Proof of the Collatz Conjecture

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Abstract

The Collatz Conjecture (or 3n+1 problem) has been explored for about 86 years. In this article, we prove the Collatz Conjecture. We will show that this conjecture holds for all positive integers by applying the Collatz inverse operation to the numbers that satisfy the rules of the Collatz Conjecture. Finally, we will prove that there are no positive integers that do not satisfy this conjecture.

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1 Introduction

The Collatz Conjecture is one of the unsolved problems in mathematics. Introduced by German mathematician Lothar Collatz in 1937 [1], it is also known as the 3n + 1 problem, 3x + 1 mapping, Ulam Conjecture (Stanislaw Ulam), Kakutani's problem (Shizuo Kakutani), Thwaites Conjecture (Sir Bryan Thwaites), Hasse's algorithm (Helmut Hasse), or Syracuse problem [2–4].

In this paper, $\mathbb{N}$ $=\{0, 1, 2, 3, 4, 5, \ldots\}$, the symbol $\mathbb{N}$ represents the natural numbers. $\mathbb{N}^+$ $=\{1, 2, 3, 4,\ 5, 6, \ldots\}$, the symbol $\mathbb{N}^+$ represents the positive integers. $\mathbb{N}_{odd}$ $=\{1, 3, 5, 7, 9,\ 11, 13, \ldots\}$, the symbol $\mathbb{N}_{odd}$ represents the positive odd integers.

2 The Conjecture and Related Conversions

Definition 2.1 Let $n, k \in \mathbb{N}^+$ and a function $f : \mathbb{N}^+ \to \mathbb{N}^+$, Collatz defined the following map:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if n is even} \\ 3n + 1, & \text{if n is odd} \end{cases}$$

The Collatz Conjecture states that the orbit formed by iterating the value of each positive integer in the function $f(n)$ will eventually reach 1. The orbit of n under f is $n; f(n), f(f(n)), f(f(f(n))), \ldots\ f^k(n) = 1$ $(k \in \mathbb{N}^+)$.
In the following sections, we will call these two arithmetic operations ($n/2$ and $3n + 1$), which we apply to any positive integer n according to the rule of assumption, Collatz operations (CO).

Remark 2.2 According to the definition of the Collatz Conjecture, if the number we choose at the beginning is an even number, then by continuing to divide all even numbers by 2, one of the odd numbers is achieved. So it is sufficient to check whether all odd numbers reach 1 by the Collatz operations.

Therefore, if we prove that it reaches 1 when we apply the Collatz operations to all the elements of the set $\mathbb{N}_{odd}$ $=\{1, 3, 5, 7, 9, 11, 13, 15, \ldots\}$, we have proved it for all positive integers.

Remark 2.3 If the Collatz operations are applied to the numbers 2^n $(n \in \mathbb{N}^+)$, then eventually 1 is reached. If we can convert all the elements of the set $\mathbb{N}_{odd}$ into 2^n numbers by applying the Collatz operations, we get the result.

2.1 Collatz Inverse Operation (CIO)

Let $n \in \mathbb{N}^{+}$and $a \in \mathbb{N}_{odd}$; for a to be converted to 2^n by the Collatz operation (CO), it must satisfy the following equation,

$$3.a + 1 = 2^n$$

then,

$$a = \frac{2^n - 1}{3} \tag{1}$$

Lemma 2.4 In (1) $a = \frac{2^n-1}{3}$, a cannot be an integer if n is a positive odd integer.
Proof. If n is a positive odd integer, we can take $n = 2m + 1$ ($m \in \mathbb{N}$), then substituting $2m + 1$ for n in (1) we get,

$$a = \frac{2^{2m+1} - 1}{3} \tag{2}$$

if we factor $2^{2m+1} + 1$,
$2^{2m+1} + 1 = (2 + 1)(2^{2m} - 2^{2m-1} + 2^{2m-2} - \ldots + 1) = \mathbf{3}.k$ ($k \in \mathbb{N}_{odd}$).
Since $(2^{2m+1} + 1)$ is a multiple of 3, $(2^{2m+1} - 1)$ is not a multiple of 3. So in (1) a is not an integer for any posiitive odd integer n.

If we substitute $2n$ for n in (1), we get equation

$$a = \frac{2^{2n} - 1}{3} \tag{3}$$

Lemma 2.5 In (3) $a = \frac{2^{2n}-1}{3}$, for each positive integer n, there exists a distinct positive odd integer a_n, ($n \in \mathbb{N}^{+}$).

Proof. When we factorize $2^{2n} - 1$ for $\forall n \in \mathbb{N}^{+}$,

$(2^{2n} - 1) = (2^{x_1} - 1)(2^{x_1} + 1)(2^{x_2} + 1)(2^{x_3} + 1)\ldots.(2^{x_{n-1}} + 1)(2^{x_n} + 1)$ or
$(2^{2n} - 1) = (2^{x_1} - 1)(2^{x_1} + 1)$ in these equations, x_1 is a positive odd integer and x_2, x_3, x_4 $\ldots$ x_n are positive even integers. Since x_1 is a positive odd number,
$(2^{x_1} + 1) = (2 + 1)(2^{x_1-1} - 2^{x_1-2} + 2^{x_1-3} - \ldots + 1) = \mathbf{3}.(\ldots)$ so,
$(2^{2n} - 1) = \mathbf{3}.(\ldots)$
Since each of these numbers has a multiplier of 3, we can find positive odd integers a for all n, and when we apply Collatz operations to these a numbers, we always get 1. $2^{2n} + 1$ is not a multiple of 3, since $2^{2n} - 1$ is a multiple of 3, for $\forall n \in \mathbb{N}^{+}$. In (3), If we replace n with positive integers, we get the set A.

$$a = \frac{2^{2n}-1}{3};$$

A= { 1,5,21,85,341,1365,5461,21845,87381,... } (Collatz Numbers)

If we can generalize the elements of the set $A = \{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381, \ldots\}$ to all positive odd numbers, we have proved the Collatz Conjecture.

2.2 Transformations in the Set A with Infinite Elements

Let the elements of the set $A = \{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381, \ldots\}$ be $\{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots\}$ respectively.

Lemma 2.6 In the set A$\setminus\{a_0\}$, if $a_n \equiv 1 \pmod 3$

$$b_n = \frac{2^{2m}.a_n - 1}{3} \tag{4}$$

$m \in \mathbb{N}^+$, if we value m from 1 to infinity, we get B_n set with infinite b_n elements (Collatz numbers) from each a_n.These numbers satisfy the conjecture.

Proof. If $a_n \equiv 1 \pmod 3$, we can take a_n as $3.p + 1$, $(p \in \mathbb{N})$
$a_n = 3.p + 1$ substituting in (4),

$$b_n = \frac{2^{2m}.(3.p+1)-1}{3} = \frac{2^{2m}3p + 2^{2m} - 1}{3} = 2^{2m}p + \frac{2^{2m}-1}{3}$$

$2^{2m} - 1$ is divisible by 3 (Lemma 2.5). So we get an infinite number of different b_n elements, which can be converted to a_n, i.e. 1, by the Collatz operation. The numbers b_n are Collatz numbers and are a sequence of the form $b_{n+1} = 4.b_n + 1$.

Example 2.7 Let $a_1 = 85$, then $a_1 \equiv 1 \pmod 3$,in (4),

$B = \{113, 453, 1813, 7253, 29013, 116053, \ldots\}$

Lemma 2.8 In the set A$\setminus\{a_0\}$, if $a_n \equiv 2 \pmod 3$,

$$b_n = \frac{2^{2m-1}.a_n - 1}{3} \tag{5}$$

$m \in \mathbb{N}^+$, if we value m from 1 to infinity, we get B_n set with infinite b_n elements (Collatz numbers) from each a_n. These numbers satisfy the conjecture.

Proof. If $a_n \equiv 2 \pmod 3$, we can take a_n as $3.p + 2$ ($p \in \mathbb{N}$)
$a_n = 3.p + 2$ substituting in (5),

$$b_n = \frac{2^{2m-1}.(3p+2)-1}{3} = \frac{2^{2m-1}.3p + 2^{2m} - 1}{3} = 2^{2m-1}p + \frac{2^{2m}-1}{3}$$

$2^{2m} - 1$ is divisible by 3 (Lemma 2.5). So we get an infinite number of different b_n elements, which can be converted to a_n, i.e. 1, by the Collatz operation. The numbers b_n are Collatz numbers and are a sequence of the form $b_{n+1} = 4.b_n + 1$.

Example 2.9 Let $a_1 = 5$, then $a_1 \equiv 2 \pmod 3$;

$$B = \{3,\ 13,\ 53,\ 213,\ 853,\ 3413,\ 13653,\ 54613,\ \ldots\}$$

Lemma 2.10 In the set A$\setminus\{a_0\}$, if $a_n \equiv 0 \pmod 3$,

$$b_n = \frac{2^m.a_n - 1}{3} \tag{6}$$

$m \in \mathbb{N}^+$, there is no such integer b_n.

Proof . If $a_n \equiv 0 \pmod 3$, we can take a_n as $3.p$ ($p \in \mathbb{N}$)

$a_n = 3.p$ substituting in (6),

$$b_n = \frac{2^m(3.p)-1}{3} = \frac{2^m 3.p - 1}{3} = 2^m.p - \frac{1}{3},$$

is not integer.
In the following sections, we will call the operations of deriving new Collatz numbers from Collatz numbers by Equations (3), (4) or (5) as Collatz inverse operations (CIO).

2.3 Conversion of all Positive Odd Integers to Collatz Numbers

In the previous sections, when we applied the Collatz operations, we called positive odd integers that reached 1 as Collatz numbers. Now let's see how all positive odd integers can be converted to these Collatz numbers.

A= { 1,5,21,85,341,1365,5461,21845,87381 ... } (Collatz Numbers)

If we apply the Collatz inverse operations [Equations (4) or (5)] continuously to each Collatz number, we get infinitely many new Collatz numbers.

$\mathbb{N}_{odd} \rightarrow$ Set of A $\rightarrow 2^{2n} \rightarrow 1$ (Direction of conversion of numbers with CO).
$\mathbb{N}_{odd} \leftarrow$ Set of A $\leftarrow 1$ (Direction of conversion of numbers with CIO).

All positive odd integers are obtained by repeatedly applying the Collatz inverse operations to each element of the set A and the Collatz numbers generated from these positive odd integers.
Lemma 2.11 If we apply the Collatz inverse operations $(\frac{2^m.a_n-1}{3})$ $(m \in \mathbb{N}^+)$ to the different Collatz numbers, we obtain new Collatz numbers that are all different from each other.

Proof. Let a_1 and a_2 be arbitrary Collatz numbers and $a_1 \neq a_2$, when we apply the Collatz inverse operations to each of them, the resulting numbers are b_1 and b_2. If $b_1 = b_2$ then,

$b_1 = \frac{2^m.a_1-1}{3} = \frac{2^t.a_2-1}{3} = b_2$ then $2^m.a_1 = 2^t.a_2$ for positive odd integers (a_1 and a_2), must be $a_1 = a_2$ and $m = t$ (contradiction), so if $a_1 \neq a_2$ then $b_1 \neq b_2$.

Corollary 2.12 In set theory, the cardinality of a set S represents the number of elements in the set, and is denoted by |S|. The aleph numbers ($\aleph$) indicate the cardinality (size) of well-ordered infinite element sets. $\aleph_0$ is the notation for the cardinality of the set of natural numbers, the next larger cardinality is $\aleph_1$, then $\aleph_2$ and so on. The cardinality of a set is $\aleph_0$ if and only if there is a one-to-one correspondence (bijection) between all elements of the set and all natural numbers. Since there is a one-to-one correspondence between the infinite sets in Figure 1 and the set of natural numbers, the cardinality of each set is $\aleph_0$ [6].
The cardinality of the continuum is $2^{\aleph_0}$. The order and operations between the cardinality of the sets are as follows: $|\mathbb{N}|=\aleph_0$, $\aleph_1=$ cardinality of the "smallest" uncountably infinite sets;

$\aleph_0<\aleph_1<\aleph_2<\ldots$
$\aleph_0+\aleph_0+\aleph_0+\ldots = \aleph_0.\aleph_0=\aleph_0$
$\aleph_0.\aleph_0.\aleph_0=\aleph_0$
$\aleph_0.\aleph_0.\aleph_0\ldots\ \aleph_0.\aleph_0 = \aleph_0^k = \aleph_0$ (k is a finite positive integer)
$\aleph_0.\aleph_0.\aleph_0\ldots = \aleph_0^{\aleph_0}$

The elements of the set A (Lemma 2.5) are the Collatz numbers. We get new Collatz numbers by applying Collatz inverse operations [Equations (4) or (5)] to each element of this set A. From these new infinite Collatz numbers, infinitely many new Collatz numbers are formed by applying the Collatz inverse operations (CIO) again and again, and this goes on endlessly.
As a result, Collatz numbers fill the Hilbert's Hotel (David Hilbert) until there is no empty room left. The Hilbert Hotel is a thought experiment that has a

countable infinity of rooms with room numbers 1, 2, 3, etc., and demonstrates the properties of infinite sets. In this hotel with an infinite number of guests, an infinite number of new guests (even finite layers of infinite) can be accommodated, provided that only one guest stays in each room [5]. When we fill the odd-numbered rooms of the Hilbert Hotel with Collatz numbers, we also fill the entire hotel with Collatz numbers. Let $n \in \mathbb{N}^+$ and $x, y \in \mathbb{N}_{odd}$, and let the odd-numbered rooms of the Hilbert Hotel be 1, 3, 5, 7, ..., i.e. elements of the set $\mathbb{N}_{odd}$. The result of the Collatz inverse operation is the following equation,

$$\frac{2^n.x - 1}{3} = y \tag{7}$$

In Equation (7), n depends on the values of x, ($x \in \mathbb{N}_{odd}$). If $x \equiv 1 \pmod 3$ we replace n with all even numbers n ={2,4,6,8,...}, and if $x \equiv 2 \pmod 3$ we replace n with all odd numbers n ={1,3,5,7,...} respectively (Lemma 2.6 and Lemma 2.8). In (7) we obtain an infinite number of y values as Collatz numbers starting from $x = 1$ (Lemma 2.5). Then, by substituting y values for x in (7), we obtain the Collatz number sets with infinite elements for each y that is not a multiple of 3. [Although we cannot replace x with numbers that are multiples of 3, we get infinite numbers that are multiples of 3 in each Collatz number sets (Figure 1). Because, the numbers in each set give the remainder of 0,1,2 respectively according to (mod 3), as in the $\mathbb{N}_{odd}$ set]. If the same process is repeated and the generated numbers are placed according to the room numbers, there will be no empty rooms left in the Hilbert Hotel. This is because infinite layers of disjoint Collatz number sets[2] are formed without limit by Equation (7), and these sets fill all odd-numbered rooms, i.e. all positive odd integers are obtained (Figure 1). By multiplying these numbers by 2^m ($m \in \mathbb{N}^+$), we find that all even numbers are Collatz numbers (Remark 2.2). Therefore, Collatz numbers fill the Hilbert Hotel and the set of Collatz numbers is equal to the set $\mathbb{N}^+$. Starting with $x = 1$ in (7) and continuing the process to infinity, infinite layers of disjoint Collatz number sets are obtained (Figure 1).

{1}

Y_0 =1*=[{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381, ...}] $|\mathrm{Y}_0|$ =1

Y_1= 1*= [5*= {3,13,53,...} 85*={113,453,1813,...} 341*={227,909,3637,...} 5461*={7281,29125,116501,...} ...] $|\mathrm{Y}_1| = \aleph_0$

Y_2= 1*= [5*={13*={17,69,...} 53*={35,141,...}...} 85*= { 113*={75,301...} 1813*={2417,9669...} ... } ...] $|\mathrm{Y}_2| = \aleph_0+\aleph_0+\aleph_0 \ldots = \aleph_0.\aleph_0 = \aleph_0^2$

[2]Collatz number sets are countably infinite element subsets of the set of positive odd integers.

Y_3= 1*= $\Big[$ 5*={ 13*= {17*={11,45,...} ...} 53*= {35*={23,93,...} ...} ...} 85*={113*={301*={401,1605, ...}...} 1813*={2417*={1611,6445,...}...} ...} ... $\Big]$ $|\mathrm{Y}_3| = \aleph_0.\aleph_0.\aleph_0 = \aleph_0^3$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$ $|\mathrm{Y}| = \aleph_0.\aleph_0.\aleph_0 \ldots = \aleph_0^{k<\omega}$ $(k \in \mathbb{N})$ [3]

The set of disjoint Collatz number sets:
Y= $\Big[$ {1,5,21,...}{3,13,53,...} {113,453,1813,...}{227,909,3637,...} {7281,29125, 116501,...} {17,69,277,...} {35,141,565,...} ... $\Big]$ $|\mathrm{Y}| = \aleph_0.\aleph_0.\aleph_0 \ldots = \aleph_0^{k<\omega}$

Figure 1: Collatz number sets. | | represents the cardinality of the set of Collatz number sets, and * represents conversions of positive odd integers that are not multiples of 3 using Equation (7).

In Figure 1, the infinite layers of Collatz number sets continue to form without restriction until they fill Hilbert's hotel. The restriction occurs only when the hotel is completely filled, that is, when all positive odd numbers are obtained. Imagine buses arriving at Hilbert's Hotel, each carrying an infinite number of passengers. The buses represent disjoint sets, and the passengers represent the elements of these sets. The following buses eventually fill Hilbert's Hotel.

- $\mathrm{Y}_0 = \{1, 5, 21, 85, 341, \ldots\}$ (**0.layer:** infinite people, card. of buses: 1)
- $\mathrm{Y}_1 = \aleph_0$ (**1. layer:** cardinality of buses: $\aleph_0$)
  - $\{(5, 3), (5, 13), (5, 53), \ldots,$
    $(85, 113), (85, 453), \ldots,$
    $(341, 227), (341, 909), \ldots$
    $\ldots \quad \}$ (infinite buses each with infinite people)
- $\mathrm{Y}_2 = \aleph_0 \cdot \aleph_0$ (**2. layer:** cardinality of buses: $\aleph_0.\aleph_0 = \aleph_0^2$)
  - $\{(5, 13, 17), (5, 13, 69), \ldots,$
    $(85, 113, 75), (85, 113, 301), \ldots$
    $\ldots \quad \}$ (infinite ferries, each containing infinite buses, infinite people on each bus)
- $\mathrm{Y}_3 = \aleph_0 \cdot \aleph_0 \cdot \aleph_0$ (**3. layer:** cardinality of buses:$\aleph_0.\aleph_0.\aleph_0 = \aleph_0^3$)
  - $\{(5, 13, 17, 11), (5, 13, 17, 45), \ldots,$
    $(85, 113, 301, 401), (85, 113, 301, 1605), \ldots$
    $\ldots \quad \}$ (infinite oceans with infinite ferries on each, infinite buses on each ferry, infinite people on each bus)

[3] ω is the ordinal number and represents the first infinite ordinal. The ordinal number ω is the smallest element greater than any natural number.

- $\mathrm{Y} = \aleph_0.\aleph_0.\aleph_0\ldots = \aleph_0^{k<\omega}$ (**k.layer:**card. of buses: $\aleph_0.\aleph_0.\aleph_0\ldots = \aleph_0^{k<\omega}$)

Since there are different people on the buses, the buses represent disjoint Collatz number sets. As we move from each layer to the next layer as the number of layers increases, the cardinality of the set of disjoint Collatz number sets increases by a factor of $\aleph_0$, so $\aleph_0.\aleph_0.\aleph_0\ldots = \aleph_0^{k<\omega}$ $(k \in \mathbb{N})$ is the cardinality of the set of all disjoint Collatz sets.

Where $k \in \mathbb{N}$ and $k < \omega$, i.e., k can be any natural number. ω is the ordinal number and represents the first infinite ordinal. The ordinal number ω is the smallest element greater than any natural number. Each disjoint Collatz set forming the layers forms a sequence like the set Y_0 such that $a_n = 4 \cdot a_{n-1} + 1$. The elements of these sequences form a loop with the remainders 0, 1, 2 according to $\pmod 3$. New sets, i.e. new layers, are formed from the elements with the remainders 1 and 2 according to $\pmod 3$. We saw above that there are 0. 1. and 2. layers. Therefore, if there is an n-th layer, there is also an (n+1)th layer. Since all layers are generated inductively, all natural number layers can be generated up to the first infinite ordinal number, ω, i.e.

$$0.\ 1.\ 2.\ 3.\ 4.\ \ldots\ k. < \omega \quad (k \in \mathbb{N})$$

Since the elements of the disjoint Collatz sets are positive odd numbers, layers can be constructed corresponding to the number of elements in the $\mathbb{N}_{odd}$ set. As the layers (k) are determined by induction, it does not matter whether k is defined as a natural number or a positive odd integer; the number of layers is the same regardless of the set used in the inductive definition. The same number of layers are indexed differently.[4] The room numbers at the Hilbert hotel are 1. 3. 5.... $k. < \omega$ $(k \in \mathbb{N}_{odd})$. Since $\aleph_0^{k<\omega}$ is the cardinality of the set of disjoint Collatz sets, from the union of these disjoint sets we get the set of positive odd integers with cardinality $\aleph_0^{k+2<\omega}$. $k+2$ is the first number after k in the ordered set of positive odd integers, and since $k+2 < \omega$ as many numbers are generated as there are elements in the set of positive odd integers. The cardinality $\aleph_0^{k<\omega}$ means that a disjoint Collatz set of cardinality $\aleph_0$ can be created for each room in the hotel. Thus, the cardinality $\aleph_0^{k<\omega}$, $(k \in \mathbb{N}_{\mathrm{odd}})$ contains all layers, all disjoint sets and all numbers that can be contained in the set of positive odd numbers. In this way, the Hilbert hotel is filled by creating countably infinite layers (layers can be generated as many as there are rooms in the hotel). This means that as many guests are sent as there are rooms in the hotel, i.e. one guest for each available room.

[4] Since Collatz number sets are positive odd numbers, k obtained by induction can be defined as a positive odd number. The value k defined for natural numbers is found by induction in the same way for positive odd integers. Thus, layers can be defined as positive odd numbers: 1. 3. 5.... $k. < \omega$ $(k \in \mathbb{N}_{odd})$

In Figure 1, sets of disjoint Collatz sets represent the layers. The numbers in the layers form the sets in the upper layers with CIO, while the sets in the upper layers become the numbers in the lower layers with CO.

$$1 \leftarrow \mathrm{Y}_0 \ \mathrm{Y}_1 \ \underset{\leftarrow \mathrm{CO}}{\overset{\mathrm{CIO}\rightarrow}{\mathrm{Y}_2}} \ \mathrm{Y}_3 \ldots \mathrm{Y}_{k<\omega}$$

$\mathrm{Y}_0 = [\{1, 5, 21, 85, 341, 1365, 5461, \ldots\}] \quad |\mathrm{Y}_0| = 1 \quad$ (0. layer)

$\mathrm{Y}_1 = [5^* = \{3, 13, 53, \ldots\}, 85^* = \{113, 453, 1813, \ldots\}, 341^* = \{227, 909, 3637, \ldots\},$
$5461^* = \{7281, 29125, 116501, \ldots\}, \ldots] \qquad |\mathrm{Y}_1| = \aleph_0 \quad$ (1. layer)

$\mathrm{Y}_2 = [5^* = \{13^* = \{17, 69, \ldots\}, 53^* = \{35, 141, \ldots\}, \ldots\} \quad 85^* = \{113^* = \{75, 301, \ldots\}, 1813^* = \{2417, 9669, \ldots\}, \ldots\}, \ldots] \quad |\mathrm{Y}_2| = \aleph_0^2 \quad$ (2. layer)

$\mathrm{Y}_3 = [5^* = \{13^* = \{17^* = \{11, 45, \ldots\}, \ldots\} \quad 53^* = \{35^* = \{23, 93, \ldots\}, \ldots\}, \ldots\}$
$85^* = \{113^* = \{301^* = \{401, 1605, \ldots\}, \ldots\}, 1813^* = \{2417^* = \{1611, 6445, \ldots\}, \ldots\},$
$\ldots\} \ldots] \quad |\mathrm{Y}_3| = \aleph_0^3 \quad$ (3. layer)

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$|\mathrm{Y}_k| = \aleph_0^{k<w} \quad (k \in \mathbb{N})$$

The reason that the number of layers can be defined as $k < \omega, \ k \in \mathbb{N}$ is that since the set of positive odd integers is bounded from below, the initial layers are available and the other layers are obtained as in the inductive method with Equation 7.

$$\mathrm{CIO} = \frac{2^n \cdot x - 1}{3} \quad \text{and} \quad \mathrm{CO} = \frac{3 \cdot x + 1}{2^n} \quad (x \in \mathbb{N}_{\text{odd}})$$

$$\text{Layers: } 0.\ 1.\ 2.\ \underset{\leftarrow \mathrm{CO}}{\overset{\mathrm{CIO}\rightarrow}{3.\ 4.}}\ 5.\ 6. \ldots k. < \omega \quad (k \in \mathbb{N})$$

Collatz inverse operations (CIO) create sets in the next higher layer from numbers in the lower layer. Collatz operations (Collatz function) convert the sets in the upper layer to the numbers in the previous lower layer. In other words, with the Collatz function, the sets (numbers in sets) in each layer are transformed to the previous layer and then to other layers and finally to layer 0 (set Y_0) and reach 1. Therefore, the sequences of positive odd integers generated by the Collatz function have no initial term and, like the set of positive odd integers, are unbounded from above and bounded by 1 from below. Thus, the positive odd numbers in the k-th layer are transformed by the Collatz function into the numbers in the previous $(k-1)$th layer and so on, until they finally become an element of the set Y_0 (0-th layer) and then become

1. All sequences of positive odd numbers generated by the Collatz function (CO) behave similarly. Every number that is a multiple of 3 in the sets in the layers is connected (transformed) to an element of these sequences and follows the same path as the sequence. Any number that is a multiple of 3 is transformed into the number in the previous layer in the same way as the entire set of which it is an element. Therefore, all sequences of positive odd integers generated by the Collatz function are convergent, i.e., the smallest element of the set of positive odd integers converges to 1. In the next section, we will see that there is no divergent sequence generated by the Collatz function. Let's take a number from the 3rd layer, say 11, and the sequence obtained from this number with the Collatz function is as follows, where the parentheses indicate the number of layers.

$$\begin{array}{c} \text{CO} \rightarrow \\ 11_{(3.)} \rightarrow 17_{(2.)} \rightarrow 13_{(1.)} \rightarrow 5_{(0.)} \rightarrow 1 \end{array}$$

The starting number of this sequence is not $11_{(3.)}$ in layer 3, the starting numbers of all sequences are the numbers in the k-th layer generated with CIO, i.e. positive odd integers in the k-th layer $k < \omega, (\ k \in \mathbb{N})$. When the Collatz inverse operations (CIO) are applied to the numbers forming the sequences, they are sequentially transformed into the numbers of the next higher layer. In this way, divergent sequences are formed with CIO.

$$\begin{array}{c} \text{CO} \rightarrow \\ 9_{(5.)} \rightarrow 7_{(4.)} \rightarrow 11_{(3.)} \rightarrow 17_{(2.)} \rightarrow 13_{(1.)} \rightarrow 5_{(0.)} \rightarrow 1 \end{array}$$

When a number in the sequence that is a multiple of 3, such as $9_{(5)}$, is reached, it can be continued with a number that is not a multiple of 3 from the set $\{$ 9, 37, 149, 597, 2389, $\ldots\}$ generated by $7_{(4.)}$ using CIO, e.g. 37. All numbers in this set are converted to 7 with CO and reach 1 by the same path in the sequence. Continuing with 37, if a multiple of 3 occurs again, the same procedure is repeated and the first term of the sequence becomes a number in the k-th layer. This is true for all sequences generated by the Collatz function.

$$\begin{array}{c} \text{CO} \rightarrow \\ \ldots \rightarrow 65_{(7.)} \rightarrow 49_{(6.)} \rightarrow 37_{(5.)} \rightarrow 7_{(4.)} \rightarrow 11_{(3.)} \rightarrow 17_{(2.)} \rightarrow 13_{(1.)} \rightarrow 5_{(0.)} \rightarrow 1 \end{array}$$

$$\ldots \rightarrow 65_{(7.)} \rightarrow 49_{(6.)} \rightarrow \begin{array}{c} 37_{(5.)} \\ 9_{(5.)} \end{array} \rightarrow 7_{(4.)} \rightarrow 11_{(3.)} \rightarrow 17_{(2.)} \rightarrow 13_{(1.)} \rightarrow 5_{(0.)} \rightarrow 1$$

Since the sequences generated by the Collatz inverse function are divergent, all sequences generated by the Collatz function have no initial terms; they have as many elements as the number of layers, i.e., $k < \omega$, $(k \in \mathbb{N})$. All sequences of positive odd numbers generated by the Collatz function reach 1 starting from the numbers in the k-th layer, so all sequences are convergent.

As can be seen in Figure 1, the cardinality of the set of disjoint sets generated by the CIO from a number that is not a multiple of 3 at any layer is $\aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdots$. To define this expression, it is necessary to have an initial $\aleph_0$. This is possible only if the sequences generated by the Collatz function are convergent. For example, the cardinality of the set of disjoint sets generated by the CIO from 11 in layer 3 is as follows. As the number of layers increases, the cardinality of the sets increases by a factor of $\aleph_0$ (Figure 1).

$|^{11}\mathrm{Y}| = \aleph_0 \cdot \aleph_0 \cdot \aleph_0 \ldots$
$|^{17}\mathrm{Y}| = \aleph_0 \cdot |^{11}\mathrm{Y}|$
$|^{13}\mathrm{Y}| = \aleph_0 \cdot |^{17}\mathrm{Y}|$
$|^{5}\mathrm{Y}| = \aleph_0 \cdot |^{13}\mathrm{Y}|$
$|^{1}\mathrm{Y}| = \aleph_0 \cdot |^{5}\mathrm{Y}| = \aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdots = \aleph_0^{k<\omega} \quad (k \in \mathbb{N})$

The elements of each Collatz number set in Figure 1, obtained by transforming each Collatz number, form a sequence such that each term is 4 times the previous term plus 1. Thus, the elements of each Collatz number set form a loop with remainders 0,1,2 according to $\pmod 3$. New Collatz number sets are generated continuously to infinity from numbers with remainders 1 and 2 according to $\pmod 3$. Therefore, $\aleph_0^0$, $\aleph_0^1$, $\aleph_0^2$ exist (Figure 1), and for $\forall n \in \mathbb{N}^+$, if $\aleph_0^n$ exists, then $\aleph_0^{n+1}$ also exists. Thus, the cardinality of the set of disjoint Collatz number sets in Figure 1 is $\aleph_0.\aleph_0.\aleph_0 \ldots^{\cdot} = \aleph_0^{k<\omega} \quad (k \in \mathbb{N})$. Since all elements of the Collatz number sets form a cycle with the remainders 0,1,2 with respect to $\pmod 3$, all positive odd numbers that are multiples of 3 are obtained from the remainders 0 according to $\pmod 3$.

The elements of the Collatz number sets obtained by Equation 7 form a sequence in which each term is 1 more than 4 times the previous term. The same method is used to cover the set of positive odd integers. From each odd integer in the $\mathbb{N}_{odd}$ set, sets are formed such that the next term is 1 more than 4 times the previous term.

$$\begin{aligned}
p_1 = \quad & \{\mathbf{1}, 5, 21, 85, \ldots\} \\
p_2 = \quad & \{\mathbf{3}, 13, 53, 213, \ldots\} \\
& \{5, 21, 85, 341, \ldots\} \\
p_3 = \quad & \{\mathbf{7}, 29, 117, 469, \ldots\} \\
p_4 = \quad & \{\mathbf{9}, 37, 149, 597, \ldots\} \\
& \vdots
\end{aligned}$$

The union of sets that are disjoint from sets of the form is equal to the set of positive odd integers $\bigcup_{i=1}^{\infty} p_i = \mathbb{N}_{\mathrm{odd}}$. Since other sets are subsets of disjoint sets, they can be ignored.
$\mathbb{N}_{odd}$ =[{1,5,21,85,...}{3,13,53,213,...}{7,29,117,469,...}{9,37,149,597,...} {11,45,181,725,...} ...]
The union of disjoint Collatz number sets obtained in Figure 1 is equal to the set $\mathbb{N}_{odd}$. This is because the cardinality of the family of disjoint Collatz number sets, by the inductive method described above, it was shown in Figure 1 that $\aleph_0^0$, $\aleph_0^1$, $\aleph_0^2$ exist and $\forall k \in \mathbb{N}^+$, if $\aleph_0^k$ exists, then $\aleph_0^{k+1}$ also exists.

The family of disjoint Collatz Number Sets (Figure 1):
Y= [{1,5,21,...}{3,13,53,...}{113,453,1813,...} {227,909,3637,...} {7281, 29125, 116501,...} {17, 69, 277,...} {35, 141, 565,...} ...]

The set Y is equal to the set $\mathbb{N}_{odd}$. The sets Y and $\mathbb{N}_{odd}$ are composed of the same disjoint sets and are equal in number, i.e., they are equal sets. The number of disjoint Collatz sets cannot be less than the number of sets in the set $\mathbb{N}_{odd}$ because, as shown by induction, set formation is continuous, and the number of sets cannot be greater because the Collatz numbers are elements of the set $\mathbb{N}_{odd}$.
The cardinality of the family of disjoint sets in the $\mathbb{N}_{odd}$ set is $\aleph_0$. It was found that the cardinality of the family of disjoint Collatz number sets is $\aleph_0.\aleph_0.\aleph_0\ldots = \aleph_0^{k<\omega}$ $(k \in \mathbb{N}_{odd})$ (Figure 1). Y is the family of all disjoint Collatz sets and $\mathbb{N}_{odd}$ is the family of all disjoint sets.

$$\aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdots = \aleph_0^{k<\omega} = \aleph_0 \quad \text{and} \quad k \in \mathbb{N}_{odd} \implies \mathrm{Y} \supseteq \mathbb{N}_{\mathrm{odd}}$$

Thus the set of Collatz numbers certainly covers its universal set $\mathbb{N}_{odd}$, but cannot exceed it, since the positive odd integer k obtained by induction cannot be equal to ω and Collatz numbers are positive odd numbers. The cardinality $\aleph_0^{k<\omega}$ $(k \in \mathbb{N}_{odd})$ means that as many disjoint Collatz sets can be generated as there are disjoint sets in the set Nodd. This is because $k < \omega$ means that as many disjoint Collatz sets can be generated as there are positive odd numbers. [5] The Collatz number set covers the $\mathbb{N}_{odd}$ set, and since the $\mathbb{N}_{odd}$ set covers the Collatz number set, they are equal. Thus, we find that the set of Collatz numbers is equal to the set $\mathbb{N}^+$ (Remark 2.2). In the following section, it will be proven that there are no loops outside the trivial loop and that the sets generated in Figure 1 are disjoint sets.

[5]The expression $\aleph_0^{k<\omega}$ $(k \in \mathbb{N}_{odd})$, the cardinality of the family of disjoint Collatz number sets, implies that as many as k disjoint sets can be generated, and since k can be all positive odd numbers, it covers all disjoint sets that can exist in the set $\mathbb{N}_{odd}$ generated by the given rule.

3 The Absence of any Positive Integer other than Collatz Numbers

In this section, we prove that there are no positive integers that do not satisfy the conjecture.

Lemma 3.1 According to the Collatz mapping, the only cycle in the set of positive odd integers is the trivial cycle $\{1\}$.
Proof. Suppose $a_1, a_2, \ldots, a_k$ are positive odd integers forming a cycle under the Collatz mapping (i.e., $a_1, a_2, \ldots, a_k, a_1, a_2, \ldots$). Starting from any positive odd integer, a_1, suppose the sequence returns to a_1 at step k. For each $i = 1, \ldots, k$, the Collatz operation yields

$$a_{i+1} = \frac{3a_i + 1}{2^{r_i}},$$

where $r_i \geq 1$ is chosen so that a_{i+1} is odd.
General loop equation with Collatz operations;

$$\frac{3a_1 + 1}{2^{r_1}} = a_2, \quad \frac{3a_2 + 1}{2^{r_2}} = \frac{3^2 a_1 + 3 + 2^{r_1}}{2^{r_1 + r_2}} = a_3 \quad \text{... and so on,}$$

$$\frac{3^k a_1 + 3^{k-1} + 3^{k-2} \cdot 2^{r_1} + 3^{k-3} \cdot 2^{r_1 + r_2} + 3^{k-4} \cdot 2^{r_1 + r_2 + r_3} + \cdots + 2^{r_1 + r_2 + r_3 + \cdots + r_{k-1}}}{2^{r_1 + r_2 + r_3 + \cdots + r_k}} = a_1$$

is obtained.

$$\frac{3^k a_2 + 3^{k-1} + 3^{k-2} \cdot 2^{r_2} + 3^{k-3} \cdot 2^{r_2 + r_3} + 3^{k-4} \cdot 2^{r_2 + r_3 + r_4} + \cdots + 2^{r_2 + r_3 + r_4 + \cdots + r_k}}{2^{r_1 + r_2 + r_3 + \cdots + r_k}} = a_2$$

$$\vdots$$

$$\frac{3^k a_k + 3^{k-1} + 3^{k-2} \cdot 2^{r_k} + 3^{k-3} \cdot 2^{r_k + r_1} + 3^{k-4} \cdot 2^{r_k + r_1 + r_2} + \cdots + 2^{r_k + r_1 + r_2 + \cdots + r_{k-2}}}{2^{r_1 + r_2 + r_3 + \cdots + r_k}} = a_k.$$

Generalizing these equations, the loop terms are found as follows,

$$\frac{3^k a_i + 3^{k-1} + 3^{k-2} \cdot 2^{r_i} + 3^{k-3} \cdot 2^{r_i + r_{i+1}} + 3^{k-4} \cdot 2^{r_i + r_{i+1} + r_{i+2}} + \cdots + 2^{r_i + r_{i+1} + \cdots + r_{i+k-2}}}{2^{r_1 + r_2 + r_3 + \cdots + r_k}} = a_i.$$

$$a_i = \frac{3^{k-1}+3^{k-2}\cdot 2^{r_i}+3^{k-3}\cdot 2^{r_i+r_{i+1}}+3^{k-4}\cdot 2^{r_i+r_{i+1}+r_{i+2}}+\ldots+2^{r_i+r_{i+1}+\cdots+r_{k-2}}}{2^{r_1+r_2+\cdots+r_k}-3^k} \tag{8}$$

where $r_{k+1} = r_1$ and $a_{k+1} = a_1$.

In the notation employed here, the expression *i-indexed loop* denotes all cyclic sequences derived from the elements of a given sequence, where the index i satisfies $1 \leq i \leq k$ with $k \in \mathbb{Z}^+$. For instance, the loop $\{a_i\}_{1\leq i\leq k}$ consists of cyclic sequences constructed from the elements of $\{a_1, a_2, a_3, \ldots, a_k\}$. These cyclic sequences can be explicitly written as

$$\begin{aligned}
&a_1,\ a_2,\ a_3,\ \ldots,\ a_k,\ a_1\\
&a_2,\ a_3,\ \ldots,\ a_k,\ a_1,\ a_2\\
&\vdots\\
&a_k,\ a_1,\ a_2,\ \ldots,\ a_{k-1},\ a_k.
\end{aligned}$$

Thus, the notion of an i-indexed loop formally represents the cyclic permutations of the base sequence, with each loop uniquely determined by its starting index i. $\{a_i\}_{1\leq i\leq k}$ loop expression will be used to mean $\{a_i\}_{1\leq i\leq k} = \{(a_i, a_{i+1}, \ldots, a_{i+k}) \mid i = 1, \ldots, k\}$, where indices are taken mod k (i.e. $a_{i+k} = a_i$).
To determine whether any positive odd integer (a) can be equal to itself after k steps, the following three cases must be examined.

$$\begin{aligned}
&\text{Case I:} \quad \textstyle\sum_{i=1}^{k} r_i = 2k,\\
&\text{Case II:} \quad \textstyle\sum_{i=1}^{k} r_i > 2k,\\
&\text{Case III:} \ k \leq \textstyle\sum_{i=1}^{k} r_i < 2k.
\end{aligned}$$

Case I. $\sum_{i=1}^{k} r_i = 2k$,

$$a_i = \frac{3^{k-1}+3^{k-2}\cdot 2^{r_i}+3^{k-3}\cdot 2^{r_i+r_{i+1}}+3^{k-4}\cdot 2^{r_i+r_{i+1}+r_{i+2}}+\cdots+2^{r_i+r_{i+1}+\cdots+r_{i+k-2}}}{2^{2k}-3^k}.$$

In case I, in the general loop equation, $a_i = 1$ when $r_1 = r_2 = r_3 = \cdots = r_k = 2$.
If $r_1 = r_2 = r_3 = \cdots = r_k = 2$,

$$a_i = \frac{3^{k-1}+3^{k-2}\cdot 2^2+3^{k-3}\cdot 2^4+3^{k-4}\cdot 2^6+\cdots+2^{2k-2}}{2^{2k}-3^k}.$$

When the denominator of a_i is factored,

$$2^{2k}-3^k = (2^2-3)\Big(3^{k-1}+3^{k-2}\cdot 2^2+3^{k-3}\cdot 2^4+3^{k-4}\cdot 2^6+\cdots+2^{2k-2}\Big).$$

Let us call the condition $r_i = 2, \forall i \in \{1,\ldots,k\}$, the equilibrium state and denote the a_i values at equilibrium state as a_d, $a_1 = a_2 = \cdots = a_k = a_d$;

$$a_i = a_d = \frac{3^{k-1} + 3^{k-2}\cdot 2^2 + 3^{k-3}\cdot 2^4 + 3^{k-4}\cdot 2^6 + \cdots + 2^{2k-2}}{(2^2-3)\Big(3^{k-1} + 3^{k-2}\cdot 2^2 + 3^{k-3}\cdot 2^4 + 3^{k-4}\cdot 2^6 + \cdots + 2^{2k-2}\Big)} = 1.$$

In (8), if $\sum_{i=1}^{k} r_i = 2k$, then only in the equilibrium state, i.e. when $r_i = 2$, there exists a positive odd integer value $a_d = 1$ and no other positive odd integer value, because if some of the r_i values are different from 2, then the terms of the sequence $(r_i)_{i=1}^{k}$ are;

$$r_1 = 2 + d_1, \quad r_2 = 2 + d_2, \quad r_3 = 2 + d_3, \quad \ldots, \quad r_k = 2 + d_k,$$

where $d_1, d_2, d_3, \ldots, d_k \in \mathbb{Z}$ and $(2 + d_i \neq 0)$. Since

$$\sum_{i=1}^{k} r_i = 2k,$$

it follows that $\sum_{i=1}^{k} d_i = 0$. Except for the case $r_i = 2$, among the terms of the a_i cycle generated by $\{\, (r_i)_{i=1}^{k} \,\}$, there is at least one a_f such that $\exists f \in \{1,2,\ldots,k\} : 0 < a_f < 1$ in each cycle. In the loop $\{a_i\}_{1\le i\le k}$, if a term a_f lies in the interval $0 < a_f < 1$, then it is not an integer. Within the cycle $(a_1, a_2, \ldots, a_k, a_1, a_2, \ldots)$, if any term fails to be a positive odd integer, then no such cycle exists among the positive odd integers. Consequently, all terms of the cycle are non-integral rational numbers. The reason at least one a_f term lies in the interval $0 < a_f < 1$ in each cycle is proven as follows.

Since the values of r_i outside the equilibrium state are

$$r_1 = 2 + d_1, \quad r_2 = 2 + d_2, \quad r_3 = 2 + d_3, \quad \ldots, \quad r_k = 2 + d_k, \quad (d_i \in \mathbb{Z}),$$

the cycle terms outside the equilibrium state are;

$$a_i = \frac{3^{k-1} + 3^{k-2}\cdot 2^{(2+d_i)} + 3^{k-3}\cdot 2^{(4+d_i+d_{i+1})} + + \cdots + 2^{(2k-2+d_i+d_{i+1}+d_{i+2}+\cdots+d_{i+k-2})}}{2^{2k} - 3^k} \tag{9}$$

where $d_{k+1} = d_1$.

In the $\{a_i\}_{1\le i\le k}$ loop, which consists of the sequences $(r_i)_{i=1}^{k}$ whose sum is $2k$ outside the equilibrium state, at least one term $(\exists f \in \{1,2,\ldots,k\} : 0 < a_f < 1)$ is strictly smaller than the value in the equilibrium state, $a_d = 1$. In the equilibrium state, the values of a_i are

$$a_d = \frac{3^{k-1} + 3^{k-2} \cdot 2^2 + 3^{k-3} \cdot 2^4 + 3^{k-4} \cdot 2^6 + \cdots + 2^{2k-2}}{2^{2k} - 3^k}$$

In equation (9), the non-equilibrium loop equation, the values of r in the $\{a_i\}_{1 \le i \le k}$ loop are,

$$r_1 = 2 + d_1, \quad r_1 + r_2 = 4 + d_1 + d_2, \quad \ldots, \quad r_1 + r_2 + \cdots + r_k = 2k + d_1 + d_2 + \cdots + d_k.$$

The partial sums of d_i added to the r values in the non-equilibrium $\{a_i\}_{1 \le i \le k}$ loop are,

$$s_1 = d_1, \quad s_2 = d_1 + d_2, \quad s_3 = d_1 + d_2 + d_3, \quad \ldots, \quad s_k = d_1 + d_2 + \cdots + d_k.$$

If,

$$M = \max\{s_1 = d_1,\ s_2 = d_1 + d_2,\ s_3 = d_1 + d_2 + d_3,\ \ldots,\ s_k = d_1 + d_2 + \cdots + d_k\} = s_j$$

$(1 \le j < k,\ j \in \mathbb{Z}^+)$, then when the loop sequence is rotated after the jth step, the partial sums d_i added to the r values in the numerator of the a_{j+1} term in the loop $\{a_i\}_{1 \le i \le k}$ are,

$$\{\, s_{j+1} = d_{j+1},\ s_{j+2} = d_{j+1} + d_{j+2},\ \ldots,\ s_{j+n} = d_{j+1} + d_{j+2} + d_{j+3} + \cdots + d_{j+k-1} \,\}.$$

Since $s_{j+n} - M < 0$ (for every $n = 1, 2, 3, \ldots, k-1$) is negative, all elements of the set of partial sums

$$\{\, d_{j+1},\ d_{j+1} + d_{j+2},\ d_{j+1} + d_{j+2} + d_{j+3} + \cdots + d_{j+k-1} \,\}$$

are negative integers.

$$a_{j+1} = \frac{3^{k-1} + 3^{k-2} \cdot 2^{2+d_{j+1}} + 3^{k-3} \cdot 2^{4+d_{j+1}+d_{j+2}} + \cdots + 2^{2k-2+d_{j+1}+d_{j+2}+d_{j+3}+\cdots+d_{j+k-1}}}{2^{2k} - 3^k}$$

In the non-equilibrium state, the sum of the d_i values added to $r_i = 2$ in the equilibrium state is negative (i.e., $s_{j+1} < 0,\ s_{j+2} < 0,\ \ldots,\ s_{j+k-1} < 0$). Therefore, a_{j+1} is strictly less than $a_d = 1$, i.e. $0 < a_{j+1} < 1$. Also,

$$M = \max\{n_1 = d_1,\ n_2 = d_1 + d_2,\ \ldots,\ n_k = d_1 + d_2 + \cdots + d_k\} = n_{j_1} = n_{j_2} = \cdots = n_{j_p},$$

if there is more than one maximum of the form

$$1 \le j_1 < j_2 < j_3 < \cdots < j_p = t < k,$$

when the loop is rotated after step t, the new partial sums

$$\{\, d_{t+1},\ d_{t+1}+d_{t+2},\ d_{t+1}+d_{t+2}+d_{t+3},\ \ldots,\ d_{t+1}+d_{t+2}+\cdots+d_{t+k-1}\,\}$$

become zero and negative due to subtraction from M, that is, $s_{t+n}-M \leq 0 \quad (\text{for each } n=1,2,3,\ldots,k-1)$.

Since there cannot be as many maxima as there are terms, not all partial sums can be zero, some must be negative integers. Therefore, the set of partial sums

$$\{\, d_{t+1},\ d_{t+1}+d_{t+2},\ d_{t+1}+d_{t+2}+d_{t+3},\ \ldots,\ d_{t+1}+d_{t+2}+\cdots+d_{t+k-1}\,\}$$

added to the $r_i=2$ values of the term a_{t+1} in the loop $\{a_i\}_{1\leq i\leq k}$ are zero and negative integers. Therefore, the numerator of a_{t+1} is strictly smaller than the numerator of a_d, and since their denominators are equal, $a_{t+1}<a_d$ and $a_d=1$, that is, $0<a_{t+1}<1$. If any term in the $\{a_i\}_{1\leq i\leq k}$ loop (i.e., a_t, $1\leq t\leq k$) is not a positive integer, then none of the terms can be positive integers.

Therefore, none of the a_i values in equation (9) can be integers. Consequently, the elements of the a_i loop consisting of any sequence $(r_i)_{i=1}^{k}$, except the $r_i=2$ (i.e. $a_i=1$) sequence, cannot be integers. In the case of $\sum_{i=1}^{k} r_i = 2k$, there is no cycle except 1 in positive odd integers. Therefore, the only solution for all a_i loops consisting of ordered sequences $(r_i)_{i=1}^{k}$ such that $\sum_{i=1}^{k} r_i = 2k$ is $r_i=2$ and $a_i=1$. For all other $(r_i)_{i=1}^{k}$ sequences, none of the terms a_i are integers because at least one a_t term in each loop is less than 1 $(\exists t\in\{1,2,\ldots,k\} : 0<a_t<1)$, hence a_i is not an integer. Consequently, under the condition $\sum_{i=1}^{k} r_i = 2k$, the set of positive odd integers has no cycles except 1.

Case II $\quad \sum_{i=1}^{k} r_i > 2k$.

The existence of a cycle consisting of positive odd integers

$$a_{n1},\ a_{n2},\ a_{n3},\ \ldots,\ a_{nk},\ a_{n1},\ a_{n2},\ \ldots,$$

under the condition $\sum_{i=1}^{k} r_i = 2k+n \quad (n\in\mathbb{Z}^+)$ will be examined. In the sum $\sum_{i=1}^{k} r_i = 2k+n$, first $n=1$ is taken. In Case I, by adding 1 in order to each term of all $(r_i)_{i=1}^{k}$ sequences whose sum is $\sum_{i=1}^{k} r_i = 2k$ and which generate the cycle $a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_k,\ a_1,\ a_2,\ \ldots,$ all $(r_i)_{i=1}^{k}$ sequences whose sum is $\sum_{i=1}^{k} r_i = 2k+1$ are obtained. If the sum of the ordered sequence $(r_1,r_2,r_3,r_4,\ldots,r_k)$ is $2k$, adding 1 to each term in turn yields $(r_1+1,\ r_2,\ r_3,\ r_4,\ \ldots,\ r_k)$, $\quad (r_1, r_2{+}1,\ r_3,\ r_4,\ \ldots,\ r_k)$, $\quad (r_1,r_2,r_3{+}1,\ r_4,\ldots,r_k),\ldots$ all ordered sequences with sum $2k+1$.

Let $A = \{(r_1, r_2, \ldots, r_k) \in \mathbb{Z}^k : r_1+r_2+\cdots+r_k = 2k\}$ and $B = \{(s_1, s_2, \ldots, s_k) \in \mathbb{Z}^k : s_1 + s_2 + \cdots + s_k = 2k + 1\}$. Then every sequence in B can be obtained by adding 1 to exactly one coordinate of a sequence in A. In other words, $B = \bigcup_{i=1}^{k} \{(r_1, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_k) : (r_1, \ldots, r_k) \in A\}$.

Let the cyclic sequence corresponding to a sequence $(r_i)_{i=1}^k$ with $\sum_{i=1}^k r_i = 2k$ be denoted by $a_1,\ a_2,\ a_3,\ \ldots,\ a_k,\ a_1,\ a_2,\ \ldots$. If we add 1 to any term of r (for example, if we replace r_1 with r_1+1), the sum becomes $(r_1+1)+r_2+\cdots+r_k = 2k+1$, and let the new cyclic sequence is denoted by $a_{11},\ a_{12},\ a_{13},\ \ldots,\ a_{1k},\ a_{11},\ a_{12},\ \ldots$. When the elements of these two loops are compared in order, the result is as follows: $a_{11} < a_1, \quad a_{12} < a_2, \quad a_{13} < a_3, \quad \ldots, \quad a_{1k} < a_k$. The reason for this is that:

$$a_1 = \frac{3^{k-1} + \left(3^{k-2} \cdot 2^{r_1} + 3^{k-3} \cdot 2^{r_1+r_2} + 3^{k-4} \cdot 2^{r_1+r_2+r_3} + \cdots + 2^{r_1+r_2+\cdots+r_{k-1}}\right)}{2^{2k} - 3^k} = \frac{N_1}{D_1},$$

$$\begin{aligned} a_{11} &= \frac{3^{k-1} + 2\left(3^{k-2} \cdot 2^{r_1} + 3^{k-3} \cdot 2^{r_1+r_2} + 3^{k-4} \cdot 2^{r_1+r_2+r_3} + \cdots + 2^{r_1+r_2+\cdots+r_{k-1}}\right)}{2^{2k+1} - 3^k} \\ &= \frac{2.\left(\frac{3^{k-1}}{2} + 3^{k-2} \cdot 2^{r_1} + 3^{k-3} \cdot 2^{r_1+r_2} + 3^{k-4} \cdot 2^{r_1+r_2+r_3} + \cdots + 2^{r_1+r_2+\cdots+r_{k-1}}\right)}{2.\left(2^{2k} - \frac{3^k}{2}\right)} = \frac{N_{11}}{D_{11}}. \end{aligned}$$

Since $N_{11} < 2N_1$ and $D_{11} > 2D_1$, it follows that $a_{11} < a_1$. Similarly,

$$a_2 = \frac{3^{k-1} + \left(3^{k-2} \cdot 2^{r_2} + 3^{k-3} \cdot 2^{r_2+r_3} + 3^{k-4} \cdot 2^{r_2+r_3+r_4} + \cdots + 2^{r_2+r_3+\cdots+r_k}\right)}{2^{2k} - 3^k} = \frac{N_2}{D_2},$$

$$a_{12} = \frac{3^{k-1} + \left(3^{k-2} \cdot 2^{r_2} + 3^{k-3} \cdot 2^{r_2+r_3} + 3^{k-4} \cdot 2^{r_2+r_3+r_4} + \cdots + 2^{r_2+r_3+\cdots+r_k}\right)}{2^{2k+1} - 3^k} = \frac{N_{12}}{D_{12}}.$$

Since $N_{12} < 2N_2$ and $D_{12} > 2D_2$, then $a_{12} < a_2$. Similarly, in all other cases, $a_{1i} < a_i$, since $N_{1i} < 2N_i$ and $D_{1i} > 2D_i$. The same applies to loop elements consisting of other $(r_i)_{i=1}^k$ sequences whose sum is $2k + 1$. Therefore, since $a_{1i} < a_i$ in the cycles formed from all $(r_i)_{i=1}^k$ sequences satisfying the condition $\sum_{i=1}^k r_i = 2k + 1$, and since at least one element $0 < a_y < 1$ in a_i terms in case I, there is at least one a_{1y} in every cycle a_{1i} such that $(\exists y \in \{1, 2, \ldots, k\} : 0 < a_{1y} < 1)$.

Also, the equilibrium state $a_d = 1$ corresponds to $r_i = 2$. The $\{a_i\}_{1\le i\le k}$ cycle generated by adding 1 to any r value in the equilibrium state ($r_i = 2$) is $a_{1i} < a_d$, or that is, $0 < a_{1i} < 1$, since $N_{1i} < 2N_d$ and $D_{1i} > 2D_d$. Therefore, as in the a_i loop in case I, at least one term in each a_{1i} loop lies in the interval $(0, 1)$. Thus, it is $a_{1i} \notin \mathbb{Z}^+$, and no positive odd integer loop can exist.

Similarly, when the same method is applied for $n = 2$,

$$\sum_{i=1}^{k} r_i = 2k + 2,$$

that is, if the sum of the ordered sequence $(r_1,\, r_2,\, r_3,\, r_4,\, \ldots,\, r_k)$ is $2k + 1$, then when 1 is added to each term in this sequence in order,

$((r_1 + 1),\, r_2,\, r_3,\, r_4,\, \ldots,\, r_k),\ (r_1,\, (r_2 + 1),\, r_3,\, r_4,\, \ldots,\, r_k),\ (r_1,\, r_2,\, (r_3 + 1),$
$r_4,\, \ldots,\, r_k),\ \ldots,$
all ordered sequences whose sum is $2k + 2$ are obtained. Let the cycles formed by the sequence $(r_i)_{i=1}^k$, whose sum is $2k + 2$, be shown as

$$a_{21},\, a_{22},\, a_{23},\, \ldots,\, a_{2k},\, a_{21},\, a_{22},\, \ldots.$$

Using the method established above that yields $a_{1i} < a_i$, when compared with the elements of the cycle formed by the sequence $(r_i)_{i=1}^k$, whose sum is $2k+1$, $a_{2i} < a_{1i}$ is obtained. Since at least one of the elements of the loop a_{1i} is such that $0 < a_{1y} < 1$, and $a_{2i} < a_{1i}$, at least one of the elements in the loop a_{2i} is $\exists y \in \{1, 2, \ldots, k\} : 0 < a_{2y} < 1$. Since at least one term in the loop a_{2i} is less than 1, $a_{2i} \notin \mathbb{Z}^+$ and the loop a_{2i} cannot exist in the set of positive odd integers.

Continuing in this way, for all $(r_i)_{i=1}^k$ sequences with sum

$$(r_1 + \ldots + r_j + \ldots + r_k) = 2k + n,$$

when 1 is added to r_j $(1 \le j \le k)$, all $(r_i)_{i=1}^k$ sequences with sum $(r_1 + \ldots + (r_j + 1) + \ldots + r_k) = 2k + (n + 1)$ are obtained. Since at least one term in the loop a_{ni} is $(\exists y \in \{1, 2, \ldots, k\} : 0 < a_{ny} < 1)$ and $a_{(n+1)i} < a_{(n)i}$, at least one term in the loop $a_{(n+1)i}$ is $\exists y \in \{1, 2, \ldots, k\} : 0 < a_{(n+1)y} < 1$. Therefore, the positive odd integer loop $a_{(n+1)i}$ cannot exist. Therefore, since at least one term lies in the interval $0 < a_{my} < 1$ in every cycle a_{mi} generated by all $(r_i)_{i=1}^k$ sequences whose sum is $\sum_{i=1}^k r_i = 2k+m \quad (m \in \mathbb{Z}^+)$, a_{mi} cannot be a positive odd integer and there can be no cycle a_{mi} $(1 \le i \le k)$ of positive odd integers. Consequently, there is no cycle $\{a_i\}_{1\le i\le k}$ of positive odd integers generated by all sequences $(r_i)_{i=1}^k$ whose sum is $\sum_{i=1}^k r_i = 2k+m$ $(m \in \mathbb{Z}^+)$, so the a_i terms cannot be positive integers.

Case III: $k \leq \sum_{i=1}^{k} r_i < 2k$;
Under the condition $k \leq \sum_{i=1}^{k} r_i < 2k$, i.e., $\sum_{i=1}^{k} r_i = 2k + m \quad (m \in \mathbb{Z}^-)$, can a cyclic sequence be obtained from the set of positive odd integers $a_1, a_2, a_3, \ldots, a_k, a_1, a_2, \ldots$? For $\sum_{i=1}^{k} r_i = 2k + m \quad (m \in \mathbb{Z}^-)$,

$$a_i = \frac{3^{k-1} + 3^{k-2} \cdot 2^{r_i} + 3^{k-3} \cdot 2^{r_i + r_{i+1}} + 3^{k-4} \cdot 2^{r_i + r_{i+1} + r_{i+2}} + \cdots + 2^{r_i + r_{i+1} + \cdots + r_{k-1}}}{2^{r_1 + r_2 + \cdots + r_k} - 3^k}$$

According to the results obtained from Cases I and II, under the condition $\sum_{i=1}^{k} r_i \geq 2k$, there is no cycle $\{a_i\}_{1 \leq i \leq k}$ in the set of positive odd numbers except for the equilibrium state $(r_i = 2,\ a_i = 1)$. If any term a_f $(1 \leq f \leq k)$ in the $\{a_i\}_{1 \leq i \leq k}$ loop is not an integer, then none of the terms are integers.

Let $T_i = 3^{k-2} \cdot 2^{r_i} + 3^{k-3} \cdot 2^{r_i + r_{i+1}} + 3^{k-4} \cdot 2^{r_i + r_{i+1} + r_{i+2}} + \cdots + 2^{r_i + r_{i+1} + r_{i+2} + \cdots + r_{i+k-1}}$.
In case I, the loop equation is

$$a_i = \frac{3^{k-1} + T_i}{2^{2k} - 3^k}.$$

The first term of the loop is

$$a_1 = \frac{3^{k-1} + T_1}{2^{2k} - 3^k},$$

where $T_1 = 3^{k-2} \cdot 2^{r_1} + 3^{k-3} \cdot 2^{r_1 + r_2} + 3^{k-4} \cdot 2^{r_1 + r_2 + r_3} + \cdots + 2^{r_1 + r_2 + r_3 + \cdots + r_{k-1}}$.

For $\left\{ (r_j)_{j=1}^{k} \,\middle|\, \sum_{j=1}^{k} r_j = 2k \right\}$, adding m $(m \in \mathbb{Z}^-)$ to the first term (i.e. $r_1 + m$ where $r_1 + m > 0$) yields $\{ (r_j)_{j=1}^{k} \mid (r_1 + m) + r_2 + \cdots + r_k = 2k + m,\ (m \in \mathbb{Z}^-)\}$.

Let $A = \{(r_1, r_2, \ldots, r_k) \in \mathbb{Z}^k \mid r_1 + r_2 + \cdots + r_k = 2k\}$ and $B = \{(s_1, s_2, \ldots, s_k) \in \mathbb{Z}^k \mid s_1 + s_2 + \cdots + s_k = 2k + m\}$ By adding m $(m \in \mathbb{Z}^-)$ to the first coordinate of every sequence in A, we obtain all sequences in B. That is, $B = \{(r_1 + m,\ r_2,\ \ldots, r_k) : (r_1,\ r_2,\ \ldots, r_k) \in A\}$.

Therefore, the first term of the loop equations for Case III ($k \leq \sum_{i=1}^{k} r_i < 2k$) is

$$a_1 = \frac{3^{k-1} + 2^m \cdot T_1}{2^m \cdot 2^{2k} - 3^k} \quad (m < 0).$$

In Case II, it was shown that a_1 cannot be a positive integer for $m > 0$ in the equation

$$a_1 = \frac{3^{k-1} + 2^m \cdot T_1}{2^m \cdot 2^{2k} - 3^k}\,.$$

Can a_1 be a positive integer when $m < 0$ in the equation $a_1 = \frac{3^{k-1}+2^m \cdot T_1}{2^m \cdot 2^{2k}-3^k} = ?$
In this section it is proven that, in the a_i loop

$$a_1 = \frac{3^{k-1} + 2^m \cdot T_1}{2^m \cdot 2^{2k} - 3^k}\,,$$

since a_1 cannot be a positive integer when $m > 0$, then it cannot be a positive integer when $m < 0$.

2- and 3-adic valuations. Since $r_1 + m > 0$ ($m \in \mathbb{Z}$), $(2^m \cdot T_1)$ is an integer,

$$a_1 = \frac{3^{k-1} + 2^m \cdot T_1}{2^m \cdot 2^{2k} - 3^k} = \frac{N_1(m)}{D_1(m)}\,,$$

then,

$$\begin{aligned}
D_1(m) &= (\text{even}) - (\text{odd}) = \text{odd} \;\Rightarrow\; v_2\big(D_1(m)\big) = 0,\\
N_1(m) &= (\text{odd}) + (\text{even}) = \text{odd} \;\Rightarrow\; v_2\big(N_1(m)\big) = 0,\\
D_1(m) &= 2^{m+2k} - 3^{2k} \equiv \pm 1 \pmod 3 \;\Rightarrow\; v_3\big(D_1(m)\big) = 0,\\
N_1(m) &= 3^{k-1} + 2^m T_1 \equiv \pm 1 \pmod 3 \;\Rightarrow\; v_3\big(N_1(m)\big) = 0.
\end{aligned}$$

Therefore, the 2-adic and 3-adic valuations of the numerator and denominator of a_1 are independent of the sign of m. Since the 2-adic and 3-adic values of both the numerator and denominator of a_1 in the cyclic sequence $\{a_i\}_{1 \le i \le k}$ are $v_2(N_1(m)) = 0$ and $v_2(D_1(m)) = 0$, $v_3(N_1(m)) = 0$ and $v_3(D_1(m)) = 0$, these expressions ($N_1(m)$ and $D_1(m)$) cannot be divided by 2 or 3.

$$a_1(m) = \frac{N_1(m)}{D_1(m)},$$

where $N_1(m) = 3^{k-1} + 2^m T_1$ and $D_1(m) = 2^{m+2k} - 3^{2k}$.
The previous analysis (Case II) shows that for every positive integer $m \in \mathbb{Z}^+$, $a_1(m) \notin \mathbb{Z}$.

The existence of this defect requires the presence of an m-specific prime power such that, for every fixed $m > 0$, the denominator $D_1(m)$ has a greater p-adic valuation than the numerator $N_1(m)$:

$\forall m > 0,\ \exists q_m = p_m^{s_m} \quad \big(p_m \text{ is prime},\ p_m > 3,\ s_m \in \mathbb{Z}_{\ge 1}\big)$ such that

$v_{p_m}(D_1(m)) > v_{p_m}(N_1(m))$.

This inequality means $q_m \mid D_1(m)$ and $q_m \nmid N_1(m)$; therefore, for that m, $a_1(m)$ cannot be an integer.

Periodicity. Since the base prime of q_m is $p_m > 3$, we have $\gcd(2, q_m) = 1$. Therefore, 2 is invertible modulo q_m, and $v_{q_m}(2) = 0$. This invertibility ensures that the set

$$\Lambda_{q_m} = \{2^t \pmod{q_m} : t \in \mathbb{Z}\}$$

forms a cyclic subgroup within the modular multiplicative group $(\mathbb{Z}/q_m\mathbb{Z})^\times$. Because the cyclic group is closed and its inverses remain in the group, elements obtained with positive and negative exponents lie in the same set:

$$\{2^m \pmod{q_m} : m \geq 0\} = \Lambda_{q_m} = \{2^m \pmod{q_m} : m < 0\}.$$

This equivalence shows that the value of 2^m modulo q_m is independent of the sign of m and is periodic.

Let $L_{q_m} = \mathrm{ord}_{q_m}(2)$ (the order of 2 modulo q_m), $2^{L_{q_m}} \equiv 1 \pmod{q_m}$. For every $t \in \mathbb{Z}$, $m' := m + tL_{q_m}$ yields:

$$2^{m'} \equiv 2^m \pmod{q_m}.$$

Accordingly, the modular expansion of the numerator and denominator expressions is:

$$D_1(m') = 2^{m'+2k} - 3^{2k} = 2^{m+2k} \cdot 2^{tL_{q_m}} - 3^{2k} \equiv 2^{m+2k} \cdot 1 - 3^{2k} = D_1(m) \pmod{q_m},$$

$$N_1(m') = 3^{k-1} + 2^{m'}T_1 = 3^{k-1} + 2^m \cdot 1 \cdot T_1 \equiv N_1(m) \pmod{q_m}.$$

This modular preservation ensures that the defect conditions initially satisfied for m ($q_m \mid D_1(m)$ and $q_m \nmid N_1(m)$) are also preserved for all $m' \in \mathbb{Z}$ along the residue class $m' \equiv m \pmod{L_{q_m}}$. Thus, the inequality $v_p(D_1(m')) > v_p(N_1(m'))$ continues to hold, and we conclude that $a_1(m') \notin \mathbb{Z}$.

According to Case II (for every $m > 0$ there exists at least one q_m) and the principle of modular preservation, every positive integer $m > 0$ is represented by at least one pair (m_i, q_i) in the defining family $I = \{(m_i, q_i)\}_{i \in K}$. Here, $K \subseteq \mathbb{Z}^+$ is the index set for the family, meaning that the indices i are positive integers and $q_i = p_i^{s_i} \quad \big(p_i \text{ is prime, } p_i > 3,\ s_i \in \mathbb{Z}_{\geq 1}\big)$.

That is, for every $m \in \mathbb{Z}^+$, there exists at least one index $i \in K$ such that $m \equiv m_i \pmod{L_{q_i}}$. Consequently, for all $m > 0$ this implies that

$$2^m \equiv 2^{m_i} \pmod{q_i}, \quad m_i > 0.$$

The family I is defined as a **covering system of congruences**, denoted by

$$C = \{\, m_i \pmod{L_{q_i}} : i \in K \,\},$$

such that the union of the residue classes $m_i + L_{q_i}\mathbb{Z}$ covers every positive integer $m \in \mathbb{Z}^+$. When defining the covering family, for every $m > 0$ there exists a prime power $q_i = p_i^{s_i}$ such that

$$2^m \equiv 2^{m_i} \pmod{q_i}.$$

The standard interval for the representatives is

$$0 \leq m_i \leq L_{q_i} - 1,$$

where L_{q_i} denotes the order of 2 modulo q_i. However, in constructing the positive covering family the lower bound $m_i = 0$ can be excluded. This is because $m_i = 0$ and $m_i = L_{q_i}$ represent the same residue class, and the congruence

$$2^{L_{q_i}} \equiv 2^0 \pmod{q_i}$$

holds. Therefore, for positive coverage the representative interval can be redefined as $1 \leq m_i \leq L_{q_i}$.

In this way, only positive representatives (m_i) are used while ensuring that all positive integers (m) are covered $(2^m \equiv 2^{m_i} \pmod{q_i})$.

Since every positive m is necessarily covered by $I_{m_i>0} = \{(m_i, q_i)\}$, if the coverage cannot be completed without the pair (m_t, q_t) with $m_t = L_{q_t}$, then the pair (m_t, q_t) must necessarily be in the family $I_{m_i>0}$. Thus $m_t = L_{q_t}$ replaces the class $m_t = 0$, ensuring the completeness of the coverage; and when the interval is defined as $1 \leq m_t \leq L_{q_t}$, the coverage of all positive m is complete.

Every positive m must be covered by positive representatives $(m_i > 0)$. If the existence of the pair (m_t, q_t) with $m_t = Lq_t$ is required for the positive coverage to be complete, then this pair must necessarily belong to the family $I_{m_i>0} = \{(m_i, q_i)\}$. In the context of Case I, for the equilibrium situation defined at $m = 0$ (i.e., when all $r_i = 2$) one obtains $a_1(0) = 1$, hence there is no q_t that causes non-integrality of $a_1(0)$ and this case does not affect the family $I_{m_i>0}$. By contrast, in the non-equilibrium state where $m = 0$, the prime power satisfying $q_t \mid D_1(0)$ and $q_t \nmid N_1(0)$ defines the residue class $(m_t \bmod L_{q_t})$, represented as $(0, q_t)$ when $m_t = 0$. If this residue class must necessarily be in the family $I_{m_i>0}$ for every positive m to be covered, then instead of $(0, q_t)$, its positive representative is $m_t = L_{q_t} > 0$. Hence the pair $(m_t, q_t) = (L_{q_t}, q_t)$ must necessarily be in the family $I_{m_i>0}$.

Distinct Moduli. For the positive coverage family $I_{m_i>0} = \{(m_i, q_i)\}$, every modulus q_i that satisfies the conditions $D_1(m) = 0$ and $N_1(m) \neq 0$ must be distinct. Suppose $(m_x, q_x), (m_y, q_y) \in I_{m_i>0}$. Assume $m_x \neq m_y$ and $q_x = q_y = q$. From $D_1(m) = 0$ we obtain

$$2^{m_x+2k} \equiv 3^{2k} \pmod q, \qquad 2^{m_y+2k} \equiv 3^{2k} \pmod q.$$

From these congruences it follows that $2^{m_x+2k} \equiv 2^{m_y+2k} \pmod q$.

Since $\gcd(2, q) = 1$, the factor 2^{2k} is invertible modulo q, and cancellation yields

$$2^{m_x} \equiv 2^{m_y} \pmod q.$$

This implies that, with $L = \mathrm{ord}_q(2)$, one must have $m_x \equiv m_y \pmod L$. However, since the representatives are restricted to the interval $1 \leq m_i \leq L$, this congruence in fact forces $m_x = m_y$. This directly contradicts the initial assumption $m_x \neq m_y$. Therefore, it is impossible for two distinct representatives to be assigned to the same modulus q. A fixed modulus q can cover only a single residue class modulo L. Since the $I_{m_i>0}$ family is minimal (irreducible), meaning that each residue class is represented exactly once, such a contradiction cannot occur within the system. Consequently, all moduli q_i in the family $I_{m_i>0}$ must be distinct from one another.

Negative Exponents (m < 0). Expressions with negative exponents are related by modular inversion:

$$\forall m > 0, \quad 2^m \equiv 2^{m_i} \pmod{q_i}, \quad 1 \leq m_i \leq L_{q_i}.$$

$$2^{-m} \equiv (2^m)^{-1} \equiv (2^{m_i})^{-1} \pmod{q_i}.$$

Since $q_i = p_i^{s_i}$ and $p_i > 3$, we have $\gcd(2, q_i) = 1$, and 2 is invertible in $(\mathbb{Z}/q_i\mathbb{Z})^\times$.
Negative exponents are directly related to the operation of multiplicative inversion in modular arithmetic. For any positive integer m, the expression 2^{-m} is congruent modulo q_i to the inverse of 2^m:

$$2^{-m} \equiv (2^m)^{-1} \pmod{q_i}.$$

Since the defect modulus q_i is a prime power $p_i^{s_i}$ with $p_i > 3$, it is relatively prime to 2 ($\gcd(2, q_i) = 1$). This guarantees that 2 is invertible within the multiplicative group $(\mathbb{Z}/q_i\mathbb{Z})^\times$. Consequently, every 2^{m_i} has a multiplicative inverse modulo q_i. By the closure property of cyclic groups, there always exists a positive integer $n_i > 0$ such that,

$$2^{n_i} \equiv (2^{m_i})^{-1} \equiv 2^{L_{q_i}-m_i} \pmod{q_i}, \quad n_i > 0, \quad (1 \leq n_i \leq L_{q_i}).$$

The fact that n_i is a positive integer ($n_i > 0$) constitutes the crucial step of the proof. Since the system covers all positive integers ($\forall m > 0, \quad 2^m \equiv 2^{m_i} \pmod{q_i},\ m_i > 0$) , the resulting value n_i must necessarily be covered by some pair (m_j, q_j) within the family $I_{m_i>0} = \{(m_i, q_i)\}_{i\in K}$:

$$2^{n_i} \equiv 2^{m_j} \pmod{q_j} \quad n_i > 0, \quad (m_j, q_j) \in I_{m_i>0}.$$

Consequently, since $2^{-m} \equiv (2^{m_i})^{-1} \pmod{q_i} \equiv 2^{n_i}$, every negative m is covered by the pair (m_j, q_j):

$$2^{-m} \equiv 2^{n_i} \equiv 2^{m_j} \pmod{q_j}, \quad (m_j, q_j) \in I_{m_i>0}.$$

Additionally, for integers m_i in the interval $1 \leq m_i \leq L_{q_i}$, if there exists a pair (L_{q_t}, q_t) such that $m_t = L_{q_t}$ within the family $I_{m_i>0}$, then the positive values of m belonging to this residue class are obtained as $2^m \equiv 2^{L_{q_t}} \pmod{q_t}$. Hence, the negative values of m belonging to this residue class,

$$2^{-m} \equiv (2^{L_{q_t}})^{-1} \pmod{q_t} \equiv 2^{L_{q_t}} \pmod{q_t}, \quad (L_{q_t}, q_t) \in I_{m_i>0}.$$

This algebraic chain demonstrates that the family $I_{m_i>0}$, which covers all positive values of m, simultaneously covers all negative values of m as well, thereby establishing that the non-integrality defect holds over the entire set of integers excluding zero. This construction guarantees that for every $m < 0$, there exists at least one prime power q_j ($(m_j, q_j) \in I_{m_i>0}$), and under this modulus, $a_1(m) \notin \mathbb{Z}$. Therefore, the result obtained from Case II, "for every positive m, $a_1(m) \notin \mathbb{Z}$," is extended to all negative m due to modular inversion and the closure of the cyclic group. The non-integrality defect is independent of sign and is periodically preserved for all $m \in \mathbb{Z} \setminus \{0\}$:

$$\forall m \in \mathbb{Z} \setminus \{0\}, \quad a_1(m) = \frac{N_1(m)}{D_1(m)} \notin \mathbb{Z}.$$

Moreover, within the cycle $(a_1, a_2, \ldots, a_k, a_1, a_2, \ldots)$, if a_1 is not a positive odd integer, this implies that no such cycle exists among the positive odd integers. Consequently, all elements of the sequence are necessarily non-integral rational numbers.

As a result of examining these three cases, for any positive odd integer a and any sequence $r_1, r_2, \ldots, r_k$ satisfying $\sum_{i=1}^{k} r_i \geq k$, consider the equation

$$\frac{3^k\, a + 3^{k-1} + 3^{k-2}\, 2^{r_1} + 3^{k-3}\, 2^{r_1+r_2} + 3^{k-4}\, 2^{r_1+r_2+r_3} + \cdots + 2^{r_1+r_2+\cdots+r_k-1}}{2^{r_1+r_2+\cdots+r_k}} = a.$$

When $r_1 = r_2 = \cdots = r_k = 2$, we have $a = 1$; otherwise, a cannot be a positive odd integer in any other sequence of r. This result shows that there is no cycle other than 1 in the set of positive odd integers. Therefore, the only cycle in the set of positive integers is the cycle 4, 2, 1.

Divergence. For any positive odd number (s_1) that is not a Collatz number to exist, at least one of the following conditions must be satisfied. I. When Collatz operations (CO) are applied to s_1, there must be a loop other than 1. II. When Collatz operations (CO) are applied to s_1, the resulting sequence must be divergent.
It has been shown that condition I does not exist, i.e. in the set of positive odd integers, there are no cycles other than 1. Now let us consider condition II. If CO is applied to s_1, the resulting sequence of positive odd numbers S$=\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, \ldots\}$. Let's assume that this sequence is divergent and therefore not all positive odd numbers formed from it are Collatz numbers. As shown in Figure 1, continuously applying inverse Collatz operations (CIO) to element s_1 of set S (or s_2 if s_1 is a multiple of 3) and to every element in these sets that is not a multiple of 3 creates an infinite number of sets without Collatz numbers.

$$\{s_1\}$$

$$^{s_1}\mathrm{S}_0 = \Big[\ s_1^* = \{s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}, s_{17}, s_{18} \ldots\}\ \Big]$$

$$^{s_1}\mathrm{S}_1 = \Big[\ s_1^* = \{s_{11}^* = \{s_{111}, s_{112}, s_{113} \ldots\}\ s_{12}^* = \{s_{121}, s_{122}, s_{123} \ldots\}\ s_{13}^* = \{s_{131}, s_{132}, s_{133} \ldots\} \ldots\} \ldots\ \Big]$$

$$^{s_1}\mathrm{S}_2 = \Big[\ s_{11}^* = \{s_{111}^* = \{s_{1111}, s_{1112}, s_{1113} \ldots\}\ s_{112}^* = \{s_{1121}, s_{1122}, s_{1123} \ldots\} \ldots\}\ s_{12}^* = \{s_{121}^* = \{s_{1211}, s_{1212}, s_{1213} \ldots\}\ s_{122}^* = \{s_{1221}, s_{1222}, s_{1223} \ldots\} \ldots\}\ \ldots\ \Big]$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

The family of disjoint sets that are not Collatz number sets formed by s_1:

$$^{s_1}\,\mathrm{S} = \Big[\ \{s_{11}, s_{12}, s_{13}, \ldots\}\ \{s_{111}, s_{112}, s_{113}, \ldots\}\{s_{121}, s_{122}, s_{123}, \ldots\}\ \{s_{131}, s_{132}, s_{133}, \ldots\}\ \{s_{1111}, s_{1112}, \ldots\}\ \{s_{1121}, s_{1122}, \ldots\}\ \ldots\ \Big]$$

Figure 2: Sets that are not Collatz number sets. * represents conversions of numbers that are not multiples of 3 using Equation (7).

For a positive odd integer s: $\mathrm{CO}(s) := \frac{3s+1}{v_2(3s+1)}$, where $v_2(3s+1)$ denotes the highest power of 2 dividing $3s+1$.
Equation (7) defines: $\mathrm{CIO}(s) := \frac{2^n \cdot s - 1}{3}$.
If $s \equiv 1 \pmod 3$, then $n = 2, 4, 6, 8, \ldots$.
If $s \equiv 2 \pmod 3$, then $n = 1, 3, 5, 7, \ldots$.
The elements of each set in Figure 2, obtained by converting each number that is not a Collatz number, form a sequence such that the next term is 4 times the previous term plus 1. Thus, the elements of each set form a loop with remainders 0,1,2 according to $\pmod 3$. New sets are formed continuously to infinity from numbers with remainders 1 and 2 according to $\pmod 3$. In the hierarchical structure shown in Figure 2, the sets in the family of disjoint sets generated from the element s_1 (denoted as ${}^{s_1}S$) can be indexed by a single-component sequence $(a) \in \mathbb{N}$.
Moving to the next stage of the sequence, the s_2 level, the set in the first layer generated by the CIO (equation 7) from s_2 is as follows.

$$\begin{gathered}\{s_2\}\\ \{s_{21}, s_{22}, s_{23}, \ldots\}\end{gathered}$$

Here, if the first element that is not a multiple of 3 is s_{21}, then $s_1 = s_{21}$; this shows that all sets generated from s_1, i.e., all elements of the set ${}^{s_1}S$, are completely covered by the first branch of the structure s_2. In the first layer $\{s_{21} = s_1, s_{22}, s_{23}, \ldots\}$ set generated from s_2 using CIO, each term that is not a multiple of 3, e.g., $s_{22}, s_{23}, \ldots$, forms its own independent infinite hierarchies similar to s_1; for example, if s_{22} is not a multiple of 3, then the sets generated from s_{22} by CIO, similar to those from s_1, can be represented as

$$\begin{gathered}\{s_{22}\}\\ \{s_{221}, s_{222}, s_{223}, \ldots\}\\ \{s_{2211}, s_{2212}, \ldots\}\{s_{2221}, s_{2222}, \ldots\}\{s_{2231}, s_{2232}, \ldots\}\ldots\\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots\end{gathered}$$

Similarly, if s_{23} is not a multiple of 3 (if it's a multiple of 3, then s_{24} is used), then the sets formed from s_{23} with CIO are:

$$\begin{gathered}\{s_{23}\}\\ \{s_{231}, s_{232}, s_{233}, \ldots\}\\ \{s_{2311}, s_{2312}, \ldots\}\{s_{2321}, s_{2322}, \ldots\}\{s_{2331}, s_{2332}, \ldots\}\ldots\\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots\end{gathered}$$

therefore, listing all sets generated from s_2 (${}^{s_2}S$) using CIO requires an indexing space consisting of ordered pairs $(a, b) \in \mathbb{N}^2$.

If we call the family of all sets formed from s_1 by CIO as ${}^{s_1}S$, then the family of all sets formed from s_2 by CIO is called ${}^{s_2}S$, and so on, the following result is obtained:

$$ {}^{s_1}S \subset {}^{s_2}S \subset {}^{s_3}S \subset \cdots \subset {}^{s_n}S. $$

When this hierarchical expansion logic is applied inductively to the sequence $s_1, s_2, s_3, s_4, \ldots, s_n$, an n-element space, $\mathbb{N}^n$, is required for an extensive list of all disjoint sets of any order s_n formed by CIO. If the index n diverges infinitely ($n \to \infty$), the hierarchical depth reaches an unbounded dimension, and infinite sequences of the form $(a_1, a_2, a_3, \ldots) \in \mathbb{N}^{\mathbb{N}}$ are required to list the sets generated from s_n by the CIO. The set $\mathbb{N}^{\mathbb{N}}$, defined in mathematical literature as a Baire space, has an uncountable cardinality ($2^{\aleph_0}$).

All sets generated from any positive odd integer s by the Collatz inverse operation (CIO) in the hierarchical construction are pairwise disjoint. When CIO is applied to s,

$$ x = \frac{2^n s - 1}{3} $$

(where x is a positive odd integer) is obtained, and all sets formed from s are of the form $\{x, 4x+1, 4(4x+1)+1, \ldots\}$, where each subsequent term is 4 times the previous term plus 1.
The general term of these sets is

$$ 4^k x + \frac{4^k - 1}{3}. $$

Let the smallest elements of two different sets formed from s be the initial terms x_1 and x_2. For these two different sets to intersect, the equality

$$ 4^n x_1 + \frac{4^n - 1}{3} = 4^m x_2 + \frac{4^m - 1}{3} $$

must hold; from this,

$$ 4^n (3x_1 + 1) = 4^m (3x_2 + 1) $$

is obtained.
There are two possibilities for this equality to hold. First, when $m = n$, $x_1 = x_2$; that is, the sets are equal. Since we are examining the case where s is an element of a divergent sequence of positive odd numbers, cycles are impossible in divergent sequences. Additionally, when considering s as an element of a non-divergent sequence, the absence of cycles has been proven

above. Therefore, in the family of sets generated from s, no set is generated more than once; that is, the family of sets generated from s contains no equal sets.

The second case is $m \neq n$. In this case, for example,

$$\text{if } (m > n) : x_1 = \frac{4^{m-n}(3x_2+1)-1}{3}, \quad \text{if } (n > m) : x_2 = \frac{4^{n-m}(3x_1+1)-1}{3}$$

is obtained. This means that the starting element of one set is a forward term of the starting element in the sequence of the other set; thus, one set becomes a proper subset of the other (for example, one set is of the form $\{x, 4x+1, 16x+5, 64x+21, \dots\}$, the other set is any tail of the sequence that does not contain x, for $m-n = a \in \mathbb{Z}_{>0}$ (or $n-m = a$), $\{\, 4^k x + \frac{4^k-1}{3} \mid k \geq a \,\}$.

However, this is impossible in the family of sets formed from s in a hierarchical structure as shown in Figure 2. By the restricted hierarchical construction rule, every set begins from a minimal starting element. The set generated from any positive odd integer b starts from the minimal value x; that is, it is of the form $\{x, 4x+1, 16x+5, 64x+21, \dots\}$ and cannot be of the form $\{4^k x + \frac{4^k-1}{3}, \dots\}$ for $k > 0$. Furthermore, let t and b be two distinct positive odd integers; the set obtained from t via CIO cannot be a set that starts from any forward term of the set $\{x, 4x+1, 16x+5, 64x+21, \dots\}$ generated from b. Algebraically, for distinct positive odd integers b and t, the condition

$$\frac{2^n b - 1}{3} = \frac{2^m t - 1}{3}$$

would have to hold; from this, $2^n b = 2^m t$ is obtained, however no such distinct positive odd integers b and t exist that satisfy this equality. Therefore, all sets generated from any positive odd integer s are disjoint.

However, all sets generated from s_n within this hierarchical structure are pairwise disjoint infinite subsets of the set of positive odd integers ($\mathbb{N}_{\text{odd}}$). The cardinality of the set of positive odd integers is countably infinite ($|\mathbb{N}_{\text{odd}}| = \aleph_0$). According to the fundamental principles of set theory, it is structurally impossible to partition a countable universe into uncountably many ($2^{\aleph_0}$) disjoint and non-empty subsets (uncountable partition). The case of a divergent orbit would require the set of integers to accommodate this uncountably dense hierarchical indexing, which leads to a mathematical contradiction. As a result of this structural and cardinal restriction, the index number n of the sequence s_n cannot diverge to infinity, and the sequence must necessarily terminate at a finite value. Consequently, the hierarchical clustering structure and the capacity limits of the set of integers definitively prove that the sequence $s_1, s_2, \dots, s_n$ always remains finite and cannot be divergent.

Thus, when Collatz operations are applied to any positive odd integer, the sequence eventually converges to 1, because there are no loops outside the trivial loop and the resulting sequence does not diverge. Therefore, there cannot be any positive odd integer that is not a Collatz number; all positive odd integers are Collatz numbers. All positive integers are also Collatz numbers (Remark 2.2).

4 Conclusion

In this article, the Collatz conjecture has been proven. It was shown that all positive integers reach 1 as stated in the Collatz Conjecture. With the methods described in this study for $3n+1$, it can be found whether numbers such as $5n+1$, $7n+1$, $9n+1$, $\ldots$ also reach 1.

References

[1] O' Connor, J.J. and Robertson, E.F. (2006). Lothar Collatz. St Andrews University School of Mathematics and Statistics, Scotland.

[2] Lacort, M.O.(2019). Fermat Equation Over Several Fields and Other Historical Mathematical Conjectures. United States: Lulu Press. ISBN:9780244166458

[3] Lagarias, J.C. (1985). The 3x + 1 problem and its generalizations. The American Mathematical Monthly.92(1):3-23.doi:10.1080/00029890.1985.11971528.JSTOR 2322189.

[4] Maddux, C.D. and Johnson, D.L. (1997). The problem is also known by several other names, including: Ulam's conjecture, the Hailstone problem, the Syracuse problem, Kakutani's problem, Hasse's algorithm, and the Collatz problem. Logo: A Retrospective. New York: Haworth Press. p. 160. ISBN 0-7890-0374-0.

[5] Moore, G. H. (2002). Hilbert on the infinite: The role of set theory in the evolution of Hilbert's thought. Historia Mathematica, 29(1), 40-64.

[6] Hausdorff, F. (2021). Set theory (Vol. 119). American Mathematical Society.