boundary residues $5, 7 \pmod{18}$, and the live residues $\{1, 11, 13, 17\} \pmod{18}$. We also keep $F(\cdot)$, $m(\cdot)$, and $k_0(\cdot)$ from the earlier setup.

4.1. Standing conventions and phase

Every odd n is written uniquely as

$$n = 18q + r, \qquad r \in \{1, 3, 5, 7, 9, 11, 13, 15, 17\}, \; q \in \mathbb{N}_{\geq 0}.$$

We call r the residue of n and define the *phase*

$$\phi(n) := q \pmod{3} \in \{0, 1, 2\}.$$

4.2. One-step Inverse lens under k_0: triads and boundary

Define the minimal Inverse step

$$m = \frac{2^{k_0(n)}\, n - 1}{3}, \qquad k_0(n) = \begin{cases} 1, & n \equiv 2 \pmod 3 \quad \text{(C1)}, \\ 2, & n \equiv 1 \pmod 3 \quad \text{(C2)}, \end{cases}$$

with m required odd. For a fixed $r \in \{1, 5, 7, 11, 13, 17\}$ set

$$\mathcal{T}(r) := \big\{\, m(18q + r) \pmod{18} : \ q \pmod 3 \in \{0, 1, 2\} \,\big\}.$$

Lemma 12 (Triads and boundary presence)**.** *For each live residue r, the set $\mathcal{T}(r)$ has exactly three elements and forms a triad. Moreover:*

- *If $r \in \{5, 7\}$, then $\mathcal{T}(r) \subseteq$ C0.*
- *If $r \in \{1, 11, 13, 17\}$, then $\mathcal{T}(r)$ contains at least one boundary residue (5 or 7 mod 18), and the other elements lie in $\{1, 11, 13, 17\}$.*

Proof. Reduce $m(18q+r)$ modulo 18; dependence is only on r and $q \pmod 3$, giving $|\mathcal{T}(r)| = 3$ and the stated boundary structure by direct casework. □

Lemma 13 (C0 is Inverse-terminal)**.** *If $p \in$ C0, then $m(p)$ is not an odd integer.*

Proof. If $3 \mid p$, then $(2^k p - 1)/3 \notin \mathbb{N}$ for all $k \geq 1$. □

4.3. Residue rotation law

Write $n = 18q + r$ with phase $j = \phi(n) = q \pmod 3$ and let $k_0(r) \in \{1, 2\}$ be the class-determined exponent. Set

$$A_r = \begin{cases} 12, & r \in \text{C1}, \\ 24, & r \in \text{C2}, \end{cases} \qquad B_r = \frac{2^{k_0(r)} r - 1}{3}, \qquad \sigma(r) = \begin{cases} -1, & r \in \text{C1}, \\ +1, & r \in \text{C2}. \end{cases}$$

Then the minimal child m satisfies

$$m \equiv B_r + 6\,\sigma(r)\, j \pmod{18}, \qquad q' = \left\lfloor \frac{A_r q + B_r}{18} \right\rfloor,$$

so the residue advances by a constant step ± 6 inside a fixed triad (sign by class), while the new phase $\phi(m) = q' \pmod 3$ is obtained from the affine quotient. Consequently the pair $(r, \phi(n))$ uniquely determines $m \pmod{18}$.

4.3.1. Generational residue–phase map and finiteness

Define the local update

$$F:(r,\phi)\longmapsto \big(r',\phi'\big),\quad r'\equiv m(18q+r) \pmod{18},\quad \phi'=q' \pmod 3,\quad q'=\left\lfloor \frac{m}{18}\right\rfloor.$$

This yields a finite, locally deterministic automaton on the space $\{(r,\phi): r\in\{1,5,7,11,13,17\},\ \phi\in\{0,1,2\}\}$ with terminal sink C0.

Interpretation. The residue rotation law establishes that every live residue r advances within a closed triad by a fixed modular step of ± 6. This motion is cyclic, but not self-sustaining indefinitely: each triad contains at least one boundary residue (either 5 or 7 mod 18) whose next image lies in the terminal set $\mathrm{C0}=\{3,9,15\}$. Thus, although the rotation within a class appears periodic, the presence of these boundary residues ensures that repeated application of the map cannot cycle endlessly within C1 or C2.

When viewed on the full residue–phase grid (r,ϕ), the update law $F:(r,\phi)\mapsto(r',\phi')$ forms a finite directed graph in which each vertex has a single outgoing edge. Every orbit therefore follows a deterministic path through a bounded set of 18 states. Because at least one state in every rotation chain transitions to C0, all paths must eventually reach a terminal residue and halt. The rotation law therefore provides the local mechanism by which the global map attains finite convergence.

Theorem 2 (Finite local dynamics)**.** *For each step, (r_t,ϕ_t) uniquely determines $(r_{t+1},\mathrm{class}(r_{t+1}))$. Every nonterminal transition type lies among $\{\mathrm{C2}\to\mathrm{C2},\ \mathrm{C2}\to\mathrm{C1},\ \mathrm{C1}\to\mathrm{C2},\ \mathrm{C1}\to\mathrm{C1}\}$, and every trajectory in this finite automaton reaches a terminal residue in* C0 *in finitely many steps.*

4.3.2. Lift microcycles and guaranteed boundary access

For a fixed live parent n, all admissible exponents have fixed parity; lifts $k=k_0+2t$ rotate the middle-even residue by a factor 4 (mod 18):

$$10 \xrightarrow{+2} 4 \xrightarrow{+2} 16 \xrightarrow{+2} 10 \qquad \pmod{18},$$

so the child classes rotate $\mathrm{C0}\to\mathrm{C2}\to\mathrm{C1}\to\mathrm{C0}$. In particular, within at most two lifts the gate 10 (mod 18) is attained, making C0 accessible.

4.3.3. Mod-54 refinement: fixing the child residue

Refining to modulus 54 splits each live residue r (mod 18) into three residues $r, r+18, r+36$; the index m (mod 3) selects the column of a lifted triad that *already* fixes the child's odd residue modulo 18 at the first admissible Inverse step. Thus $(n \pmod{54})$ determines the child residue.

4.4. The $n = 1$ self-loop

Remark 2 (The trivial self-loop and phase stability)**.** The integer $n = 1$ is the unique odd fixed point of the odd-to-odd map: $T(1) = (3 \cdot 1 + 1)/2^{\nu_2(4)} = 1$. In the 18-lens we have $1 = 18 \cdot 0 + 1$, so $\phi(1) = 0$ and both residue and phase remain unchanged. On the Inverse side, the minimal lift for $r = 1$ is $k_0 = 2$, and $R(1; 2) = (4 \cdot 1 - 1)/3 = 1$. Hence $n = 1$ is the only state that self-loops while staying phase-stable at every lens; all other live residues either change residue at the first minimal step or exhaust their corridor in finitely many steps.

4.5. Affine Arithmetic Decomposition

Lemma 14 (Affine drift of one odd-to-odd inverse step)**.** *For any odd n and admissible exponent k,*

$$R(n; k) = \frac{2^k n - 1}{3} = \frac{2^k}{3}\, n - \frac{1}{3}.$$

Thus every odd-to-odd inverse step has the same constant drift term $-\frac{1}{3}$.

Lemma 15 (Accumulated affine drift under inverse iteration)**.** *Let $t \geq 1$ and let $k_1, \ldots, k_t$ be admissible exponents. Define the accumulated scale*

$$\Lambda(k_1, \ldots, k_t) \;:=\; \prod_{i=1}^{t} \frac{2^{k_i}}{3},$$

and define the accumulated drift

$$\mathcal{D}(k_1, \ldots, k_t) \;:=\; \frac{1}{3} \sum_{s=0}^{t-1} \prod_{i=s+2}^{t} \frac{2^{k_i}}{3},$$

where s is the summation index, i is the product index, and the empty product is interpreted as 1*.*

Let n_0 be odd and let $k_1, \ldots, k_t$ be admissible exponents. Define the inverse iterates by

$$n_i := R(n_{i-1}; k_i) \qquad (1 \leq i \leq t).$$

Then

$$n_t = \Lambda(k_1, \ldots, k_t)\, n_0 - \mathcal{D}(k_1, \ldots, k_t),$$

and $\mathcal{D}(k_1, \ldots, k_t) > 0$.

Lemma 16 (Lift-by-2 rail)**.** *The Inverse map is affine (scale $2^k/3$, subtract $1/3$). In particular,*

$$R(n; k + 2) = 4\, R(n; k) + 1,$$

so each admissible parity class generates the rail $m \mapsto 4m + 1$.

Proof.

$$R(n;k+2)=\frac{2^{k+2}n-1}{3}=4\frac{2^kn-1}{3}+1=4R(n;k)+1.$$

□

Proposition 2 (Uniqueness and disjointness of inverse odd-to-odd iteration)**.** *For any odd initial value n_0 and any finite admissible exponent sequence $(k_1,\dots,k_t)$, the inverse iterates*

$$n_i:=R(n_{i-1};k_i)$$

are uniquely determined. Moreover, distinct admissible exponent sequences produce disjoint odd-to-odd inverse trajectories.

Proof. By Lemma 15, each inverse step contributes a fixed negative affine drift, so distinct exponent sequences yield distinct affine coefficients $(\Lambda,\mathcal{D})$ and therefore cannot coincide. □

4.6. Consistency of aligned steps

4.6.1. The Trivial Loop from $n=1$: Inverse and Forward Views

Lemma 17 (1 is C_2 and has even admissible doublings)**.** *Since $1\equiv 1 \pmod 6$, the integer 1 lies in class C_2. Admissibility for the Inverse step $m=\frac{2^kn-1}{3}\in\mathbb{N}$ requires $2^kn\equiv 1 \pmod 3$. With $n=1$ and $2\equiv -1 \pmod 3$, this gives $(-1)^k\equiv 1$, hence k is even. The minimal admissible doubling count is $k_0=2$.*

Proposition 3 (Base child of 1 equals 1)**.** *With $k_0=2$, the Inverse child of $n=1$ is*

$$R_{\min}=\frac{2^{k_0}\cdot 1-1}{3}=\frac{4-1}{3}=1,$$

so the base child of 1 is 1 again. Consequently, under the Inverse map with minimal admissible doubling, $n=1$ is a fixed point in class C_2.

Remark 3 (Consistency with the Forward picture: the $4\to 2\to 1$ loop)**.** From the Forward side, starting at 1,

$$3\cdot 1+1=4 \quad\longrightarrow\quad 2 \quad\longrightarrow\quad 1,$$

which is the well-known $4\to 2\to 1$ loop. Thus the Inverse fixed point at $n=1$ (with minimal $k=2$) corresponds exactly to the unique Forward cycle.

Lemma 18 (Forward–Inverse locked step)**.** *Let n be odd and set*

$$n\;=\;T(m)\;=\;\frac{3m+1}{2^{k_m(m)}},\qquad k_m(m)=\nu_2(3m+1).$$

Then

$$R\big(m;\, k_m(n)\big) \;=\; n.$$

Conversely, for any odd m and any admissible *k,*

$$m \;:=\; R(n;k) \;=\; \frac{2^k n - 1}{3} \quad\Longrightarrow\quad T(m) \;=\; n \quad \textit{and} \quad \nu_2(3m+1) = k.$$

Proof. If $n = T(m)$ then $3m+1 = 2^{k_m} n$ with n odd, hence $R(n; k_m(n)) = (2^{k_m(n)} n - 1)/3 = m$.

Conversely, if $m = (2^k n - 1)/3$ with n odd, then $3m+1 = 2^k n$ so $\nu_2(3m+1) = k$ and $T(m) = (3m+1)/2^k = n$. □

Corollary 4 (Forward uniqueness, Inverse branching)**.** *For each odd n, the Forward step $T(m) = (3m+1)/2^{k_m(m)}$ is unique (the maximal 2-power is forced). For a fixed odd m, k_m yields a (distinct) parent $T(m) = n$. Conversely, the Inverse map branches because admissible inverse steps $R(n;k)$ exist for all $k = k_0(n) + 2e$ with $e \geq 0$, yielding distinct inverse children.Thus the Inverse tree branches, while the Forward trajectory is locked; following Lemma 18, the* edge-aligned *Inverse choice at each node reproduces the Forward path exactly.*

Theorem 3 (Forward equivalence for all admissible Inverse lifts)**.** *Let $T(m) = (3m+1)/2^{\nu_2(3m+1)}$ be the odd-to-odd map of the system, and let*

$$R(n;k) := \frac{2^k n - 1}{3}$$

denote the Inverse map whenever $2^k n \equiv 1 \pmod 3$. For each odd $n \geq 1$ let $k_0(n)$ be the minimal admissible exponent such that $m_0 := R(n; k_0(n))$ is an odd integer.

Then for every integer $e \geq 0$ such that $k := k_0(n) + 2e$ is admissible and $m_e := R(n;k)$ is odd, the Forward map satisfies

$$T(m_e) = n.$$

Equivalently,

$$T\big(R(n;k)\big) = n \qquad \textit{for every admissible } k \equiv k_0(n) \pmod 2.$$

In particular, all odd Inverse lifts of n obtained by admissible exponents $k_0(n) + 2e$ collapse under T to the same odd value n.

Proof. Fix an odd $n \geq 1$, and let $k_0 := k_0(n)$ be the minimal admissible exponent with $m_0 = R(n; k_0)$ odd. For any integer $e \geq 0$ such that $k := k_0 + 2e$ is admissible and $m_e := R(n;k)$ is odd, we have by definition

$$3m_e + 1 = 2^k n = 2^{k_0 + 2e} n.$$

Hence $\nu_2(3m_e+1) = k_0 + 2e$, and therefore

$$T(m_e) = \frac{3m_e+1}{2^{\nu_2(3m_e+1)}} = \frac{2^{k_0+2e}n}{2^{k_0+2e}} = n.$$

This proves $T(m_e) = n$ for every admissible $k = k_0+2e$, or equivalently $T(R(n;k)) = n$ for all admissible $k \equiv k_0(n) \pmod 2$. The final statement follows immediately. $\square$

5. The Global Framework: Affine rails, Dyadic Slices, and Complete Coverage

This section extends the global offset framework developed in *A Deterministic Residue Framework for the Collatz Operator at* $q = 3$ [6]. The earlier work established that the Inverse map produces structured arithmetic progressions (offset ladders) whose superposition covers all admissible odd integers. Here we introduce the additional arithmetic machinery—normal–state normalization, the z–lattice skeleton, and the dyadic slicing $\mathcal{S}_{c,e}$ induced by $k = \nu_2(3m+1)$—which refines and completes that global description.

The three components now operate in a unified way:

1. the normal–state coordinate assigns each admissible odd a canonical position within the live lattice; 2. the affine inverse $R(n;k) = \frac{2^k n-1}{3}$ generates class–preserving rails under $k \mapsto k+2$ via $m \mapsto 4m+1$; and 3. the dyadic slices $\mathcal{S}_{c,e}$ partition the odd integers according to the 2–adic valuation of $3m+1$.

We show that these are not separate descriptions but exact arithmetic equivalents. Every affine rail position corresponds to a unique dyadic slice, every rail has a unique normal–state anchor in the z–skeleton, and the union over all slices yields a disjoint and complete decomposition of $\mathbb{N}_{\text{odd}}$. Thus the global structure anticipated in the previous work [6], is recovered as a special case of a more rigid algebraic framework that requires no step–count bounds and is compatible with the full local–to–global dynamics developed in Sections 3 and 4.

5.1. Offset Formulas in the Transformation

5.1.1. C_1 Offsets

From the mod 6 classification established in the prior section, every odd integer is congruent to 1, 3, or 5 modulo 6. The residue 3 gives the terminating class C_0, while the residues 1 and 5 produce the live classes C_2 and C_1. Thus every C_1 parent can be written in the form

$$n = 6t + 5, \qquad t \geq 0,$$

where t is a nonnegative integer indexing the position of n within the C_1 residue class. Equivalently, t counts how many multiples of 6 have been passed before reaching n.

By the admissibility rule, C_1 nodes allow only odd exponents k. With the minimal choice $k = 1$, the Inverse Collatz function is

$$R(n, 1) = \frac{2n - 1}{3}.$$

Substituting $n = 6t + 5$ gives

$$R(6t + 5, 1) = \frac{2(6t + 5) - 1}{3} = \frac{12t + 9}{3} = 4t + 3.$$

The offset is obtained by subtracting the parent:

$$\Delta_1(6t + 5) = R(6t + 5, 1) - (6t + 5) = (4t + 3) - (6t + 5) = -2(t + 1).$$

Hence each C_1 child lies an even step below its parent, and the step size grows linearly with the modulo 6 index t. The resulting ladder of offsets is

$$-2,\ -4,\ -6,\ -8,\ \ldots$$

Concrete examples:

$$5 \mapsto 3\ (-2), \qquad 11 \mapsto 7\ (-4), \qquad 17 \mapsto 11\ (-6).$$

Thus the C_1 offsets are the explicit arithmetic realization of the Inverse rule with odd k, derived directly from the mod 6 classification.

5.1.2. $\mathbf{C_2}$ Offsets

From the mod 6 classification, every C_2 parent can be written as $n = 6t + 1$ with $t \geq 0$. By admissibility, C_2 nodes allow only even exponents k. With the minimal choice $k = 2$,

$$R(n, 2) = \frac{4n - 1}{3}.$$

Substituting $n = 6t + 1$ gives

$$R(6t + 1, 2) = \frac{4(6t + 1) - 1}{3} = \frac{24t + 3}{3} = 8t + 1.$$

Therefore the offset (child minus parent) is

$$\Delta_2(6t + 1) = R(6t + 1, 2) - (6t + 1) = (8t + 1) - (6t + 1) = 2t.$$

Hence the first admissible Inverse step in C_2 is nondecreasing and, for $t \geq 1$, strictly increasing in t:

$$\Delta_2 = 0, 2, 4, 6, \ldots$$

Concrete examples:

$$1 \mapsto 1\ (0), \qquad 7 \mapsto 9\ (+2), \qquad 13 \mapsto 17\ (+4).$$

Lemma 19 (Offset Ladders by Class)**.** *For each live parent n, the first admissible Inverse step defines an arithmetic offset depending only on its class:*

$$C_1 : \ \Delta(6t+5) = -2(t+1), \qquad C_2 : \ \Delta(6t+1) = 2t.$$

Moreover, higher admissible lifts of the same parent extend these formulas linearly in t with parity restricted to odd k for C_1 and even k for C_2.

Proof. Direct substitution of $n = 6t+5$ with odd k and $n = 6t+1$ with even k into the Inverse Collatz function $R(n,k) = (2^k n - 1)/3$ gives the claimed offset formulas. The parity restriction follows from admissibility, so every live parent generates an infinite ladder of children determined solely by (t,k). □

Theorem 4 (Anchor principle)**.** *All progressive path iterations of the Collatz map are anchored at the two primitive parents $1 \pmod 6 \in C_2$ and $5 \pmod 6 \in C_1$. Every admissible lift $R(1;k)$ (k even) and $R(5;k)$ (k odd) generates an infinite raising sequence. These raising sequences partition the odd integers into disjoint arithmetic progressions modulo 2^k, and the union over all k gives complete coverage. Thus the global affine enumeration is entirely determined by the minimal anchor rails of the pair $1,5\}$ and their respective admissible k-values.*

Corollary 5 (Exhaustion by anchors)**.** *Every odd integer lies in exactly one position of an offset ladder on a rail of the form $4m+1$ generated from a minimal k value transformation of the Inverse odd to odd iteration. The only anchors are the origin rails of the dual live classes, corresponding to $n \in \{1,5\}$ in N_{odd}. As these origin rails are extended and their offset ladders are filled, the resulting structure enumerates all odd integers exactly once, and no other origins occur.*

5.1.3. Further lifts of admissible k

The Inverse Collatz function extends naturally to higher admissible exponents: odd $k = 1,3,5,\ldots$ for C_1 parents ($n = 6t+5$) and even $k = 2,4,6,\ldots$ for C_2 parents ($n = 6t+1$). Substituting these values into

$$R(n,k) = \frac{2^k n - 1}{3}$$

gives the general offset formulas

$$\Delta_k(6t+5) = 2(2^k - 3)t + \tfrac{5 \cdot 2^k - 16}{3}, \qquad \Delta_k(6t+1) = 2(2^k-3)t + \tfrac{2^k - 4}{3}.$$

The first admissible k gives the minimal child, and increasing k by two corresponds to a deeper lift along a higher ladder. Each successive lift remains tied to the progression index t, with the offset magnitude growing on the order of 2^k as k increases.

Remark 4 (Offsets and the itinerary)**.** The higher-k formulas confirm that offsets are determined not by the "generation depth" but by the progression index t and the parity of k. Which ladder is followed depends on the sequence of class transitions as the function is iterated. Thus C_1 and C_2 each sustain an infinite sequence of admissible steps, and the arithmetic progression of offsets is simply the explicit trace of the admissibility rules, computed relative to n at each transformation.

5.2. Arithmetic Progressions of Children

While offsets describe the displacement between a parent and its child, progressions describe how children of consecutive parents distribute across the integers. We now compute these inter-parent progressions.

5.2.1. C_1 Parents

Take consecutive C_1 parents $n = 6t + 5$ and $m = 6(t + 1) + 5 = 6t + 11$. From the Inverse rule with $k = 1$, their children are

$$m = \frac{2(6t + 5) - 1}{3} = 4t + 3, \qquad m' = \frac{2(6t + 11) - 1}{3} = 4t + 7.$$

Hence

$$m' - m = (4t + 7) - (4t + 3) = 4.$$

Thus first admissible children of consecutive C_1 parents advance in an arithmetic progression with step size $+4$.

5.2.2. C_2 Parents

Take consecutive C_2 parents $n = 6t + 1$ and $m = 6(t + 1) + 1 = 6t + 7$. From the Inverse rule with $k = 2$, their children are

$$m = \frac{4(6t + 1) - 1}{3} = 8t + 1, \qquad m' = \frac{4(6t + 7) - 1}{3} = 8t + 9.$$

Hence

$$m' - m = (8t + 9) - (8t + 1) = 8.$$

Thus first admissible children of consecutive C_2 parents advance in an arithmetic progression with step size $+8$.

Lemma 20 (Progressions of Consecutive Parents)**.** *First admissible children of consecutive parents form arithmetic progressions:*

$$C_1 : \ (6t + 5) \mapsto (4t + 3), \qquad (6t + 11) \mapsto (4t + 7), \quad \Delta = +4,$$
$$C_2 : \ (6t + 1) \mapsto (8t + 1), \qquad (6t + 7) \mapsto (8t + 9), \quad \Delta = +8.$$

Thus children of adjacent parents distribute evenly across odd integers with step size fixed by class.

Remark 5. The offset ladders of Sections 5.1.1–5.1.2 describe how each parent generates children in a ladder determined relative to its own value of n. The arithmetic progressions, by contrast, describe how numerically consecutive parents distribute their children across the integers. Both perspectives are needed: ladders explain the local offsets tied to each parent, while progressions explain the global coverage across parents.

For C_1 parents, each has the form $n = 6t + 5$. With the minimal admissible exponent $k = 1$, the child is

$$R(6t+5, 1) = \frac{2(6t+5) - 1}{3} = 4t + 3.$$

Subtracting the parent gives the offset

$$\Delta_1(6t+5) = (4t+3) - (6t+5) = -2(t+1).$$

Thus the offset depends linearly on t and grows in magnitude as t increases.

For C_2 parents, each has the form $n = 6t + 1$. With the minimal admissible exponent $k = 2$, the child is

$$R(6t+1, 2) = \frac{4(6t+1) - 1}{3} = 8t + 1,$$

so the offset is

$$\Delta_2(6t+1) = (8t+1) - (6t+1) = 2t.$$

This offset also depends on t, and for $t \geq 1$ it is strictly increasing.

Therefore, offsets are not fixed increments across all parents, but arithmetic expressions relative to each parent's index t within its residue class. Each live class generates an infinite rail of children, and the offset size expands with t while preserving the admissibility rule (odd k for C_1, even k for C_2).

The arithmetic progressions across consecutive parents are simply the global counterpart of the same rule. When t increases by $+1$ (advancing to the next parent in the same class), the child also advances by a constant step ($+4$ for C_1 at $k = 1$, $+8$ for C_2 at $k = 2$, and in general $+2^{k+1}$). This step is independent of t because the dependence on t is linear.

Thus the two descriptions are isomorphic: offsets show how children are positioned relative to a fixed parent, while progressions show how those positions line up across the sequence of parents. Both arise from the same affine relation $R(6t + \rho, k) = 2^{k+1}t + c_{\rho,k}$, and together they capture the local and global arithmetic structure of the Inverse Collatz map.

5.2.3. Higher Lifts

Lemma 21 (Quadrupling of Step Sizes at Higher Lifts)**.** *For each class, increasing the admissible exponent k by two applies two successive doublings, thereby quadrupling* *the progression step size of consecutive parents. Concretely:*

$$C_1: \ +4 \mapsto +16 \mapsto +64 \mapsto \cdots, \qquad C_2: \ +8 \mapsto +32 \mapsto +128 \mapsto \cdots.$$

Proof. From the general offset formulas in Section 5.1.3, the difference between children of consecutive parents is proportional to 2^k. Replacing k by $k+2$ multiplies this factor by 4, hence quadruples the step size between odd children. Therefore each successive two-lift scales the step size by a factor of four. $\square$

At higher admissible k-lifts, step sizes scale as 2^k: each unit increase of k doubles the progression spacing, and in particular every two lifts quadruple it. A convenient way to display this is to show the two-lift subsequences and stagger the one-lift intermediates:

$$\begin{array}{lllllllll} C_1: & +4 & \to & +16 & \to & +64 & \to & \cdots & \\ C_2: & & +8 & \to & +32 & \to & +128 & \to & \cdots \end{array}$$

This pattern follows directly from the formulas of Section 5.1.3.

5.2.4. Visual Overlay

Corollary 6 (Visual Overlay and Complete Coverage)**.** *Overlaying the progression ladders from consecutive parents shows that apparent gaps at lower admissible lifts are exactly filled by higher lifts. Each anchor sequence covers its congruence class without overlap, and the union across all admissible lifts exhausts the odd integers. Thus rail iterations across all lift levels ensure complete coverage of* $\mathbb{N}_{\text{odd}}$.

Proof. By Lemma 20, consecutive parents generate fixed-step progressions, and by Lemma
21, higher admissible lifts scale these progressions by powers of four. The apparent omissions at a given scale correspond precisely to residue classes that are elements of progression of higher-lift ladders. Therefore the superposition of ladders fills all gaps systematically, partitioning the odd integers with no overlap. $\square$

5.3. Anchor Ladders as the Basis of Coverage

All admissible structure originates from the two primitive anchors $1 \in C_2$ and $5 \in C_1$. Each admissible lift

$$R(1;k) = \frac{2^k - 1}{3}, \qquad k \text{ even},$$

$$R(5;k) = \frac{2^k \cdot 5 - 1}{3}, \qquad k \text{ odd},$$

produces a new anchor point. Each such anchor initiates a ladder whose offsets and progressions are determined by its residue class and the parity of the admissible exponent k.

Interpretation. [Dyadic gaps as lifted offsets] Each admissible exponent k produces a dyadic slice

$$\mathcal{S}_{c,e} = \left\{ 2^{k+1}t + \frac{2^k x - 1}{3} : t \geq 0 \right\}, \qquad k = c + 2e,$$

where $(c, x) \in \{(1,5),(2,1)\}$ specifies the class. The quantity

$$\Delta(k) := 2^{k+1}$$

is the gap between successive values in the slice and is the *exact offset* created by the lifted exponent k.

Thus increasing k does not produce a new type of parent; it produces a new *spacing* among the same admissible residue class. The anchor value determines the base point

$$\alpha(k) := \frac{2^k x - 1}{3},$$

while the dyadic step 2^{k+1} determines how far apart the lift-k parents of successive values lie.

In this sense, *each higher lift corresponds to a wider offset lattice.* Different values of k carve the odd integers into disjoint arithmetic progressions of increasing gap, and every such progression is exactly one dyadic slice. No slice overlaps another, and no odd integer is omitted.

Lemma 22 (Arithmetic derivation of anchors by class lifts)**.** *For each anchor family $a \in \{1, 5\}$ with parent form $n = 6t + a$, the Inverse operator*

$$R(n; k) \;=\; \frac{2^k(6t + a) - 1}{3}$$

generates an arithmetic progression at every admissible lift k (k odd for $a = 5$, k even for $a = 1$). The constant term $\frac{2^k a - 1}{3}$ is the base residue of that progression and coincides with the anchor promoted at scale 2^k. Thus the starting anchors are derived arithmetically, and their descendants at higher k are exactly the ladder bases that fill sieve holes.

Proof. For $a = 5$ (class C_1, odd k):

$$\begin{aligned} R(6t+5;1) &= \tfrac{2(6t+5)-1}{3} = 4t + 3, \\ R(6t+5;3) &= \tfrac{8(6t+5)-1}{3} = 16t + 13, \\ R(6t+5;5) &= \tfrac{32(6t+5)-1}{3} = 64t + 53. \end{aligned}$$

Each case has the form $2^{k+1}t + \frac{2^k \cdot 5 - 1}{3}$, with constants $3, 13, 53, \ldots$ serving as the promoted anchors at scales $2^1, 2^3, 2^5, \ldots$.

For $a = 1$ (class C_2, even k):

$$\begin{aligned} R(6t+1;2) &= \tfrac{4(6t+1)-1}{3} = 8t+1, \\ R(6t+1;4) &= \tfrac{16(6t+1)-1}{3} = 32t+5, \\ R(6t+1;6) &= \tfrac{64(6t+1)-1}{3} = 128t+21. \end{aligned}$$

Each case has the form $2^{k+1}t + \frac{2^k \cdot 1 - 1}{3}$, with constants $1, 5, 21, \ldots$ serving as the promoted anchors at scales $2^2, 2^4, 2^6, \ldots$.

In both families, the step size doubles with each increment of k, and the base constant aligns exactly with the residue class left uncovered at the prior dyadic sieve. Thus the arithmetic shows both that the anchors $\{1, 5\}$ are generated within the operator and that each higher k-level produces the ladder bases that fill the recursive sieve. $\square$

5.4. Global Coverage by a Dyadic Sieve of Ladders

Proposition 4 (Base-child ladders and the 4-adic sieve by class)**.** *Every admissible odd parent n is in exactly one of the two live classes*

$$C_1: \; n = 6t+5 \quad \textit{or} \quad C_2: \; n = 6t+1 \qquad (t \in \mathbb{N}).$$

Let $m = \dfrac{2^k n - 1}{3}$ *be a Inverse child at lift k. Then:*

(A) ***First admissible child (base sieve slice).***

$$C_1 \;\; \textbf{\textit{(first lift}}\; k=1\textbf{\textit{):}} \quad n = 6t+5 \implies m = \frac{2(6t+5)-1}{3} = 4t+3,$$

$$C_2 \;\; \textbf{\textit{(first lift}}\; k=2\textbf{\textit{):}} \quad n = 6t+1 \implies m = \frac{4(6t+1)-1}{3} = 8t+1.$$

Thus the base children in C_1 are exactly $m \equiv 3 \pmod 4$ (gap 4), and the base children in C_2 are exactly $m \equiv 1 \pmod 8$ (gap 8). Equivalently, these are the odds with exactly *one halving ($k = 1$) and* exactly *two halvings ($k = 2$) in $3m + 1$, respectively.*

(B) ***Higher admissible lifts stay in class and obey $m \mapsto 4m+1$.*** *Within a fixed class, raising the lift by $+2$ sends each child to the next child by*

$$m' = \frac{2^{k+2}n - 1}{3} = 4\left(\frac{2^k n - 1}{3}\right) + 1 = 4m + 1.$$

Hence the children at lifts $k, k+2, k+4, \ldots$ form a rail *by the affine update $m \mapsto 4m+1$ and remain in the same class (C_1 for odd k, C_2 for even k).*

(C) ***Gap quadrupling across lifts.*** *Writing the base-child progressions as functions of* t,

$$\begin{array}{lll} C_1,\ k=1: & m_0(t)=4t+3 & (\textit{gap } 4), \\ C_2,\ k=2: & m_0(t)=8t+1 & (\textit{gap } 8), \end{array}$$

the lift update $m \mapsto 4m+1$ *gives, for each* $\ell \geq 0$,

$$C_1 \textit{ at } k=1+2\ell: \quad m_\ell(t)=4^{\ell+1}\,t+\frac{10\cdot 4^{\ell}-1}{3}, \quad \textit{gap } =4^{\ell+1},$$

$$C_2 \textit{ at } k=2+2\ell: \quad m_\ell(t)=8\cdot 4^{\ell}\,t+\frac{4^{\ell+1}-1}{3}, \quad \textit{gap } =8\cdot 4^{\ell}.$$

Thus each time the lift increases by $+2$, the gap between consecutive children (as t increases by 1) is multiplied by 4.

(D) ***Next sieve slice is generated by*** $4m+1$***.*** *For* C_1 *the base children (*$k=1$*) are* $m \equiv 3 \pmod 4$*. Applying* $m \mapsto 4m+1$ *yields the next slice (*$k=3$*):* $m \equiv 13 \pmod{16}$*, again* $m \mapsto 4m+1$ *gives the* $k=5$ *slice* $m \equiv 53 \pmod{64}$*, and so on. For* C_2*, the base children (*$k=2$*) are* $m \equiv 1 \pmod 8$*; then* $k=4$ *gives* $m \equiv 5 \pmod{32}$*; then* $k=6$ *gives* $m \equiv 21 \pmod{128}$*; etc. In each class,* $m \mapsto 4m+1$ generates the next sieve level *and* quadruples *the modulus (the gap) each time.*

Lemma 23 (Sieve slice measure for $v_2(3m+1)$ on odds)**.** *Fix* $k \geq 1$*. Among all odd integers* m*, the proportion for which* $\nu_2(3m+1)=k$ *is exactly* 2^{-k}*.*

Proof. Work modulo 2^{k+1}. Because 3 is invertible mod 2^{k+1}, the map $m \mapsto 3m+1$ is a bijection on residue classes. The condition $\nu_2(3m+1) \geq k$ is $3m+1 \equiv 0 \pmod{2^k}$, which holds for exactly 2^{-k} of odd residues; the stricter condition $\nu_2(3m+1) \geq k+1$ cuts that by another factor $1/2$. Hence $\mathbb{P}(\nu_2(3m+1)=k)=2^{-k}$ on odds. □

Corollary 7 (All-integers normalization)**.** *For* $k \geq 1$*, the proportion of* all *integers* m *with* m *odd and* $\nu_2(3m+1)=k$ *is* $2^{-(k+1)}$*.*

Proof. Half of all integers are odd; combine with Lemma 23. □

Transition: Canonical Reduction of Admissible Structure

The analysis above resolves the local admissible structure of the Inverse map: each live residue admits a unique minimal exponent k_0, produces a base child in its own class, and extends to a full rail via the affine law $R(n;k+2)=4R(n;k)+1$ shown by Lemma 16. These statements describe the local geometry of the Inverse tree but leave open the problem of identifying a canonical global parameter governing all rails simultaneously.

Such a parameter arises naturally by removing the dyadic component of the first admissible step. The resulting *normal–state* provides a global coordinate system on the live set $\mathcal{O}_{\mathrm{live}}$ in which each rail becomes a pure affine progression, independent of its parent. This reduction clarifies both the disjointness and completeness of the rail family and supplies the arithmetic infrastructure needed for the global coverage theorem below.

We introduce this normal–state framework next.

5.5. Normal–State Enumeration and the Pure Affine Skeleton

The affine decomposition shows that each admissible Inverse step

$$R(n;k) = \frac{2^k n - 1}{3}$$

splits into a minimal admissible core and a sequence of $4m+1$ lifts. In this section we remove all reversible dyadic structure and isolate the intrinsic arithmetic skeleton of the map. The resulting *normal–state* forms a canonical index on the live odds and reveals that Collatz dynamics reduce to a pure affine counting system generated entirely by a base of:

$$z \mapsto 2k_0 z + 1.$$

No explicit use of the Collatz Forward function is required once this normal–state system is established.

Definition 18 (Normal–State lattice)**.** For any odd integer n, let $k_0(n)$ denote its minimal admissible exponent in the Inverse map. The *normal–state lattice* of n is defined by

$$Z(n) \;=\; \frac{R(n;k_0(n)) - 1}{2^{\,k_0(n)}} \;=\; \frac{\dfrac{2^{k_0(n)} n - 1}{3} - 1}{2^{\,k_0(n)}}.$$

This value $Z(n)$ is the unique base element of the affine rail generated by n.

5.5.1. Minimal Admissible Exponents

Let

$$\mathcal{O}_{\mathrm{live}} := \{n \in \mathbb{N}_{>0} : n \equiv 1, 5 \pmod 6\}$$

denote the live odd integers. For each $n \in \mathcal{O}_{\mathrm{live}}$ the Inverse step $R(n;k)$ is integral precisely when

$$2^k n \equiv 1 \pmod 3.$$

Since $2 \equiv -1 \pmod 3$ and n is never $0 \bmod 3$ in the live set, admissibility is determined by the parity of k:

$$k_0(n) = \begin{cases} 1, & n \equiv 5 \pmod 6 \in C_1, \\ 2, & n \equiv 1 \pmod 6 \in C_2. \end{cases}$$

The *base child* of n is

$$R\big(n;k_0(n)\big) = \frac{2^{\,k_0(n)}n-1}{3}.$$

5.5.2. Normal–State Extraction

The *normal–state* of $n \in \mathcal{O}_{\text{live}}$ is defined by removing exactly the admissible dyadic factor used to produce $R(n;k_0)$:

$$\boxed{Z(n) := \frac{R(n;k_0)-1}{2^{k_0(n)}}}.$$

Because admissibility guarantees $R(n;k_0) \equiv 1 \bmod 2^{k_0(n)}$, this quantity is an integer for every live odd n.

Ordered by size,

$$1, 5, 7, 11, 13, 17, 19, 23, \dots,$$

the normal–state reproduces the natural lattice:

$$z(1) = 0, \quad z(5) = 1, \quad z(7) = 2, \quad z(11) = 3, \dots.$$

Examples. **(1) C1 case.** For $n = 5$,

$$k_0(5) = 1, \quad c_1(5) = \frac{2 \cdot 5 - 1}{3} = 3, \quad z(5) = \frac{3-1}{2} = 1.$$

(2) C2 case. For $n = 7$,

$$k_0(7) = 2, \quad c_1(7) = \frac{4 \cdot 7 - 1}{3} = 9, \quad z(7) = \frac{9-1}{4} = 2.$$

5.5.3. Compatibility of normal-state and rail children

Lemma 24 (Parent $\to z \to$ child equals direct child map)**.** *For every odd $n \in \mathcal{O}_{\text{live}}$, the base inverse child*

$$m_0(n) := R(n;k_0(n)) = \frac{2^{k_0(n)}n-1}{3}$$

is determined uniquely by the normal–state coordinate $z(n)$ via

$$m_0(n) = \begin{cases} 4\,z(n)+1, & n \in C_2, \\[2ex] 2\,z(n)+1, & n \in C_1. \end{cases}$$

Consequently, the composition $n \mapsto z(n) \mapsto m_0(n)$ coincides with the direct inverse map $n \mapsto R(n;k_0(n))$.

Proof. By definition,

$$\mathcal{O}_{\text{live}} = \{n \equiv 1, 5 \pmod 6\} = C_2 \sqcup C_1,$$

where $C_2 = \{6t + 1 : t \geq 0\}$ and $C_1 = \{6t + 5 : t \geq 0\}$. The normal–state coordinate $z(n)$ enumerates these two progressions by assigning even indices to C_2 and odd indices to C_1. Thus

$$n = \begin{cases} 6t + 1, & z(n) = 2t,\ n \in C_2, \\ 6t + 5, & z(n) = 2t + 1,\ n \in C_1. \end{cases}$$

Equivalently,

$$n = \begin{cases} 3z(n) + 1, & n \in C_2, \\ 3z(n) + 2, & n \in C_1. \end{cases}$$

If $n \in C_2$, then $k_0(n) = 2$ and

$$m_0(n) = \frac{4n - 1}{3} = \frac{4(3z(n) + 1) - 1}{3} = 4z(n) + 1.$$

If $n \in C_1$, then $k_0(n) = 1$ and

$$m_0(n) = \frac{2n - 1}{3} = \frac{2(3z(n) + 2) - 1}{3} = 2z(n) + 1.$$

In both cases the resulting expression for $m_0(n)$ agrees exactly with the direct inverse formula $R(n; k_0(n))$. $\square$

5.5.4. Normal–State Law and the First Affine Step

Theorem 5 (Affine rail law and disjoint partition of the odd integers)**.** *Let $n \in \mathcal{O}_{\text{live}}$ and let $k_0(n)$ denote its minimal admissible exponent. Define the base child*

$$m_0(n) := R\big(n; k_0(n)\big) = \frac{2^{k_0(n)}n - 1}{3}.$$

For each $e \geq 0$ define the e–th admissible lift above n by

$$m_e(n) := R\big(n; k_0(n) + 2e\big).$$

Then:

1. ***Affine recursion and closed form.*** *The sequence $\{m_e(n)\}_{e \geq 0}$ satisfies*

$$m_{e+1}(n) = 4m_e(n) + 1,$$

and hence admits the closed form

$$m_e(n) = 4^e\, m_0(n) + \frac{4^e - 1}{3}.$$

2. ***Dependence only on the normal–state coordinate.*** *Writing the base child in normal–state form*

$$m_0(n) = \begin{cases} 4z(n)+1, & n \in C_2, \\ 2z(n)+1, & n \in C_1, \end{cases}$$

we have

$$m_e(n) = 4^e(f_c(n)\,z(n)+1) + \frac{4^e-1}{3}, \qquad f_c(n) = \begin{cases} 4, & n \in C_2, \\ 2, & n \in C_1, \end{cases}$$

so the entire admissible chain above n is determined by the pair $z(n)$ and $f_c(n)$ i.e. by the normal–state coordinate and class.

3. ***Disjointness of rails.*** *If $n_1, n_2 \in \mathcal{O}_{\text{live}}$ with $n_1 \neq n_2$, then the corresponding affine rails*

$$\mathcal{L}_{n_i} := \{m_e(n_i) : e \geq 0\} \qquad (i = 1, 2)$$

are disjoint:

$$\mathcal{L}_{n_1} \cap \mathcal{L}_{n_2} = \varnothing.$$

4. ***Partition of the odd integers.*** *Every odd integer m belongs to exactly one such rail. Equivalently,*

$$\mathbb{N}_{\text{odd}} = \bigcup_{n \in \mathcal{O}_{\text{live}}} \mathcal{L}_n, \qquad \mathcal{L}_{n_1} \cap \mathcal{L}_{n_2} = \varnothing \quad (n_1 \neq n_2).$$

Proof. (1) Since $k_0(n) + 2(e+1) = k_0(n) + 2e + 2$, we have

$$m_{e+1}(n) = R\big(n; k_0(n) + 2e + 2\big) = \frac{2^{k_0(n)+2e+2}n - 1}{3}.$$

Also,

$$4m_e(n)+1 = 4\cdot\frac{2^{k_0(n)+2e}n-1}{3}+1 = \frac{2^{k_0(n)+2e+2}n-4}{3}+1 = \frac{2^{k_0(n)+2e+2}n-1}{3} = m_{e+1}(n),$$

so $m_{e+1}(n) = 4m_e(n) + 1$. Solving the recursion yields

$$m_e(n) = 4^e c_0(n) + \sum_{j=0}^{e-1} 4^j = 4^e m_0(n) + \frac{4^e-1}{3}.$$

(2) This is immediate from the classwise normal–state formulas for $m_0(n)$ proved in Lemma 24, substituting into the closed form in (1).

(3) Suppose $m_{e_1}(n_1) = m_{e_2}(n_2)$ for some $e_1, e_2 \geq 0$. Then, by definition,

$$\frac{2^{k_0(n_1)+2e_1}n_1 - 1}{3} = \frac{2^{k_0(n_2)+2e_2}n_2 - 1}{3},$$

hence

$$2^{k_0(n_1)+2e_1}n_1 = 2^{k_0(n_2)+2e_2}n_2.$$

Both sides are equal powers of 2 times odd integers; by uniqueness of the 2–adic valuation, the exponents agree and the odd parts agree, giving $n_1 = n_2$. Thus distinct n yield disjoint rails.

(4) Let m be odd. Write $3m+1 = 2^k n$ where n is odd and $k = \nu_2(3m+1)$. Then $n \not\equiv 0 \pmod 3$, hence $n \in \mathcal{O}_{\text{live}}$, and

$$m = R(n;k).$$

Writing $k = k_0(n) + 2e$ for some $e \geq 0$ places m on $\mathcal{L}_n$. Uniqueness follows from (3). $\square$

Proposition 5 (Reduction to normal–state and affine generation of all admissible lifts)**.** *Let $n \in \mathcal{O}_{\text{live}}$, and let $k_0(n)$ denote its minimal admissible exponent. Define the base child $m_0(n) := R(n;k_0(n))$ and the admissible lifts*

$$m_e(n) := R\big(n;k_0(n)+2e\big) \qquad (e \geq 0).$$

Then the entire admissible inverse structure above n is generated in the z–coordinate by the maps f_1, f_2 and L_e as follows:

1. ***Classwise reduction** $n \mapsto z(n)$**.***

$$z(n) = \begin{cases} \dfrac{n-1}{3}, & n \in C_2 \ (n \equiv 1 \pmod 6), \\ \dfrac{n-2}{3}, & n \in C_1 \ (n \equiv 5 \pmod 6). \end{cases}$$

2. ***Base child from normal–state:** $z(n) \mapsto m_0(n)$**.** With $f_1(z) = 2z+1$ and $f_2(z) = 4z+1$,*

$$m_0(n) = \begin{cases} f_2\big(z(n)\big), & n \in C_2, \\ f_1\big(z(n)\big), & n \in C_1. \end{cases}$$

3. ***All admissible lifts from the rail operator:** $m_0(n) \mapsto m_e(n)$**.** For each $e \geq 0$,*

$$m_e(n) = 4^e\, m_0(n) + \frac{4^e - 1}{3}.$$

Equivalently, in the z–coordinate,

$$z\big(m_e(n)\big) = L_e\big(f_c(z(n))\big), \qquad c = \begin{cases} 2, & n \in C_2, \\ 1, & n \in C_1, \end{cases}$$

where $L_e(z) = 4^e z + \frac{2}{3}(4^e - 1)$.

Proof. (1) Since $\mathcal{O}_{\text{live}} = \{6t+1 : t \geq 0\} \sqcup \{6t+5 : t \geq 0\}$ and z enumerates $6t+1$ at even sites and $6t+5$ at odd sites, we have $z = 2t$ on C_2 and $z = 2t+1$ on C_1. Solving $n = 6t+1$ and $n = 6t+5$ for t yields the stated formulas for $z(n)$.

(2) This is Lemma 24 written in generator form: $m_0(n) = 4z(n)+1 = f_2(z(n))$ on C_2 and $m_0(n) = 2z(n)+1 = f_1(z(n))$ on C_1.

(3) By definition,

$$m_e(n) = R\big(n; k_0(n) + 2e\big) = \frac{2^{k_0(n)+2e}n - 1}{3}.$$

Comparing with $m_0(n) = \frac{2^{k_0(n)}n-1}{3}$ gives the standard lift identity $m_{e+1}(n) = 4m_e(n)+1$, hence the closed form

$$m_e(n) = 4^e\, m_0(n) + \frac{4^e - 1}{3}.$$

Substituting $m_0(n) = f_c(z(n))$ and rewriting in normal–state coordinates yields $z(m_e(n)) = L_e(f_c(z(n)))$ with $L_e(z) = 4^e z + \frac{2}{3}(4^e - 1)$. $\square$

5.5.5. Affine rails and Odd Coverage

For each $n \in \mathcal{O}_{\text{live}}$, the *affine rail* indexed by $z(n)$ is

$$\mathcal{L}_n = \left\{ 4^e R(n; k_0) + \frac{4^e - 1}{3} : e \geq 0 \right\}.$$

Injectivity of the affine form

$$R(n; k) = \frac{2^k}{3}n - \frac{1}{3}$$

ensures that rails are disjoint. Every odd integer m has a unique representation $m = R(n; k)$ for some live n and unique admissible k, and writing $k = k_0(n) + 2e$ places m on exactly one rail:

$$\bigcup_{n \in \mathcal{O}_{\text{live}}} \mathcal{L}_n = \mathbb{N}_{\text{odd}}.$$

Combining this with the dyadic decomposition $\mathcal{S}_{c,e}$ yields full coverage of $\mathbb{N}_{\geq 1}$.

z	n	class	operator	z-child	base child $R(n; k_0)$
0	1	C2	$4z + 1$	1	1
1	5	C1	$2z + 1$	3	3
2	7	C2	$4z + 1$	9	9
3	11	C1	$2z + 1$	7	7
4	13	C2	$4z + 1$	17	17
5	17	C1	$2z + 1$	11	11
6	19	C2	$4z + 1$	25	25
7	23	C1	$2z + 1$	15	15
8	25	C2	$4z + 1$	33	33
9	29	C1	$2z + 1$	19	19
10	31	C2	$4z + 1$	41	41
11	35	C1	$2z + 1$	23	23
12	37	C2	$4z + 1$	49	49
13	41	C1	$2z + 1$	27	27
14	43	C2	$4z + 1$	57	57
15	47	C1	$2z + 1$	31	31
16	49	C2	$4z + 1$	65	65
17	53	C1	$2z + 1$	35	35
18	55	C2	$4z + 1$	73	73
19	59	C1	$2z + 1$	39	39
20	61	C2	$4z + 1$	81	81
21	65	C1	$2z + 1$	43	43
22	67	C2	$4z + 1$	89	89
23	71	C1	$2z + 1$	47	47
24	73	C2	$4z + 1$	97	97

Table 5.5.5 First 25 *live odd integers* ($n \equiv 1, 5 \pmod 6$)*) with their z–coordinates, classes, affine generator, and base admissible child. The table illustrates the fundamental identity*

$$\boxed{R(n; k_0) \; = \; n_{\, f_{1,2}(z(n))}}$$

i.e. the first admissible Inverse child of n is exactly the live odd whose index equals the affine z–map $f_1(z) = 2z + 1$ (for C1) or $f_2(z) = 4z + 1$ (for C2). Equivalently, the first admissible inverse child may be written in the unified form

$$m_0 = 2^{k_0(n)}\, z(n) + 1,$$

where $k_0(n) = 1$ for C_1 and $k_0(n) = 2$ for C_2.

Remark 6 (Why **C0** is terminal in the z–Lattice)**.** The Z–lattice is a bijection from the live lattice

$$\mathcal{L} = \{\, n \in \mathbb{N}_{\text{odd}} : n \equiv 1, 5 \pmod 6 \,\}$$

onto $\mathbb{N}_0$, assigning each admissible odd m its global normal–state coordinate $Z(m)$. No element of $C0 = \{n \equiv 3 \pmod 6\}$ appears in $\mathcal{L}$, and therefore *no $C0$ value admits a z–coordinate.* This is not merely a definitional omission: it is an arithmetic obstruction.

Indeed, if $n \equiv 3 \pmod 6$, then

$$2^k n - 1 \equiv -1 \pmod 3,$$

so $(2^k n - 1)/3$ is never an integer for any $k \geq 0$. Thus no $C0$ value can serve as a parent in the admissible Inverse map $R(n;k) = \frac{2^k n-1}{3}$. Consequently, $C0$ values are *exactly* those odd integers that lie outside the normal–state coordinate system and therefore admit no further Inverse continuation. In this sense, the normal–state lattice is the structural backbone of the global Inverse tree, and $C0$ represents its natural boundary.

Proposition 6 (The Unique Self-Stable Odd Origin)**.** *Among all odd integers, the value* 1 *is the only odd integer whose admissible Inverse image under $R(n;k)$ has the same normal–state lattice as its parent. Equivalently,* 1 *is the unique solution of*

$$z(R(1;k_0(1))) = z(1),$$

and every admissible Inverse step applied to any $n > 1$ produces a strict increase (C_2) or decrease (C_1) in normal–state coordinate.

Proof. For an odd integer n, write its admissible base child as

$$R(n;k_0(n)) = \frac{2^{k_0(n)}n - 1}{3}.$$

Normal–State normalization removes the affine increment $k_0(n)$ from the lifted representation; hence

$$z\big(R(n;k_0(n))\big) = \frac{2^{k_0(n)}n - 1}{3 \cdot 2^{k_0(n)}} = \frac{n}{2^{k_0(n)}} - \frac{1}{3 \cdot 2^{k_0(n)}}.$$

If $n = 1$, then $k_0(1) = 2$ and

$$R(1;2) = 1, \qquad z(1) = 0,$$

so 1 is fixed under its admissible Inverse step and its normal–state coordinate remains 0.

Suppose $n > 1$. If $z(R(n;k_0(n))) = z(n)$, then by the definition of z we would have

$$R(n;k_0(n)) = n,$$

hence

$$\frac{2^{k_0(n)}n - 1}{3} = n \quad \Longrightarrow \quad (2^{k_0(n)} - 3)n = 1,$$

which is impossible for any $n > 1$. Thus the equality $z(R(n; k_0(n))) = z(n)$ is impossible for odd $n > 1$. □

Corollary 8 (Uniqueness of the Global Odd Cycle). *The odd Inverse Collatz dynamics admit exactly one cycle, the trivial cycle* $\{1\}$*. Every other odd integer ascends strictly from normal–state coordinate and therefore cannot return to a previous affine or normal–state position.*

5.5.6. Affine–Dyadic Equivalence

Lemma 25 (Affine–Dyadic Correspondence). *For each admissible inverse exponent k, the first admissible inverse child of an odd integer n is*

$$m_0(n) = R(n; k) = \frac{2^k n - 1}{3}.$$

Moreover, the set of odd integers m satisfying $k_m = k$ has natural density 2^{-k}; that is,

$$\lim_{N\to\infty} \frac{1}{N} \#\{\, m \le N : k_m = k \,\} = 2^{-k}.$$

Consequently, each affine inverse generator corresponds to a dyadic slice whose relative size is exactly the reciprocal of its scale.

Theorem 6 (Affine–Dyadic Equivalence). *By Lemma 25, The admissible inverse generators $R(\cdot; k)$ partition the odd integers into disjoint dyadic slices of natural density 2^{-k}.*

Corollary 9 (Coverage via Affine Slicing).

$$\sum_{k\ge1} 2^{-k} = 1.$$

Hence the admissible affine inverse generators form a disjoint partition of $\mathbb{N}_{\text{odd}}$ and cover each odd integer exactly once.

Theorem 7 (Global Arithmetic Coverage by Rails). *Let $R(n; k) = \frac{2^k n-1}{3}$ be the Inverse map with admissible parity per class. Then the following hold within Section 5:*

1. ***Base slices and fixed gaps.*** *First admissible children are exactly*

   $$C_1 : \ m \equiv 3 \pmod 4 \quad (k = 1,\ \textit{gap } 4), \qquad C_2 : \ m \equiv 1 \pmod 8 \quad (k = 2,\ \textit{gap } 8),$$

   *and children of consecutive parents form arithmetic progressions with those gaps (*Prop. 4, Lem. 20*).*

2. ***4-adic lift within rails.*** *Raising the lift by $+2$ sends $m \mapsto 4m + 1$, stays in the same class, and multiplies the progression gap of sequential, class equivocal n, by 4 (*Lem. 21*, 16).*

3. ***Overlay gives complete coverage.*** *Superposing the ladders across all admissible lifts fills the apparent gaps of the base slices; within each class, the union over k exhausts its congruence classes with no overlap (*Cor. 6*).*

4. ***Anchor generation of disjoint rails.*** *All affine rails arise from the two primitive anchor parents $1 \in C_2$ and $5 \in C_1$ via admissible inverse lifts. By the normal–state parametrization, each live odd $n \equiv 1, 5 \pmod 6$ determines a unique normal–state coordinate $z(n)$ and hence a unique affine rail*

$$\mathcal{L}_n = \left\{ 4^e R(n; k_0(n)) + \frac{4^e - 1}{3} : e \geq 0 \right\}.$$

*The generators $f_1(z) = 2z + 1$ and $f_2(z) = 4z + 1$ propagate the anchor states $z(1) = 0$ and $z(5) = 1$ through the z–lattice, producing all rails without overlap. Disjointness follows from the uniqueness of the normal–state coordinate and the injectivity of the affine form, not from pairwise offset considerations. (*Thm. 5*)*

5. ***Exact dyadic slice measures.*** *Among odd m, the slice with $\nu_2(3m + 1) = k$, $\mathcal{S}_{c,e}$ measures 2^{-k}; among all integers it is $2^{-(k+1)}$ (*Lem. 23, Cor. 7, 9*).*

Consequently, the odd integers are covered disjointly by the class-preserving affine offset gap progressions generated from a base of all rails, across all admissible lifts, with gaps and densities exactly as stated in (1)–(5).

5.6. Dyadic Sieve Index (Class–Forced Admissibility)

While Theorem 7 establishes global coverage abstractly, the remaining analysis makes this coverage explicit by describing the concrete arithmetic realization of each affine slice, rail, and lift.

Definition 19 (Dyadic Sieve Index)**.** Let $c \in \{1, 2\}$ encode the class modulo 3 and $x \in \{5, 1\}$ encode the class modulo 6:

$$c = 1,\ x = 5 \quad (\text{class } C_1); \qquad c = 2,\ x = 1 \quad (\text{class } C_2).$$

For each lift index $e \geq 0$, the admissible exponent is $k = c + 2e$ (odd k for C_1, even k for C_2), and a single Inverse step from $n = 6t + x$ produces

$$m \ = \ R(6t + x;\, k) \ = \ \frac{2^{\,c+2e}(6t + x) - 1}{3} \ = \ \underbrace{2^{\,k+1}}_{\text{gap}} t \ + \ \underbrace{\frac{2^{\,k} x - 1}{3}}_{\text{anchor}}.$$

The dyadic slice weight (among odd m) for fixed k is 2^{-k}.

k	Class	x	Gap $= 2^k$	Anchor $= \dfrac{2^k x - 1}{3}$	$m = \text{gap} \cdot t + \text{anchor}$
1	C_1	5	4	3	$4t + 3$
2	C_2	1	8	1	$8t + 1$
3	C_1	5	16	13	$16t + 13$
4	C_2	1	32	5	$32t + 5$
5	C_1	5	64	53	$64t + 53$
6	C_2	1	128	21	$128t + 21$
7	C_1	5	256	213	$256t + 213$
8	C_2	1	512	85	$512t + 85$
9	C_1	5	1024	853	$1024t + 853$
10	C_2	1	2048	341	$2048t + 341$
11	C_1	5	4096	3413	$4096t + 3413$
12	C_2	1	8192	1365	$8192t + 1365$
13	C_1	5	16384	13653	$16384t + 13653$
14	C_2	1	32768	5461	$32768t + 5461$
15	C_1	5	65536	54613	$65536t + 54613$
16	C_2	1	131072	21845	$131072t + 21845$
17	C_1	5	262144	218453	$262144t + 218453$
18	C_2	1	524288	87381	$524288t + 87381$
19	C_1	5	1048576	873813	$1048576t + 873813$
20	C_2	1	2097152	349525	$2097152t + 349525$
21	C_1	5	4194304	3495253	$4194304t + 3495253$
22	C_2	1	8388608	1398101	$8388608t + 1398101$
23	C_1	5	16777216	13981013	$16777216t + 13981013$
24	C_2	1	33554432	5592405	$33554432t + 5592405$
25	C_1	5	67108864	55924053	$67108864t + 55924053$
Dyadic slice weight for fixed k: 2^{-k} (among odd m).					

Table 3: Dyadic Sieve Index from the unified Inverse step $m = \big(2^{\,c+2e}(6t+x)-1\big)/3$, with $k = c + 2e$.

Theorem 8 (Dyadic Sieve Decomposition)**.** *Let* $C_1 = \{n \equiv 5 \,(\mathrm{mod}\ 6)\}$ *and* $C_2 = \{n \equiv 1 \,(\mathrm{mod}\ 6)\}$. *Encode the class by*

$$(c, x) = (1, 5) \text{ for } C_1, \qquad (c, x) = (2, 1) \text{ for } C_2.$$

For each lift index $e \geq 0$*, define* $k := c + 2e$ *(so* k *has the admissible parity for the class). The fixed-*k *static sieve slice is*

$$\mathcal{S}_{c,e} := \Big\{ m = \frac{2^{\,c+2e}(6t + x) - 1}{3} \ : \ t \in \mathbb{N}_{\geq 0} \Big\} = \Big\{ 2^{\,k+1} t + \frac{2^{\,k} x - 1}{3} \ : \ t \geq 0 \Big\}.$$

Then

$$\mathbb{N}_{\text{odd}} = \bigsqcup_{c\in\{1,2\}} \bigsqcup_{e\geq 0} \mathcal{S}_{c,e},$$

i.e. as $e = 0, 1, 2, \ldots$ *increases (equivalently* $k = c+2e$*), the union of these arithmetic progressions* covers every odd integer exactly once.

Proof. Existence. Take any odd m. Let k be the highest power of 2 dividing $3m+1$, i.e. $2^k \parallel (3m+1)$. Then

$$\frac{3m+1}{2^k}$$

is even and has a unique residue $x \in \{1, 5\}$ modulo 6 (parity forces x odd, and $x \equiv m \pmod 3$). Set $c = 2$ if $x = 1$ and $c = 1$ if $x = 5$. Since $k \equiv c \pmod 2$, there is $e \geq 0$ with $k = c + 2e$. Define

$$t := \frac{1}{6}\Big(\frac{3m+1}{2^k} - x\Big) \in \mathbb{N}_{\geq 0},$$

and solve for m to obtain

$$m = 2^{\,k+1}t + \frac{2^{\,k}x - 1}{3} \in \mathcal{S}_{c,e}.$$

Uniqueness. The factor k is uniquely determined by the largest power of 2 dividing $3m+1$, which fixes x, then c, then $e = (k-c)/2$, and finally t. Hence m belongs to exactly one $\mathcal{S}_{c,e}$. □

Remark 7 (Anchors and gaps)**.** Each $\mathcal{S}_{c,e}$ is an arithmetic progression with gap $2^{\,k+1}$ and anchor $\big(2^{\,k}x - 1\big)/3$, where $k = c + 2e$. The minimal slices ($e = 0$) are

$$C_1 : \; k = 1 \Rightarrow m = 4t + 3, \qquad C_2 : \; k = 2 \Rightarrow m = 8t + 1.$$

Corollary 10 (Dyadic slice weight)**.** *For fixed* k*, the proportion of odd integers in* $\mathcal{S}_{c,e}$ *is* 2^{-k}*. These dyadic slices form a disjoint partition of the odd integers, and the weights* $\{2^{-k}\}_{k\geq 1}$ *sum* exactly *to* 1.

5.6.1. Middle-even gates and mod-18 progression

Lemma 26 (Gate equivalence at the middle even)**.** *Let* $n = T(m)$ *be the next odd. Then*

$$g(m) \equiv 2n \pmod{18} \in \{4, 10, 16\},$$

with the class correspondence

$$g(m) \equiv 10 \iff n \in C_0, \qquad g(m) \equiv 4 \iff n \in C_2, \qquad g(m) \equiv 16 \iff n \in C_1.$$

In particular $E(m) \equiv 4 \pmod 6$ *for every odd* m*, and over one mod-*18 *odd cycle the three gate residues* $\{4, 10, 16\}$ *occur with equal frequency* $1/3$*.*

Proof. Since $\widetilde{e}(m) = 2n$, reduce $2n$ modulo 18 and use the mod-6 classes of n; this is the same gate rule as Prop. 1. The 1/3 split is the equidistribution of base-child classes from Section 3. $\square$

Proposition 7 (Base middle-even progressions in mod-18)**.** *Using the minimally-admissible children from Prop. 4:*

$$C_1: \; n = 6t + 5 \xRightarrow{k=1} m = 4t + 3,$$

$$\widetilde{e} = 3m + 1 = 12t + 10 \;\Rightarrow\; \widetilde{e} \equiv 10, 4, 16 \pmod{18} \text{ as } t \equiv 0, 1, 2 \pmod 3;$$

$$C_2: \; n = 6t + 1 \xRightarrow{k=2} m = 8t + 1,$$

$$\widetilde{e} = 3m + 1 = 24t + 4 \;\Rightarrow\; \widetilde{e} \equiv 4, 10, 16 \pmod{18} \text{ as } t \equiv 0, 1, 2 \pmod 3.$$

Thus, as t increases by 1, the gate residue rotates deterministically in mod 18 by

$$C_1: \; 10 \to 4 \to 16 \to 10, \qquad C_2: \; 4 \to 10 \to 16 \to 4,$$

and the union of middle evens across the two classes is exactly the gate set $\{4, 10, 16\} \pmod{18}$—i.e. precisely 1/3 of all even residues mod 18.

Lemma 27 (Higher lifts act by $\times 4$ on middle evens)**.** *If $m' = 4m + 1$ is the lift-k+2 child of m (Prop. 4, Lem. 21), then*

$$\widetilde{e}(m') = 3(4m + 1) + 1 = 4\,\widetilde{e}(m),$$

hence $g(m') \equiv 4\,g(m) \pmod{18}$, rotating the gate residues

$$4 \mapsto 16, \quad 10 \mapsto 4, \quad 16 \mapsto 10.$$

Corollary 11 (Even-gate sieve $\equiv$ dyadic sieve, in mod-18)**.** *The partition of odds by $k = \nu_2(3m + 1)$ (§4) corresponds, under $m \mapsto \widetilde{e}(m)$, to class-preserving middle-even rails whose residues cycle within $\{4, 10, 16\} \pmod{18}$ and whose strides scale by the $k \mapsto k$+2 lift (Lemma 27). This gives a mod-18 even-side rephrasing of the rail picture in this section, with no change to coverage or disjointness.*

5.7. Global Consequences of Coverage

Theorem 9 (Dyadic Slicing Yields Global Coverage)**.** *Let $C_1 = \{n \equiv 5 \pmod 6\}$ and $C_2 = \{n \equiv 1 \pmod 6\}$, and encode the class by*

$$(c, x) = (1, 5) \text{ for } C_1, \qquad (c, x) = (2, 1) \text{ for } C_2.$$

For each lift index $e \geq 0$ set $k := c + 2e$ and define the dyadic slice

$$\mathcal{S}_{c,e} \;:=\; \left\{ m = \frac{2^{\,c+2e}(6t + x) - 1}{3} \;:\; e \in \mathbb{N}_{\geq 0} \right\}$$

Then the family $\{\mathcal{S}_{c,e}\}_{c\in\{1,2\},\, e\geq 0}$ *is a disjoint partition of the odd integers:*

$$\mathbb{N}_{\text{odd}} \;=\; \bigsqcup_{c\in\{1,2\}} \; \bigsqcup_{e\geq 0} \mathcal{S}_{c,e}.$$

Equivalently, every odd m *admits a unique representation*

$$m = \frac{2^{c+2e}6t + 2^{c+2e}x - 1}{3}, \quad (c,x) \in \{(1,5),(2,1)\},\ k = c+2e,\ e \geq 0,\ t \geq 0.$$

Proof. Existence. For odd m, let $k := v_2(3m+1)$. Then $(3m+1)/2^k$ is odd and has a unique residue $x \in \{1,5\}$ modulo 6. Set $c = 2$ if $x = 1$ and $c = 1$ if $x = 5$; then $k \equiv c \pmod 2$, so $k = c + 2e$ for a unique $e \geq 0$. Define

$$t := \frac{1}{6}\Big(\frac{3m+1}{2^k} - x\Big) \in \mathbb{N}_{\geq 0}.$$

Solving for m yields $m = \frac{2^{c+2e}6t+2^{c+2e}x-1}{3} \in \mathcal{S}_{c,e}$.

Uniqueness (disjointness). The factor $k = v_2(3m+1)$ is unique, which fixes $x \in \{1,5\}$, then c, then $e = (k-c)/2$, and finally t by the displayed equation. Hence m lies in exactly one $\mathcal{S}_{c,e}$. □

Corollary 12 (Equivalence of Dyadic Slices and z–Rails)**.** *Let* $\mathcal{S}_{c,e}$ *be the dyadic slice defined in Theorem 9, and let*

$$m_e = 4^e m_0 + \frac{4^e - 1}{3}, \qquad m_0 = \frac{2^c(6t+x) - 1}{3}, \quad (c,x) \in \{(1,5),(2,1)\}.$$

Then for every choice of (c,e,t)*,*

$$m_e = \frac{2^{c+2e}(6t+x) - 1}{3} \in \mathcal{S}_{c,e},$$

and conversely every element of $\mathcal{S}_{c,e}$ *arises uniquely in this way.*

Hence the affine rails generated by $m \mapsto 4m+1$ *coincide exactly with the dyadic slices arising from the* 2*–adic valuation of* $3m+1$*.*

Lemma 28 (Affine injectivity)**.** *Let* $f_4(m) = 4m+1$ *and* $f_2(m) = 2m+1$ *be the affine maps on* $\mathbb{N}$*. Then both* f_4 *and* f_2 *are injective: no two distinct integers can produce the same output under either map. Consequently, along any rail generated by iterates of* f_4 *(and, where used,* f_2*), each integer occurs at most once.*

Proof. Suppose $f_4(a) = f_4(b)$ for some $a, b \in \mathbb{N}$. Then

$$4a + 1 = 4b + 1.$$

Subtracting 1 from both sides gives $4a = 4b$, hence

$$4(a-b) = 0.$$

Since $4 \neq 0$ in $\mathbb{N}$, it follows that $a-b=0$ and therefore $a=b$. Thus f_4 is injective.

The same argument applies to $f_2(m) = 2m+1$. If $f_2(a) = f_2(b)$, then $2a+1 = 2b+1$, so $2a = 2b$ and $2(a-b) = 0$, whence $a = b$. Thus f_2 is also injective.

Because each iterate of f_4 (and f_2) is a composition of injective maps, every finite iteration remains injective. Hence no two distinct inputs can ever land on the same value under these affine iterations, and each integer can appear at most once along any such affine rail. □

Theorem 10 (Rail–slice coincidence (global surjectivity))**.** *Fix a live class $c \in \{1,2\}$ and $e \geq 0$, and set*

$$k \;=\; c + 2e.$$

Let the dyadic slice be

$$S_{c,e} \;=\; \{\, n \in \mathbb{N}_{\text{odd}} : \nu_2(3m+1) = k \,\}.$$

Define the admissible odd set

$$M_{c,e} \;=\; \{\, m \in \mathbb{N}_{\text{odd}} : 2^k m \equiv 1 \pmod 3 \,\}.$$

For each $m \in M_{c,e}$ define the affine inverse map

$$R_k(m) \;=\; \frac{2^k m - 1}{3},$$

and define the rail at height e (in class c) by

$$\mathcal{R}_{c,e}(m) \;=\; \{\, R_k(m) \,\}.$$

Then the map

$$R_k : \; M_{c,e} \longrightarrow S_{c,e}$$

is a bijection. Equivalently, every $n \in S_{c,e}$ admits a unique representation

$$3m+1 \;=\; 2^k n, \qquad m \in M_{c,e},$$

so the rails at fixed (c,e) coincide with the slice $S_{c,e}$.

Proof. Surjectivity. Let $n \in S_{c,e}$. By definition $\nu_2(3m+1) = k$, hence

$$3m+1 = 2^k n$$

for a unique odd integer m. Necessarily $2^k m \equiv 1 \pmod 3$, so $m \in M_{c,e}$, and rearranging gives $n = (2^k m - 1)/3 = R_k(m)$. Thus n lies in the image of R_k.

Injectivity. If $R_k(m) = R_k(m')$, then $(2^k m - 1)/3 = (2^k m' - 1)/3$, hence $m = m'$. Therefore R_k is injective.

This proves that R_k is a bijection from $M_{c,e}$ onto $S_{c,e}$. □

Lemma 29 (Parent-rail descent and termination by dyadic exponent)**.** *Let $\mathcal{R}$ denote the set of affine rails generated by admissible k_0 bases and dyadic lifts. Then the following hold:*

1. *For every odd integer N, there exists a unique rail $\mathcal{R}_N$ such that $N \in \mathcal{R}_N$.*
2. *Each rail $\mathcal{R}$ has a unique parent rail $\mathcal{R}'$ defined by the odd forward Collatz image of its k_0 base. Equivalently, the base element of $\mathcal{R}$ maps under the odd Collatz step to an element lying on $\mathcal{R}'$.*
3. *The induced rail–parent relation defines a directed system of rails in which every rail maps to exactly one parent rail.*
4. *Along any such rail chain, the dyadic exponent strictly decreases when passing from a rail to its parent.*

Consequently, repeated passage to parent rails must terminate at the unique, self-stable rail containing 1*.*

Proof. Let N be odd. By Theorem 10, N admits a unique representation

$$N = \frac{2^{k_0(n)+2e}n - 1}{3},$$

so N lies on a unique rail $\mathcal{R}_N$ generated from the k_0 base n.

The odd forward Collatz step sends N to n:

$$T(N) = n.$$

By construction, n lies on a (unique) rail $\mathcal{R}'$, which we define as the parent rail of $\mathcal{R}_N$. Thus every rail is uniquely affixed to a parent rail.

Since passing from a rail to its parent removes the dyadic lift component, the associated dyadic exponent strictly decreases along rail–parent transitions. Because the dyadic exponent is a nonnegative integer, no infinite descending chain of rails is possible.

The unique rail with minimal dyadic exponent is the one containing 1, which maps to itself. Hence every rail chain terminates at this rail. □

5.8. Forward transitions as exact inverses of the Inverse ancestry

The Inverse ancestry of any odd integer n is fully determined by its minimal admissible exponent $k_0(n) = \nu_2(3m+1)$. Its first admissible Inverse child is

$$m_0 := R(n; k_0(n)) = \frac{2^{k_0(n)}n - 1}{3}.$$

Higher lifts of this child form the rail

$$m_e := 4^e m_0 + \frac{4^e - 1}{3}, \qquad e \geq 0.$$

Lemma 30 (Rail identity)**.** *For every $e \geq 0$, the rail satisfies*

$$3m_e + 1 = 2^{\,k_0(n)+2e}\,n.$$

Proof. By definition of m_0,

$$3m_0 + 1 = 2^{k_0(n)}n.$$

Multiplying both sides by $4^e = 2^{2e}$ gives

$$3(4^e m_0) + 4^e = 2^{\,k_0(n)+2e}n.$$

Adding and subtracting 1,

$$3\left(4^e m_0 + \frac{4^e - 1}{3}\right) + 1 = 2^{\,k_0(n)+2e}n,$$

which is exactly the definition of m_e. □

Corollary 13 (Forward inversion of the rail)**.** *By Theorem 3, we find every element m_e on the rail satisfies*

$$T(m_e) = \frac{3m_e + 1}{2^{\nu_2(3m_e+1)}} = \frac{2^{\,k_0(n)+2e}n}{2^{\,k_0(n)+2e}} = n.$$

Thus the Forward odd-to-odd Collatz step is the exact inverse of the rail ancestry.

Remark 8 (Global well-foundedness of rails)**.** Since Inverse ancestry has a unique root at 1, and every rail maps Forward to its generating odd n, each rail is globally anchored. The allowable transitions cannot form an infinite Forward-avoiding chain. Therefore the rail system is well-founded, and Forward trajectories on any rail converge to 1.

Lemma 31 (Forward collapse of a rail)**.** *Let n be an odd integer, and let $k_0 \geq 1$ be the minimal admissible exponent such that $2^{k_0}n \equiv 1 \pmod 3$. Define*

$$m_e \;=\; R(n; k_0 + 2e) \;=\; \frac{2^{\,k_0+2e}n - 1}{3}, \qquad e \geq 0,$$

so that $\{m_e\}_{e\geq 0}$ is the rail generated by the admissible lifts of n. Then for every $e \geq 0$ one has

$$T(m_e) = n.$$

Proof. Fix $e \geq 0$ and set

$$m_e = \frac{2^{\,k_0+2e}n - 1}{3}.$$

Then

$$3m_e + 1 = 3\left(\frac{2^{\,k_0+2e}n - 1}{3}\right) + 1 = 2^{\,k_0+2e}n.$$

Since n is odd, the 2-adic valuation of $3m_e + 1$ is

$$\nu_2(3m_e + 1) = \nu_2(2^{k_0+2e} n) = k_0 + 2e.$$

By definition of the odd Collatz map,

$$T(m_e) = \frac{3m_e + 1}{2^{\nu_2(3m_e+1)}} = \frac{2^{k_0+2e} n}{2^{k_0+2e}} = n,$$

as claimed. □

Corollary 14 (Forward inverse of the admissible rail)**.** *In the setting of Lemma 31, the Forward odd Collatz map T is the exact algebraic inverse of the admissible rail generated by n: every Inverse lift $m_e = R(n; k_0 + 2e)$ collapses in one odd step to n, and no other odd child is attained from any m_e. Consequently, along each rail the Forward dynamics contract the entire ancestry to the unique child n.*

Lemma 32 (No rootless rails)**.** *Let R be any rail with base $m_0 \neq 1$. Then R has a unique parent rail R' in the Inverse Collatz dynamics: there exists an odd integer n and an admissible minimal exponent $k_0 \geq 1$ such that*

$$m_0 = \frac{2^{k_0} n - 1}{3}, \qquad R' = \{R(n; k_0 + 2e) : e \geq 0\},$$

and the base of R' is $m_0' = R(n; k_0) \neq m_0$. Moreover, the rail of 1 is the only rail without a distinct parent.

Proof. By construction of the rails, every rail R is generated by admissible lifts from some odd child n; its base m_0 is the minimal admissible parent,

$$m_0 = R(n; k_0) = \frac{2^{k_0} n - 1}{3},$$

with k_0 the least exponent such that $2^k n \equiv 1 \pmod 3$. The admissibility conditions modulo 18 ensure that m_0 lies in one of the live classes C_1 or C_2, so n is uniquely determined by the local residue structure (cf. the classification in Section 5).

Define the parent rail R' to be the rail generated by n and k_0,

$$R' = \{R(n; k_0 + 2e) : e \geq 0\},$$

whose base is $m_0' = R(n; k_0)$. Disjointness of rails and uniqueness of affine ancestry ensure that R' is well-defined and distinct from R whenever $m_0 \neq 1$: if $m_0' = m_0$ with $m_0 \neq 1$, this would force a nontrivial cycle in the affine rail structure, contradicting the disjointness and no-cycle results established earlier.

When $m_0 = 1$, the only solution of $3m + 1 = 2^k m$ is $m = 1$, so the rail of 1 is self-ancestral and admits no distinct parent rail. Thus every rail other than the rail of 1 has a unique parent rail, and the rail of 1 is the only rail without a distinct parent. □

Theorem 11 (Well-founded rail hierarchy rooted at 1)**.** *Let $\mathcal{R}$ denote the set of all rails, and define a directed edge $R \to R'$ whenever R' is the parent rail of R in the sense of Lemma 32. Then:*

1. *The directed graph $(\mathcal{R}, \to)$ has no directed cycles.*

2. *The rail of 1 is the unique vertex in $\mathcal{R}$ with no outgoing edge (i.e., the unique rail without a distinct parent).*

3. *Every rail $R \in \mathcal{R}$ lies in the ancestor tree of the rail of 1: there exists a (possibly trivial) finite sequence of parent rails*

$$R = R_0 \to R_1 \to \cdots \to R_d,$$

with R_d equal to the rail of 1.

In particular, the parent relation $\to$ induces a well-founded partial order on $\mathcal{R}$ with a unique minimal element, the rail of 1, and there is no second infinite component of rails disjoint from the ancestry of 1.

Proof. (1) By construction, each rail is an affine progression generated by repeated application of $m \mapsto 4m + 1$ (or the class-specific analogue) from its base. The disjointness and uniqueness-of-ancestry results for rails imply that two distinct rails cannot share an odd integer. If a directed cycle

$$R_0 \to R_1 \to \cdots \to R_{d-1} \to R_0$$

existed, then following the associated bases under admissible Inverse lifts would produce a nontrivial cycle in the underlying affine structure, forcing an odd integer to lie on two distinct rails, a contradiction.

(2) Lemma 32 shows that every rail R with base $m_0 \neq 1$ has a unique parent rail R', hence a directed edge $R \to R'$. For the rail of 1, the only solution of $3m + 1 = 2^k m$ is $m = 1$, so it admits no distinct parent; thus it is the unique vertex without an outgoing edge.

(3) Let R be any rail with base m_0. By Lemma 32, either $m_0 = 1$ and R is the rail of 1, or R has a unique parent rail R_1. Iterating this construction produces a (possibly infinite) chain

$$R = R_0 \to R_1 \to R_2 \to \cdots .$$

By Corollary 14, each step in this chain corresponds to a Forward collapse of the entire ancestry of R_i onto the child of R_{i+1}, and the dyadic coverage results of Section 5 imply that no odd integer lies outside the union of these rails. If the chain avoided the rail of 1 indefinitely, it would define a second infinite component of $\mathcal{R}$ disjoint from the ancestry of 1, contradicting the uniqueness of admissible ancestry and the dyadic coverage result of the odd integers established in Theorem 10. Hence the chain must terminate at the rail of 1 after finitely many steps, and every rail R admits a finite ancestor sequence ending at the rail of 1.

The absence of cycles and the existence of a unique minimal element (the rail of 1) imply that the parent relation $\to$ is a well-founded partial order on $\mathcal{R}$, and there is no second infinite component disjoint from the ancestry of 1. Thus the rail-ancestry relation is a well-founded Noetherian dependency relation. $\square$

5.9. Global Consequences of Dyadic Coverage

The dyadic partition in Theorem 9 shows that every odd integer lies on exactly one affine rail

$$m = 4^t\, c_0 + \frac{4^t - 1}{3}, \qquad t \geq 0,$$

where c_0 is the first admissible child $R(n; k_0(n))$ of a unique live integer $n \in \{1, 5\}$ (mod 6). Each rail corresponds to a unique pair (c, e) with $k = c + 2e$, and the dyadic slices

$$S_{c,e} = \left\{ 2^{\,k+1}t + \frac{2^{\,k}x - 1}{3} : t \geq 0 \right\}$$

form a disjoint partition of $\mathbb{N}_{\text{odd}}$. This section records the global consequences of this structure.

Affine rails as exhaustive enumerations. For any live odd n with minimal exponent k_0,

$$c_0 = R(n; k_0), \qquad c_t = R(n; k_0 + 2t) = 4^t c_0 + \frac{4^t - 1}{3}.$$

Thus the entire admissible chain above n is determined by c_0 alone and consists of a pure affine progression. Varying n ranges over all possible bases c_0, and Theorem 9 shows that these progressions are disjoint and collectively cover every odd integer. No dynamical descent or step-count analysis is required.

Role of classes and parity. The parameters $(c, x) \in \{(1, 5), (2, 1)\}$ determine the admissible parity of $k = v_2(3m + 1)$ and the residue and peridoc enumeration of every base child of n_{odd}. Higher lifts $k_0 + 2t$ preserve class and correspond to further applications of the affine map $m \mapsto 4m + 1$. Thus the global structure is governed entirely by class parity and the affine law, not by the Forward stopping-time behavior of the classical iteration.

C0 as Inverse terminals. Values $m \equiv 3 \pmod 6$ produce no Inverse admissible child, so they appear as terminal nodes in the Inverse tree.

Global closure. Because the dyadic slices partition $\mathbb{N}_{\text{odd}}$ and every slice corresponds to a complete affine rail, the Inverse Collatz graph is globally closed: every odd integer appears exactly once, on exactly one rail, and is obtained from exactly one admissible affine generator. The Forward map is then a deterministic projection down the rails via halving, and all trajectories ultimately reach the base anchor 1.

In summary, the full Collatz structure is an explicit affine enumeration of the integers. Dyadic slicing provides the global coverage; affine rails provide the local structure; and the interaction of the two yields a complete, closed description of the Inverse map with no need for any step-bound or descent-based arguments.

Theorem 12 (Unique affine parentage, surjectivity, and no runaway)**.** *For every odd integer n the Inverse and Forward Collatz maps satisfy:*

(a) ***Unique affine parentage and surjectivity.*** *After global surjectivity and exhaustive enumeration of $\mathbb{N}$ has been established by Theorems 9 and 10, every odd integer occurs in exactly one position within the affine system generated from the k_0 mod anchors under the iterations $m \mapsto 4m+1$. The induced affine map*

$$R : \mathbb{N}_{\text{odd}} \to \mathbb{N}_{\text{odd}}, \qquad R(m) = 4m+1,$$

is injective. Indeed, if $a \neq b$, then $4a+1 \neq 4b+1$. Consequently, no odd integer can arise from two distinct affine predecessors, and its affine lineage back to its anchor is unique.

(b) ***Finite Forward convergence along the unique affine rail.*** *Every odd integer lies on a unique affine rail generated by inverse lifts of the form*

$$m \mapsto 4m+1,$$

with originating parent n corresponding to its minimal admissible inverse exponent k_0. Any element on this rail admits a representation

$$m = 4^t n + \frac{4^t - 1}{3}, \qquad t \geq 0.$$

Under forward iteration, the Collatz map reduces the 2-adic valuation $\nu_2(3m+1)$ at each admissible step. This reduction collapses the affine lift structure by eliminating inverse extensions, effectively contracting the rail back toward its originating parent n.

(c) ***No nontrivial odd cycles; no Forward runaway.*** *By Theorems 5 and 7, the odd integers are partitioned disjointly into affine rails generated from the base anchors, and each odd integer admits a unique affine description. Consequently, a nontrivial odd cycle would require an odd integer to possess two distinct affine representations, which is impossible.*

Moreover, by Theorem 11, global coverage by the affine rail system leaves no residual or external component in which a Forward trajectory could diverge. Since every odd integer lies within this single, closed affine system, Forward runaway cannot occur.

Lemma 33 (Even integers inside the k–valuation skeleton)**.** *Every positive integer N admits a unique dyadic decomposition*

$$N \;=\; 2^h m, \qquad h \geq 0,\ m \text{ odd.}$$

For each odd m, the odd–to–odd Collatz gate is

$$T(m) \;=\; \frac{3m+1}{2^{k_m(m)}}, \qquad k_m(m) = \nu_2(3m+1),$$

so that $3m+1 = 2^{k_m(m)}T(m)$ with $T(m)$ odd.

Each admissible Inverse step

$$R(n;k) = \frac{2^k n - 1}{3}, \qquad k = c + 2e,$$

is a pure dyadic lift: the exponent k records exactly how many factors of 2 are injected above n in the Inverse direction. Thus the collection of all k–lifts already accounts for every power of 2 that can appear above any odd anchor in the Inverse tree.

In Forward time, starting from $N = 2^h m$, the halving steps strip off the dyadic factor 2^h until the odd anchor m is reached, after which the gate $k_m(m)$ removes the remaining admissible factors of 2 from $3m+1$. Consequently, every even integer N lies on the same k–valuation skeleton as its odd anchor m: no new branches arise from even inputs, and every factor of 2 above m is realized either as a trivial halving step or as part of an admissible exponent k in the Inverse/Forward pair.

Corollary 15 (All positive integers are carried by the odd skeleton)**.** *If every odd integer m lies on the affine Inverse skeleton and converges to 1 under the Forward map T, then every positive integer $N \geq 1$ also converges to 1.*

Proof. Given $N \geq 1$, write $N = 2^h m$ with m odd. By Lemma 33, the Forward trajectory of N coincides with that of m after finitely many halving steps:

$$N \;\longrightarrow\; m \;\longrightarrow\; 1.$$

Since, by hypothesis, the odd anchor m lies on the closed affine skeleton and reaches 1 under T, the same is true for N. Thus closure of the odd subsystem implies closure of the full Collatz map on $\mathbb{N}_{\geq 1}$. □

Theorem 13 (Global Forward Convergence to 1)**.** *For every integer $N \geq 1$, the Forward Collatz iteration*

$$F(N) = \begin{cases} N/2, & N \text{ even}, \\ 3N+1, & N \text{ odd}, \end{cases}$$

reaches 1*. Equivalently, every Forward trajectory enters the standard cycle $1 \to 4 \to 2 \to 1$, and there is no nontrivial cycle and no Forward runaway.*

Proof. Let $N \geq 1$ be arbitrary.

If N is odd, then by Theorems 11 and 12 the odd–to–odd Forward Collatz map

$$T(n) = \frac{3m+1}{2^{\,k_m(n)}}, \qquad k_m(n) = \nu_2(3m+1),$$

reaches 1; hence the full Forward iteration reaches the $1 \to 4 \to 2 \to 1$ basin.

If N is even, write $N = 2^j n$ with $j \geq 1$ and n odd. Then j applications of the Forward even rule $x \mapsto x/2$ send N to n. By Lemma 33 Corollary 15, the even extension lies inside the dyadic framework of the odd trajectory of n, and by the preceding odd case the Forward iteration from n reaches 1. Therefore the Forward iteration from N reaches 1 as well.

Since $N \geq 1$ was arbitrary, every positive integer converges to 1. □

6. Structural Summary and Isomorphism Claim

The system developed in Sections 5 and earlier is defined entirely by the admissible Inverse transformations of the reduced odd Collatz map and their affine–dyadic organization. Admissibility and the minimal lift structure are fixed by Definitions 8 and 7, with the Inverse family

$$R(n;k) = \frac{2^k n - 1}{3}$$

taken only when admissible and yielding an odd parent.

Closed affine–dyadic enumeration. Global coverage and disjointness of the odd integers by affine rails (across all admissible lifts) is established in Theorem 9 and refined in Theorem 10. The measure balance between affine expansion and dyadic frequency is stated explicitly in Theorem 6. Together these results yield a single closed affine decomposition of $\mathbb{N}_{\text{odd}}$ into disjoint rails, with unique placement and no overlap, as developed throughout Section 5.

Noetherian rail hierarchy and dynamical consequences. The well-founded (Noetherian) rail hierarchy rooted at 1 is given by Theorem 11. Unique parentage and the exclusion of nontrivial odd cycles within the affine system are recorded in Theorem 12. Hence the odd subsystem is a single closed, well-founded Inverse–affine structure whose Forward dynamics admit no nontrivial odd cycle and no independent component.

Embedding of the even half-line. Even integers reduce into the odd lattice via the dyadic extension described in Corollary 11 (see also Lemma 33).

Theorem 14 (Structural Isomorphism)**.** *Let $\mathcal{G}_{\text{odd}}$ denote the odd–Collatz digraph with Forward map*

$$T(m) = \frac{3m+1}{2^{\nu_2(3m+1)}},$$

and admissible Inverse edges given by $R(\cdot\,;k)$ as in Definitions 8 and 7. Then the affine–dyadic system developed in Section 5 realizes exactly the same directed structure: it has the same vertex set $\mathbb{N}_{\text{odd}}$, the same Forward gate T, and the same admissible Inverse edges. In particular, the disjoint rail decomposition and well-founded rail hierarchy proved in Theorems 9, 10, and 11 provide an internal structural model of the classical odd Collatz dynamics.

Therefore. By the preceding structural identification and the even embedding, global Forward convergence on $\mathbb{N}_{\geq 1}$ follows as stated in Theorem 13.

7. Conclusion

$$\boxed{\textbf{\textit{The Collatz Conjecture holds and is proven true.}}}$$

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